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AGRICULTURAL RESEARCH INSTITUTE
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PHILOSOPHICAL TRANSACTIONS

OF THE

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OF

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FOR THE YEAR MDCCCLXIII.

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VOL. 153.—PART I.

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MDCCCLXIII.

ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions*, take this opportunity to acquaint the Public, that it fully appears, as well from the Council-books and Journals of the Society, as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgement of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

The Meteorological Journal hitherto kept by the Assistant Secretary at the Apartments of the Royal Society, by order of the President and Council, and published in the Philosophical Transactions, has been discontinued. The Government, on the recommendation of the President and Council, has established at the Royal Observatory at Greenwich, under the superintendence of the Astronomer Royal, a Magnetical and Meteorological Observatory, where observations are made on an extended scale, which are regularly published. These, which correspond with the grand scheme of observations now carrying out in different parts of the globe, supersede the necessity of a continuance of the observations made at the Apartments of the Royal Society, which could not be rendered so perfect as was desirable, on account of the imperfections of the locality and the multiplied duties of the observer.

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PHILOSOPHICAL TRANSACTIONS.

I. *On the Relation of Radiant Heat to Aqueous Vapour.* *By* JOHN TYNDALL, *F.R.S., Member of the Academies and Societies of Holland, Geneva, Göttingen, Zürich, Halle, Marburg, Breslau, Upsala, la Société Philomathique of Paris, Cam. Phil. Soc. &c.; Professor of Natural Philosophy in the Royal Institution.*

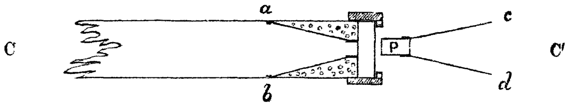
Received November 20,—Read December 18, 1862.

I HAVE already placed before the Royal Society an account of some experiments which brought to light the remarkable fact that the body of our atmosphere, that is to say the mixture of oxygen and nitrogen of which it is composed, is a comparative vacuum to the calorific rays, its main absorbent constituent being the aqueous vapour which it contains. It is very important that the minds of meteorologists should be set at rest on this subject—that they should be able to apply, without misgiving, this newly revealed physical property of aqueous vapour; for it is certain to have numerous and important applications. I therefore thought it right to commence my investigations this year with a fresh series of experiments upon atmospheric vapour, and I now have the honour to lay the results of these experiments before the Royal Society.

Rock-salt is a hygroscopic substance. If we breathe on a polished surface of rock-salt, the affinity of the substance for the moisture of the breath causes the latter to spread over it in a film which exhibits brilliantly the colours of thin plates. The zones of colour shrink and finally disappear as the moisture evaporates. Visitors to the International Exhibition may have witnessed how moist were the pieces of rock-salt exhibited in the Austrian and Hungarian Courts. This property of the substance has been referred to by Professor MAGNUS as a possible cause of error in my researches on aqueous vapour; a film of brine deposited on the surface of the salt would produce the effect which I had ascribed to the aqueous vapour. I will, in the first place, describe a method of experiment by which even an inexperienced operator may avoid all inconvenience of this kind.

In the Plate which accompanies my former paper, the thermo-electric pile is figured with two conical reflectors, both outside the experimental tube; in my present experiments the reflector which faced the experimental tube is placed *within the latter*, its narrow aperture, which usually embraces the pile, abutting against the plate of rock-salt which stops the tube. Fig. 1 is a sketch of this end of the experimental tube. The edge of the inner reflector fits tightly against the interior surface of the tube at *a b*; *c d* is the diameter of the wide end of the outer reflector, supposed to be turned towards the "compensating cube" situated at C′*. The naked face of the pile P is turned towards the plate of salt, being separated from the latter by an interval of about $\frac{1}{20}$th of an inch. The space between the outer surface of the interior reflector and the inner surface of the experimental tube is filled with fragments of freshly-fused chloride of calcium, intended to keep the circumferential portions of the plate of salt perfectly dry. The flux of heat coming from the source C being converged upon the central portion of the salt, completely chases every trace of humidity from the surface on which it falls.

Fig. 1.



With this arrangement I repeated all my former experiments on humid and dry air. The result was the same as before. *On a day of average humidity the quantity of vapour diffused in London air produced upwards of* 60 *times the absorption of the air itself.*

It has been suggested to me that the air of our laboratory might be impure; the suspended carbon particles in a London atmosphere have also been mentioned to me as a possible cause of the absorption which I had ascribed to aqueous vapour. With regard to the first objection, I may say that the same results were obtained when the apparatus was removed to a large room at a distance from the laboratory; and with regard to the second cause of doubt, I met it by procuring air from the following places:—

1. Hyde Park.
2. Primrose Hill.
3. Hampstead Heath.
4. Epsom race-course.
5. A field near Newport, Isle of Wight.
6. St. Catharine's Down, Isle of Wight.
7. The sea-beach near Black Gang Chine.

The aqueous vapour of the air from these localities exerted absorptions from 60 *to* 70 *times that of the air in which the vapour was diffused.*

I then purposely experimented with smoke, by carrying air through a receiver in which ignited brown paper had been permitted to smoulder for a time, and drying it afterwards. It was easy, of course, in this way to intercept the calorific rays; but, confining myself to the lengths of air actually experimented on, I convinced myself that, even when the east wind blows, and pours the carbon of the city upon the west end of London, the heat intercepted by the suspended carbon particles is but a minute fraction of that absorbed by the aqueous vapour.

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* I here assume an acquaintance with my two last contributions to the Philosophical Transactions, in which the method of compensation is described.

Further, I purified the air of the laboratory so well that its absorption was less than unity; the purified air was then conducted through two U-tubes filled with fragments of clean glass moistened with distilled water. Its neutrality when dry proved that all prejudicial substances had been removed from the air; and in passing through the U-tubes it could have contracted nothing save the pure vapour of water. *The vapour thus carried into the experimental tube exerted an absorption* 90 *times as great as that of the air which carried it.*

I have had the pleasure of showing the experiments on atmospheric aqueous vapour to several distinguished men, and among others to Professor MAGNUS. After operating with common undried air, which showed its usual absorption, and while the undried air remained in the experimental tube, I removed the plates of rock-salt from the tube and submitted them to the inspection of my friend. They were as dry as polished rock-crystal, or polished glass; their polish was undimmed by humidity; and a dry handkerchief placed over the finger and drawn across the plates left no trace behind it*.

I would make one additional remark on the above experiments. A reference to the Plate which accompanies my two last papers will show the thermo-electric pile standing, with its two conical reflectors, at some little distance from the end of the experimental tube. Hence, to reach the pile after it had quitted the tube, the heat had to pass through a length of air somewhat greater than the depth of the reflector. It has been suggested to me that the calorific rays may be entirely sifted in this interval—that all rays capable of being absorbed by air may be absorbed in the space of air intervening between the experimental tube and the adjacent face of the pile. If this were the case, then the filling of the experimental tube itself with dry air would produce no sensible absorption. Thus, it was imagined, the neutrality of dry air which my experiments revealed might be accounted for, and the difference between myself and Professor MAGNUS, who obtained an absorption of 12 per cent. for dry air, explained. But I think the hypothesis is disposed of by the foregoing experiments; for here the reflector which separated the pile from the tube no longer intervenes, and it cannot be supposed that in an interval of $\frac{1}{20}$th of an inch of air an absorption of 12 per cent. has taken place. If, however, a doubt on this point should exist, I can state that I have purposely sent radiant heat through an interval of 24 inches of dry air previous to permitting it to enter my experimental tube, and found the effects to be the same as when the beam had traversed 24 inches of a vacuum.

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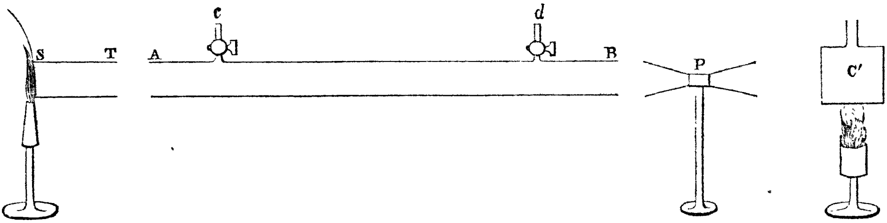
* The present Number of the 'Monatsbericht' of the Academy of Berlin contains an account of some experiments executed with plates of rock-salt by Professor MAGNUS. The plates which stopped the ends of a tube were so far wetted by humid air that the moisture trickled from them in drops. As might be expected, the plates thus wetted cut off a large amount of heat. The experiments are quite correct, but they have no bearing on my results. In the earlier portions of my journal many similar cases are described. In fact, it is by making myself, in the first place, acquainted with the anomalies adduced by Professor MAGNUS, that I have been able to render my results secure. I may add that the communication above referred to was made to the Academy of Berlin before my friend had an opportunity of examining my rock-salt plates. I do not think he would now urge this objection against my mode of experiment.

In confirmation of the results obtained when my tube was stopped by plates of rock-salt, I have recently made the following experiments with a tube in which no plates were used. S is the source of heat, and S T the front chamber which is usually kept exhausted, being connected with the experimental tube at T. This chamber is now left open. A B is the experimental tube, with both its ends also open. P is the thermo-electric pile, the anterior face of which receives rays from the source S, while its posterior surface is warmed by the rays from the compensating cube C′. At *c* and *d* are two stopcocks—that at *c* being connected with an india-rubber bag containing air, while that at *d* is connected with an air-pump.

Fig. 2.



My aim in this arrangement was to introduce at pleasure, into the portion of the tube between *c* and *d*, dry air, the common laboratory air, or air artificially moistened. The point *c*, at which the air entered, was 18 inches from the source S; the point *d*, at which the air was withdrawn, was 12 inches from the face of the pile. By adopting these dimensions, and thus isolating the central portion of the tube, one kind of air may with ease and certainty be displaced by another without producing any agitation either at the source on the one hand, or at the pile on the other.

The tube A B being filled by the common air of the laboratory, and the needle of the galvanometer pointing steadily to zero, dry air was forced gently from the india-rubber bag through the cock *c*; the pump was gently worked at the same time, the dry air being thus gradually drawn towards *d*. On the entrance of the dry air, the needle commenced to move in a direction which showed that a greater quantity of heat was now passing through the tube than before. The dry air proved more transparent than the common air, and the final deflection thus obtained was 41 degrees. Here the needle stopped, and beyond this point it could not be moved by the further entrance of dry air.

Shutting off the india-rubber bag and stopping the action of the pump, the apparatus was abandoned to itself; the needle returned with great slowness to zero, thus indicating a correspondingly slow diffusion of the aqueous moisture through the dry air within the tube. By working the pump the descent of the needle was hastened, and it finally came to rest at zero.

Dry air was again admitted; the needle moved as before, and reached a final limit of 41 degrees; common air was again substituted, and the needle descended to zero.

The tube being filled with the common air of the laboratory, which was not quite saturated, and the needle pointing to zero, air from the india-rubber bag was now forced through two U-tubes filled with fragments of glass wetted with distilled water. The common air was thus displaced by air more fully charged with vapour. The needle moved in a direction which indicated augmented absorption; the deflection obtained in this way was 15 degrees.

I have repeated these experiments hundreds of times, and on days widely distant from each other. I have also subjected them to the criticism of various eminent men, and altered the conditions in accordance with their suggestions. The result has been invariable. The entrance of each kind of air is always accompanied by its characteristic action. The needle is under the most complete control, its motions are steady and uniform. In short, no experiments hitherto made with solids and liquids are more free from caprice, or more certain in their execution, than are the foregoing experiments with dry and humid air.

The quantity of heat absorbed in the above experiments, expressed in hundredths of the total radiation, was found by screening off one of the sources of heat, and determining the full deflection produced by the other and equal source.

By a careful calibration, repeatedly verified, this deflection was proved to correspond to 1200 units of heat,—the unit being, as before, the quantity of heat necessary to move the needle of the galvanometer from 0° to 1°. According to the same standard, a deflection of 41° corresponds to an absorption of 50 units. From these data we immediately calculate the number of rays per hundred absorbed by the aqueous vapour,

$$1200 : 100 = 50 : 4{\cdot}2.$$

An absorption of 4·2 per cent. was therefore effected by the atmospheric vapour which occupied the tube between the points *c* and *d*. Air *perfectly saturated* on the day in question gave an absorption of 5½ per cent.

These results were obtained in the month of September, and on the 27th of October I determined the absorption of aqueous vapour with the above tube when stopped with plates of rock-salt. Three successive experiments gave the deflections produced by the aqueous vapour as 46°·6, 46°·4, 46°·8. Of this concurrent character are all the experiments on the aqueous vapour of the air. The absorption corresponding to the mean deflection here is 66. The total radiation through the exhausted tube was on this day 1085; hence we have

$$1085 : 100 = 66 : 6{\cdot}1;$$

that is to say, the absorption of the aqueous vapour of the air contained in a tube 4 feet long, was on this day 6 per cent. of the total radiation.

The tube with which these experiments were made was of brass, polished within; and it was suggested to me that the vapour of the moist air might have precipitated itself on the interior surface of the tube, thus diminishing its reflective power, and producing an effect apparently the same as absorption. In reply to this objection, I would remark that the air on many of the days on which my experiments were made was at least 25 per cent. under its point of saturation. It can hardly be supposed that air in this condition would deposit its vapour upon a polished metallic surface, against which, moreover, the rays from our source of heat were impinging. More than this, the absorption was exerted even when only a small fraction of an atmosphere was made use of, and found to be proportional to the quantity of atmospheric vapour present in the tube. The following Table shows the absorptions of humid air at tensions varying from 5 to 30 inches:—

Humid Air.

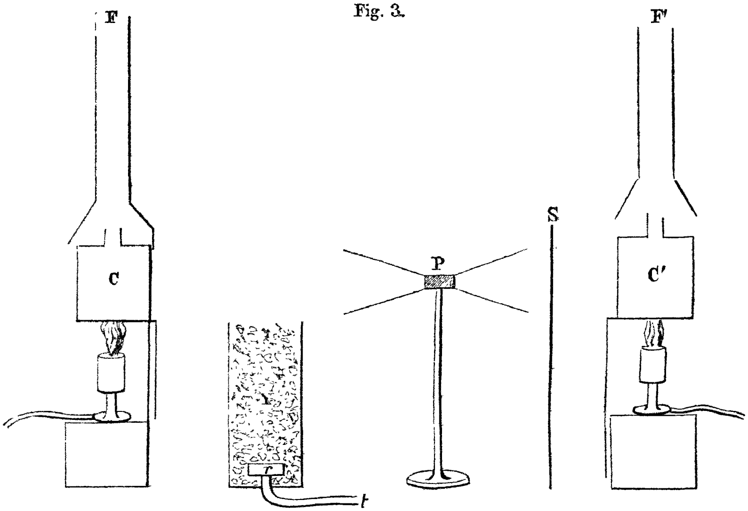
Tensions in inches.	Absorption.	
	Observed.	Calculated.
5	16	16
10	32	32
15	49	48
20	64	64
25	82	80
30	98	96

The third column here is calculated on the assumption that the absorption, within the limits of the experiment, is sensibly proportional to the quantity of matter in the tube. The agreement with observation is almost perfect. It cannot be supposed that results so regular as these, agreeing so completely with those obtained with small quantities of other vapours, and even with small quantities of the permanent gases, can be due to the condensation of vapour on the surface of the tube. When 5 inches were in the tube it had less than one-sixth of the quantity of vapour necessary to saturate the space. Condensation under these circumstances is not to be assumed, and more especially a condensation which should produce such regular effects as those above recorded.

The subject, however, is so important that I thought it worth while to make the following additional experiments:—

C is a cube of boiling water, intended for our source of heat; Y is a hollow brass cylinder, 3·5 inches in diameter and 7·5 inches in depth; P is the thermo-electric pile, and C′ the compensating cube; S is an adjusting screen, used to regulate the amount of heat falling on the posterior surface of the pile. The apparatus was entirely surrounded by boards, the space within being divided by tin screens into compartments which were loosely stuffed with paper or horsehair. The formation of air-currents near the cubes or the pile was thus prevented, and irregular motions of the external air were intercepted. A roof, moreover, was bent over the pile, and this was flanked by sheets of tin. The action here sought I knew must be small, and hence the necessity of excluding every disturbing influence.

Fig. 3.



The cylinder Y was first filled with fragments of quartz moistened with distilled water. A rose burner *r* was placed at the bottom of the cylinder, and from it the tube *t* led to a bag containing air. The bag being subjected to gentle pressure, the air passed upwards amid the fragments of quartz, imbibing moisture from them, and finally discharged itself in the open space between the cube C and the pile. The needle moved and assumed a permanent deflection of 5 degrees, indicating that the opacity of the intervening space to the rays of heat was augmented by the discharge of the saturated air.

The moist quartz fragments were now removed, and the vessel Y was filled with fragments of the chloride of calcium. The rose burner being, as before, connected with the india-rubber bag, air was gently forced up among the calcium fragments and discharged in front of the pile. The needle moved and assumed a permanent deflection of 10 degrees, indicating that the transparency of the space between the pile and source was augmented by the presence of the dry air. By timing the discharges the swing of the needle could be augmented to 20 degrees. Repetition showed no deviation from this result—the saturated air always augmented the opacity, and the dry air always augmented the transparency of the space between the source and the pile.

Not only, therefore, have the plates of rock-salt been abandoned, but also the experimental tube itself, the displacement between dry and humid air being effected in the open atmosphere. The experiments are all perfectly concurrent as regards the action of the aqueous vapour upon radiant heat.

The power of aqueous vapour being thus established, meteorologists may, I think, apply the result without fear. That 10 per cent. of the entire terrestrial radiation is absorbed by the aqueous vapour which exists within ten feet of the earth's surface on a day of average humidity, is a moderate estimate. In warm weather and air approaching to saturation, the absorption would probably be considerably greater. This single fact at once suggests the importance of the established action as regards meteorology. I am persuaded that by means of it many difficulties will be solved, and many familiar effects, which we pass over without sufficient scrutiny because they are familiar, will have a novel interest attached to them by their connexion with the action of aqueous vapour on radiant heat. While leaving these applications to be made in all their fullness by meteorologists, I would refer, by way of illustration, to one or two points on which I think the experiments bear.

And first it is to be remarked that the vapour which absorbs heat thus greedily radiates it very copiously. This fact must, I think, come powerfully into play in the tropical region of calms, where enormous quantities of vapour are raised by the sun, and discharged in deluges upon the earth. This has been assigned to the chilling consequent on the rarefaction of the ascending air. But if we consider the amount of heat liberated in the formation of those falling torrents, the chilling due to rarefaction will hardly account for the entire precipitation. The substance quits the earth as vapour, it returns to it as water; how has the latent heat of the vapour been disposed of? It has in great part, I think, been radiated into space. But the radiation which disposes of such enormous quantities of heat subsequent to condensation, is competent, in some measure at least, to dispose of the heat possessed prior to condensation, and must therefore hasten the act of condensation itself. Saturated air near the surface of the sea is in circumstances totally different from those in which it finds itself in the higher atmospheric regions. Aqueous vapour is a powerful radiant, but it is an equally powerful absorbent, and its absorbent power is a maximum when the body which radiates into it is vapour like itself. Hence, when the vapour first quits the equatorial ocean and ascends, it finds, for a time, a mass of vapour above it, into which it pours its heat, and by which that heat is intercepted and in part returned. Condensation in the lower regions of the atmosphere is thereby prevented. But as the mass ascends it passes through successive vapour-strata which diminish far more speedily in density than the associated strata of air, until finally our ascending body of vapour finds itself lifted above the screen which for a time protected it. It now radiates freely into space, and condensation is the necessary consequence. The heat liberated by condensation is, in its turn, spent in space, and the mass thus deprived of its potential energy returns to the earth as water. To what precise extent this power of aqueous vapour as a radiant comes into play as a promoter of condensation, I will not now inquire; but it must be influential in producing the torrents which are so characteristic of the tropics.

The same remarks apply to the formation of cumuli in our own latitudes. They are the heads of columnar bodies of vapour which rise from the earth's surface and are condensed to cloud at a certain elevation. Thus the visible cloud forms the capital of an invisible pillar of saturated air. Certainly the top of the column, piercing the sea of vapour which hugs the earth, and offering itself to space, must lose heat by the radiation from its vapour, and in this act alone we should have the necessity for condensation. The "vapour plane" must also depend, to a greater or less extent, on the chilling effects of radiation.

The action of mountains as condensers must, I think, be connected with these considerations. When a moist wind encounters a mountain-range it is tilted upwards, and condensation is no doubt to some extent due to the work performed by the expanding air; but the other cause cannot be neglected; for the air not only performs work, but it is lifted to a region where its vapour can freely lose its heat by radiation into space. During the absence of wet winds the mountains themselves also lose their heat by radiation, and are thus prepared for actual surface condensation. We must indeed take into account the fact that this radiant quality of water is persistent throughout its three states of aggregation. As vapour it loses its heat and promotes condensation; as water it loses its heat and promotes congelation; as solid it loses its heat and renders the surfaces on which it rests more powerful refrigerators than they would otherwise be. The formation of a cloud before the air which contains it *touches* a cold mountain, and indeed the formation of a cloud anywhere over a cold tract of land, where the cloud is caused by the cold of the tract, is due to the radiation from the aqueous vapour. The uniformly diffused fogs which sometimes fill the atmosphere in still weather may be due to cold generated by uniform radiation throughout the mass, and not to the mixture of currents of different temperatures. The cloud by which the track of the Nile and Ganges (and sometimes the rivers of our own country) may be followed on a clear morning is, I believe, due to the chilling of the saturated air above the river by radiation from its vapour.

Observation proves the radiation to augment as we ascend a mountain. MARTINS and BRAVAIS, for example, found the lowering of a radiation-thermometer 5°·7 Cent. at Chamouni; while on the Grand Plateau, under the same conditions, it was 13°·4 Cent. The following remarkable passage from HOOKER'S Himalayan Journals, 1st edit. vol. ii. p. 407, bears directly upon this point:—"From a multitude of desultory observations I conclude that, at 7400 feet, 125°·7 or 67° above the temperature of the air, is the average maximum effect of the sun's rays on a black-bulb thermometer. . . . . . These results, though greatly above those obtained at Calcutta, are not much, if at all, above what may be observed on the plains of India [because of the dryness of the air.—J. T.]. The effect is much increased with the elevation. At 10,000 feet, in December, at 9 A.M. I saw the mercury mount to 132° [in the sun], with a difference [above the shaded air] of 94°, while the temperature of shaded snow hard by was 22°. At 13,100 feet, in January, at 9 A.M. it has stood at 98°, with a difference of 68°·2, and at 10 A.M. at 114°, with a difference of 81°·4, *whilst the radiating thermometer on the snow had fallen at sunrise to 0°·7.*" This enormous chilling is fully accounted for by the absence of aqueous vapour overhead. I never under any circumstances suffered so much from heat as in descending on a sunny day from the so-called Corridor to the Grand Plateau of Mont Blanc. The air was perfectly still, and the sun literally blazed against my companion and myself. We were hip deep in snow; still the heat was unendurable. Immersion in the shadow of the Dôme du Goûté soon restored our powers, though the *air* of the shade was not sensibly colder than that through which the sunbeams passed. Notwithstanding the enormous daily accession of heat from the sun, terrestrial radiation at these altitudes preserves an extremely low temperature at the earth's surface.

Without quitting Europe we find places where, even when the day temperature is high, the hour before sunrise is intensely cold. I have often experienced this even in Germany; and the Hungarian peasants, if exposed at night, take care, even in hot weather, to prepare for the nocturnal chill. The *range* of temperature augments with the dryness, and an "excessive climate" is certainly in part caused by the absence of aqueous vapour.

Regarding Central Australia, Mr. MITCHELL publishes extremely valuable tables of observations, from which we learn that, when the days are at the same time calm and clear, the daily thermometric range is exceedingly large. The temperature at noon being 68° on the 2nd of March 1835, that at sunrise next morning was 20°, showing a difference of 48°. The 7th and 8th were also clear and calm; the difference between noon and sunrise on the former day was 38°, while on the latter it was 41°. Indeed between April and September a range of 40° in clear weather was quite common—or more than double the amount which it is in London at the corresponding season of the year.

A freedom of escape similar to that from bodies at great elevations would occur at any other level were the vapour removed from the air above it. Hence the withdrawal of the sun from any region over which the atmosphere is dry, must be followed by quick refrigeration. This is simply an *à priori* conclusion from the facts established by experiment; but I believe all the experience of meteorology confirms it. The winters in Tibet are almost unendurable from this cause. The isothermals dip deeply from the north into Central Asia during the winter, the earth's heat being wasted without impediment in space, and no sun existing sufficiently powerful to make good the loss. I believe the fact is well established that the desert of Sahara, which during the day is burning hot, is often extremely cold at night. This effect has been hitherto referred in a general way to the "purity of the air;" but purity, as judged by the eye, is a very imperfect test of radiation, for the existence of large quantities of vapour is consistent with a transparent atmosphere. The purity really consists in the absence of aqueous vapour from those so-called rainless districts, which, when the sun is withdrawn, enables the hot surface of the earth to run speedily down to a freezing temperature.

On the most serene days the atmosphere may be charged with vapour; in the Alps, for example, it often happens that skies of extraordinary clearness are the harbingers of rain. On such days, no matter how pure the air may seem to the eye, terrestrial radiation is arrested. And here we have the simple explanation of an interesting fact noticed by Sir JOHN LESLIE, which has remained without explanation up to the present time. This eminent experimenter devised a modification of his differential thermometer, which he called an *Æthrioscope*. The instrument consisted of two bulbs united by a vertical tube, of a bore small enough to retain a little liquid index by its own adhesion. The lower bulb was protected by a metallic coating; the upper or sentient bulb was blackened, and was placed in the concavity of a polished metal cup, which protected it completely from terrestrial radiation. "This instrument," says its inventor, "will at all times during the day and night indicate an impression of cold shot downwards from the higher regions. . . . . But the cause of its variations does not always appear so obvious. Under a fine blue sky the *Æthrioscope* will sometimes indicate a cold of 50 millesimal degrees; yet on other days, *when the air is equally bright*, the effect is hardly 30°." It is, I think, certain that these anomalies were due to differences in the amount of aqueous vapour in the air, which escaped the sense of vision. LESLIE himself connects the effect with aqueous vapour by the following remark:—"The pressure [apparently a misprint for *presence*]. of hygrometric moisture in the air probably affects the indications of the instrument." In fact, the moisture opened and closed an invisible door for the radiation of the "sentient bulb" of the instrument into space. The following observation in reference to radiation-experiments with POUILLET's pyrheliometer, now receives its explanation. "In making such experiments," says M. SCHLAGINTWEIT, "deviations in the transparency are often recognized which are totally inappreciable to the telescope or the naked eyes, but which afterwards announce themselves in the presence of thin clouds," &c.

In his beautiful essay on Dew, WELLS gives the true explanation of the formation of ice in India, by ascribing the effect to radiation. I think, however, his theory needs supplementing. Given the same day-temperature here as at Benares, could we, even in clear weather, obtain a sufficient fall of temperature to produce ice? I think not. The interception of the calorific rays by our humid air would too much retard the chill. It is apparent, from the descriptions given of the process, that a dry still air was the most favourable for the formation of the ice. The nights when it was formed in greatest abundance were those during which the dew was not copious. The flat pans used in the process were placed on dry straw, and if the straw became wetted it was necessary to have it removed. WELLS accounts for this by saying that the wetted straw is more dense than the dry, and hence more competent to transfer heat from the earth to the basins. This may be to some extent true; but it is also certain that the evaporation from the moist straw, by throwing over the pans an atmosphere of aqueous vapour, would check the radiation and thus tend to diminish the cold.

MELLONI, in his excellent paper "On the Nocturnal Radiation of Bodies," gives a theory of the *serein*, or excessively fine rain which sometimes falls in a clear sky a few moments after sunset. Several authors, he says, attribute this effect to the cold resulting from radiation of the air during the fine season immediately on the departure of the sun. "But," writes MELLONI, "as no fact is yet known which distinctly proves the emissive power of pure transparent elastic fluids, it appears to me more conformable to the principles of natural philosophy to attribute this species of rain to the radiation and subsequent condensation of a thin veil of vesicular vapour distributed through the higher strata of the atmosphere"*. Now, however, that the power of aqueous vapour as a radiant is known, the difficulty experienced by MELLONI disappears. The former hypothesis, however, though probably correct in ascribing the effect to radiation, was incorrect in ascribing it to the radiation of "*the air.*"

Dr. HOOKER encourages me to hope that this newly discovered action may throw some light on the formation of hail. The wildest and vaguest theories are afloat upon this subject. But the same action which produces *serein* must, if augmented, freeze the minute rain, and the aggregation of the small particles thus frozen would form hail. I cannot think the hail that I have had an opportunity of examining to be due to the freezing of drops of water, each hailstone being merely the ice of the drop. The "stones" are granular aggregates. the components of which may, I think, be produced by the chill of radiation. I will not, however, dwell further on this subject, but will now commit the entire question to those who are more specially qualified for its investigation.

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* TAYLOR's Scientific Memoirs, vol. v. p. 551.

II. *On the Volumes of Pedal Surfaces.* *By* T. A. HIRST, *F.R.S.*

Received August 25,—Read November 20, 1862.

1. IN accordance with the proposition recently made by Dr. SALMON in his excellent treatise on Surfaces*, the term *pedal surface* is here adopted, as the English equivalent of the French *surface-podaire* and the German *Fusspuncts-Fläche*, to indicate the locus of the feet of perpendiculars, let fall from one and the same point in space, upon all the tangent planes of a given *primitive surface.*

The point of contact of the tangent plane, and the foot of the perpendicular upon the latter, are said to be *corresponding points* on the primitive and its pedal. The point whence perpendiculars are let fall may be termed the *pedal-origin.* It is obvious that the pedal surface may likewise be regarded as the envelope of spheres having for their diameters the radii vectores from this origin to the several points of the primitive†.

The primitive surface remaining unaltered, the magnitude and form of the pedal will of course vary with the position of its origin. Between the volumes of all such pedals, however, certain very general and remarkable relations exist. The object of the present paper is to establish these relations.

2. Twenty-four years ago‡ Professor STEINER, in one of his able and purely geometrical memoirs presented to the Academy of Berlin, established analogous relations between the areas of *pedal curves* corresponding to different origins in the plane of the primitive. I am not aware, however, of any attempt having been made to extend his results to surfaces, although such an extension can scarcely have failed to suggest itself, not only to STEINER himself, but to many of his readers§. For the sake of comparison I will here state a few of these results.

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* A Treatise on the Analytic Geometry of Three Dimensions, by G. SALMON, D.D., 1862, p. 369.

† The pedal origin being the same, the surface derived from the pedal, in the same manner as it was derived from the primitive, would be called the *second pedal*; the pedal of this, again, the *third pedal*, and so on. It has, further, been found convenient to apply the term *positive* to the pedals of this series, in order to distinguish them from another series of surfaces obtained by reversing the above process of derivation. Thus the surface of which the primitive is the pedal is termed the *first negative pedal*, and so on. I may also remark that the whole series of positive and negative pedals is identical with the series of *derived surfaces* which forms the subject of papers published by Messrs. TORTOLINI and W. ROBERTS, as well as by myself, in TORTOLINI's 'Annali' and the 'Quarterly Journal of Math.' for 1859. In the present paper first positive pedals are alone considered, though it would no doubt be interesting to examine the volumes of pedals of higher order.

‡ See CRELLE's Journal, vol. xxi. p. 57.

§ Dr. BORCHARDT has quite recently (April 1863) apprized me of the existence of an Inaugural Dissertation, entitled "De superficierum pedalium theorematibus quibusdam," whose publication was sanctioned, in 1859, by the University of Berlin, and in which the two fundamental theorems of art. 3 are established. To English mathematicians, however, the theorems in question will probably be still new, since, so far as I can ascertain, their discoverer, Dr. FISCHER, has never given full publicity to the results of his investigations.

"The primitive curve being closed, but otherwise perfectly arbitrary, the locus of the origins of pedals of constant area is a circle. The several circular loci, corresponding to different areas, are concentric, and their common centre is the origin of the pedal of minimum area."

STEINER signalizes, as a very remarkable mechanical property of this common centre, the fact that it always coincides with the *Krümmungs-Schwerpunct* of the primitive curve,—that is to say, with the centre of gravity of that primitive, regarded as a material curve whose density is everywhere proportional to the curvature.

In 1854, sixteen years after the appearance of STEINER's memoir, Professor RAABE of Zürich[*] extended STEINER's theorem so as to embrace the pedals of unclosed curves. The general definition of the area of a pedal being the space swept by the perpendicular as the point of contact of the tangent describes the primitive arc, RAABE found that "the origins of all pedals of the same area lie on a conic." The several quadric loci, corresponding to different areas, are concentric and co-axal; their common centre is again the origin of the pedal of least area; and though it no longer coincides with the *Krümmungs-Schwerpunct* of the primitive arc, it is intimately connected therewith, as has been more recently shown by Dr. WETZIG of Leipzig[†].

3. With respect to surfaces, the volume of the pedal may be stated, in general terms, to be that of the cone whose vertex is the pedal-origin and whose base is that portion of the pedal surface which corresponds to the given portion of the primitive. This definition being accepted, it will be shown in the sequel that, *whatever the nature of the primitive surface may be, the origins of pedals of equal volume always lie on a surface of the third order;* and further, that *when the primitive surface is closed, but otherwise perfectly arbitrary, this cubic locus degenerates to a quadric, the whole of the loci, corresponding to all possible volumes, then forming a system of similar, similarly placed, and concentric quadrics whose common centre is the origin of the pedal of least volume.*

4. For the sake of comparison it is desirable to treat, by a uniform method, the two analogous questions respecting pedal curves and pedal surfaces. I commence, therefore, with a brief consideration of STEINER's theorem.

Let (C) represent the primitive curve, (P) the pedal whose origin A has the coordinates x, y, and (P_0) the pedal whose origin O coincides with that of the coordinate axes. The curve (C) may be regarded as dividing the plane into two parts, distinguishable as external and internal; let α and β then be the angles, each positive and less than π, between the positive directions of the coordinate axes and that of the normal at any point M of (C), this normal being always supposed to be drawn from the curve into the external part of the plane. Further, let p and p_0 be the perpendiculars let fall respectively from the point A, and from the origin O upon the tangent at M, so that their feet m and m_0 are the points on the pedals (P) and (P_0) which correspond to M on the primitive. The direction-angles of each perpendicular will be

$$\alpha,\ \beta,\ \text{or}\ \pi-\alpha,\ \pi-\beta,$$

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[*] CRELLE's Journal, vol. l. p. 193.

[†] Zeitschrift für Mathematik und Physik, 1860, vol. v. p. 81.

according as its direction coincides with, or is opposed to that of the normal; so that if we regard p and p_0 as positive or negative according as the one or the other of these circumstances occurs, we shall have, generally,

$$p=p_0-x\cos\alpha-y\sin\alpha.$$

If we, further, denote by $d\theta$ the arc of the unit-circle, around the origin, intercepted between radii whose directions coincide with those of the normals at the extremities of the element ds of the primitive arc at M, and agree to consider the parallel elements ds and $d\theta$ as alike or unlike in sign according as their directions coincide with or are opposed to each other, the corresponding elements $d\mathrm{P}$ and $d\mathrm{P}_0$ of the areas of the pedals (P) and (P_0) will be

$$d\mathrm{P}=\frac{p^2d\theta}{2},\quad d\mathrm{P}_0=\frac{p_0^2d\theta}{2},$$

and, by the preceding relation, we shall have

$$2d\mathrm{P}=(p_0-x\cos\alpha-y\sin\alpha)^2d\theta;$$

whence, by integration, we deduce the equation

$$\mathrm{P}=\mathrm{P}_0-\mathrm{A}_1x-\mathrm{A}_2y+\tfrac{1}{2}(\mathrm{A}_{11}x^2+2\mathrm{A}_{12}xy+\mathrm{A}_{22}y^2),\quad\ldots\ldots\quad(\mathrm{A}.)$$

wherein P and P_0 denote the areas of the two pedals, and the coefficients have the values

$$\mathrm{A}_1=\int p_0d\theta.\cos\alpha,\quad \mathrm{A}_2=\int p_0d\theta.\cos\beta,$$

$$\mathrm{A}_{11}=\int d\theta\cos^2\alpha,\quad \mathrm{A}_{12}=\int d\theta\cos\alpha\cos\beta,\quad \mathrm{A}_{22}=\int d\theta\cos^2\beta,$$

dependent only on the position of the origin O, and on the curvature of the primitive curve. The integration in each case is, of course, to be extended to all points of the primitive arc.

5. The above formula, by means of which the area of any pedal (P) may be found when the area of any other (P_0) is known, shows at once that the locus (A) of the origin A of a pedal (P) of constant area is a conic, and that all such loci constitute a system of similar, similarly placed, and concentric conics, the common centre of the loci being the point at which the integrals A_1, A_2 vanish. If we suppose the origin of our coordinate axes to coincide with this point, the equation of the locus (A) may be written thus:

$$\mathrm{P}=\mathrm{P}_0+\tfrac{1}{2}\int(x\cos\alpha+y\cos\beta)^2d\theta,$$

whence we learn that the common centre of all the quadric loci (A) is the origin of the pedal of least area.

6. This is RAABE'S theorem; in order to deduce STEINER'S from it let us consider, in the first place, the pedals of a primitive arc containing a point of inflexion and having parallel normals at its extremities. The normals along such an arc will consist of pairs of like-directed parallels; but in passing from one extremity to the other the sign

of $d\theta$ will change, so that the integrals A_{11}, A_{12}, A_{22} will each consist of equal and opposite elements and vanish in consequence*.

If, now, the primitive be a closed curve, but otherwise perfectly arbitrary, we may always conceive it to consist of arcs (C′) of the kind just considered, and of other arcs (C″) the directions of whose normals represent exactly all possible directions round a point. But it has already been shown that for every arc (C′) the integrals A_{11}, A_{12}, A_{22} vanish, and it is easy to see that, extended over the arcs (C″), these integrals have the values

$$A_{11}=A_{22}=n\pi,\ A_{12}=0,$$

where n represents the number of such arcs, in other words, the number of convolutions of the primitive curve. In this case, therefore, the equation of art. 5 becomes

$$P=P_0+\frac{n\pi}{2}(x^2+y^2)=P_0+\frac{n\pi}{2}r^2,$$

and for constant values of P represents a circle around the origin of the least pedal.

7. In order to illustrate by an example what is meant by the area of a pedal, let us consider for a moment the case of an ellipse with the semiaxes a, b. The focal pedal, as is well known, is a circle whose diameter is the major axis; so that putting for P, n, r^2 the values πa^2, 1, a^2-b^2 respectively, we find, for the area of the central pedal, the value

$$P_0=\frac{\pi}{2}(a^2+b^2),$$

equal to the area of the semicircle whose radius is the line joining the extremities of the axes; and the area of any other pedal is

$$P=\frac{\pi}{2}(a^2+b^2+r^2).$$

For the circle $a=b$, we have

$$P=\pi a^2+\frac{\pi}{2}r^2,$$

which clearly represents the *sum* of the areas of the two loops of which the pedal consists when its origin is without the circle. When a vanishes, the pedal is well known to be the circle on r as diameter. Our last formula shows, however, that we must conceive this circle to be doubled. A glance at the expressions for p and dP in art. 4 explains this distinctive feature of pedal areas. It will be there seen that the sign of the increment dP does not depend upon that of p, which latter changes according as the pedal-origin lies on one or the other side of the tangent. For pedal surfaces, to which we will now proceed, the case is otherwise.

8. Let x, y, z be the coordinates of the origin A of a pedal (P) of a surface (S); and, as before, let (P_0) denote the pedal of the same surface whose origin O coincides with that of the coordinate axes. Then if α, β, γ be the direction-angles of the *external*

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* The locus (A) of equal pedal origins coincides, in this case, with the right line $P=P_0-A_1x-A_2y$, as was first shown by WETZIG.

normal at a point M of (S), and p, p_0 the perpendiculars from the origins of (P) and (P_0) upon the tangent plane at M, we shall, again, have the general relation

$$p=p_0-x\cos\alpha-y\cos\beta-z\cos\gamma,$$

provided the sign of p be understood to depend upon the side of the tangent plane upon which the pedal origin is situated.

Further, let $d\sigma$ be the surface-element of the unit-sphere intercepted by radii having precisely the same directions as the external normals at the contour of the element ds at M on the primitive surface. According to GAUSS's definition $d\sigma$ will also be the *total curvature* of the element ds, and will have the value kds, where k is the *measure of curvature* at M, in other words, the reciprocal of the product of the principal radii of curvature. The volume-element of the pedal P will, obviously, have the value

$$dP=\tfrac{1}{3}p^3d\sigma,$$

and will change sign with p as well as with $d\sigma$. By means of the preceding relation, then, we have

$$3dP=(p-x\cos\alpha-y\cos\beta-z\cos\gamma)^3d\sigma,$$

which expression, when developed and integrated, assumes the form

$$\left.\begin{aligned}P=P_0-(A_1,\ A_2,\ A_3)(x,\ y,\ z)+(A_{11},\ A_{22},\ A_{33},\ A_{23},\ A_{31},\ A_{12})(x,\ y,\ z)^2\\ -\tfrac{1}{3}(A_{111},\ A_{222},\ A_{333},\ A_{112},\ A_{113},\ A_{223},\ A_{221},\ A_{331},\ A_{332},\ A_{123})(x,\ y,\ z)^3,\end{aligned}\right\}\quad .\ .\ .\ \ (A.)$$

where the nineteen coefficients are independent of the position of the pedal origin A, and represent double integrals to be extended to all points of the primitive surface. Of these coefficients it will suffice to write the values of the following six, the remaining thirteen being deducible therefrom by permutations of α, β, γ in accordance with those of the suffixes $_1$, $_2$, $_3$.

$$A_1=\int p_0^2d\sigma\cos\alpha\quad,\quad A_{11}=\int p_0d\sigma\cos^2\alpha\quad,\quad A_{111}=\int d\sigma\cos^3\alpha,$$

$$A_{12}=\int p_0d\sigma\cos\alpha\cos\beta,\quad A_{112}=\int d\sigma\cos^2\alpha\cos\beta,\quad A_{123}=\int d\sigma\cos\alpha\cos\beta\cos\gamma.$$

The above formula for the volume of the pedal (P) at any point A shows at once, as stated in art. 3, that *the origins of pedals of equal volume are situated on a surface of the third order.*

9. The analogy between the cases of pedal curves and surfaces will be evident on observing that the above cubic locus proceeds essentially from the three dimensions of space, just as the quadric locus, in the case of pedal curves, was due to the two dimensions of a plane. It is interesting to note, however, that whilst the hypothesis of a closed primitive *curve* had merely the effect of altering the species, not the order, of the locus (A), the hypothesis of a closed primitive *surface* leads to a reduction of this locus from a cubic to a quadric. The former effect was produced by the equalization of the coefficients of x^2 and y^2, and the vanishing of that of xy (art. 6); the latter is due to

the vanishing of each of the ten integrals A_{111}, A_{112}, &c.... which, not involving p_0, have values dependent solely upon the *curvature* of the primitive surface.

I do not attempt any complete discussion of all possible singularities of curvature, but merely observe that the above-mentioned property of the ten integrals is easily recognized when the primitive surface is not only closed, but everywhere convex; for since all directions round a point are then exactly represented by its normals, the integrals in question each represent a sum of pairs of equal and opposite elements. In the more general case, where certain directions are represented more than once, and consequently an odd number of times, by the normals of the primitive, the property in question may be verified by a method similar to that employed in art. 6.

10. The primitive being a closed surface, the form to which the equation (A.) of art. 8 becomes reduced, at once shows that *the several quadric loci corresponding to pedals* (P) *of different, but constant volumes, constitute a system of similar, similarly situated, and concentric quadrics, their common centre being the origin of the pedal of least volume.* For if this centre, which is determined by the conditions

$$A_1=\int p_0^2 d\sigma \cos\alpha=0, \quad A_2=\int p_0^2 d\sigma \cos\beta=0, \quad A_3=\int p_0^2 d\sigma \cos\gamma=0,$$

were chosen as origin of coordinate axes, the equation (A.) of art. 8 would assume the form

$$P=P_0+(A_{11}, A_{22}, A_{33}, A_{23}, A_{31}, A_{12}\between x, y, z)^2,$$

which may be also written thus,

$$P=P_0+\int p_0(x\cos\alpha+y\cos\beta+z\cos\gamma)^2 d\sigma,$$

in which form it renders apparent the minimum property in question.

When the closed primitive has itself a centre, the latter will also be the common centre of the loci (A); for, the centre of the primitive being taken as origin of coordinate axes, each of the integrals A_1, A_2, A_3 will again consist of pairs of equal and opposite elements.

11. To illustrate the foregoing principles, as well as to facilitate future applications, we will consider for a moment the simplest of all cases—where the primitive is a sphere with radius a. Taking its centre for origin, sixteen of the integrals of art. 8 will be found to vanish, and the remaining ones, A_{11}, A_{22}, A_{33}, to acquire the common value $\frac{4}{3}\pi a$; so that the volume of any pedal (P) becomes

$$P=\tfrac{4}{3}\pi a(a^2+x^2+y^2+z^2)=\tfrac{4}{3}\pi a(a^2+r^2).$$

When the origin of (P) is without the sphere, the pedal consists, of course, of two distinct sheets, each passing through the origin and touching the primitive; the volume of the pedal, as above given, is the *difference* of the volumes enclosed by these sheets. When the sphere diminishes to a point, the volumes of all pedals vanish according to the formula; so that we must regard the pedal of a point as consisting of *two* coincident spherical sheets.

In like manner the pedal of a tubular surface would, in general, consist of distinct sheets which would coincide when the primitive degenerated to a line. Although the pedal surface, therefore, still exists when two dimensions of the primitive are supposed to vanish—being, in fact, still the envelope of spheres whose diameters are the radii vectores of the curve—its volume must be regarded as evanescent.

The case is otherwise, however, when one only of the three dimensions of the primitive is supposed to vanish. Such a surface (S′) would consist of two coincident sheets, and would, therefore, enclose no space; to the eye, in fact, it would not be distinguishable from some definite portion of an ordinary surface. Its pedal, however, would be of a compound nature—consisting, *first*, of a surface (P′) of the same nature as (S′), undistinguishable to the eye from a portion of its ordinary pedal, and, *secondly*, of the simple pedal (P) of the curve (C) forming the contour of the primitive (S′). The *volume* of the compound pedal, however, would be simply that of the pedal (P) of the contour (C). This volume, therefore, properly interpreted, ought to be deducible from our general formulæ.

It must be observed, however, that although the *form* of the pedal of a curve (C) is invariable, its volume must be differently estimated according to the nature of the two-dimensional surface (S′) of which the curve is supposed to form the contour. To render this more evident, it will be convenient to regard the pedal of a curve, not only as the envelope of a sphere, but also as the locus of a circle whose magnitude varies at the same time that its plane rotates about a fixed point, the pedal-origin. This circle, in fact, is the *characteristic* of the pedal; its plane is perpendicular to the tangent at a point M on the curve (C), and its chords, though the origin, are the perpendiculars upon the *several* tangent planes of (S′) at the point M of its contour.

Remembering now the convention of art. 8 with respect to the signs of these perpendiculars, and the relation between the same and those of the corresponding volume-elements, we easily conclude that the volume of the pedal (P) will be the difference of the volumes of the surfaces generated by the segments into which the characteristic circle is divided by the perpendicular p' upon the *ordinary* tangent plane of (S′) at the point M of its contour.

The most interesting case, and the only one we shall examine further, is when the surface (S′) coincides with the developable of which (C) is the cuspidal edge. The perpendicular p' then coincides with that let fall on the osculating plane of the primitive curve (C); through it pass the planes of two consecutive characteristics, and the locus of its extremity is the cuspidal edge of the pedal (P), and at the same time the curve to which, as is well known, the pedal surface of the developable (S′) resolves itself. The volume of the pedal (P) has now the simplest possible definition, and the double integrals of art. 8, by means of which this volume may be expressed, are easily reducible to single ones.

12. To effect this reduction, we will first express any perpendicular p_0 by means of p' the perpendicular on the osculating plane of (C)—parallel therefore to the binormal—and

p the perpendicular on the rectifying plane; the latter will of course be at right angles to p' and parallel to the principal normal. Since p' and p are perpendicular chords of a circle, passing through the same point of its circumference, we have at once

$$p_0 = p' \cos \varphi + p \sin \varphi,$$

where φ is the angle between p' and p_0.

Further, the direction-cosines of p_0, that is to say $\cos \alpha$, $\cos \beta$, $\cos \gamma$, may in like manner be expressed by means of those of p' and p, which we will denote respectively by λ', μ', ν' and λ, μ, ν. For the projections on p_0, p', p of the linear unit, set off on any line through the origin, are clearly, again, chords of a circle, so that, operating successively on the three coordinate axes, we readily deduce the relations

$$\cos \alpha = \lambda' \cos \varphi + \lambda \sin \varphi,$$

$$\cos \beta = \mu' \cos \varphi + \mu \sin \varphi,$$

$$\cos \gamma = \nu' \cos \varphi + \nu \sin \varphi.$$

Lastly, representing by $d\theta$ the angle between the planes of two consecutive characteristics, in other words the *angle of contact* of the primitive curve (C), the surface-element $d\sigma$ of the unit-sphere will have the value

$$d\sigma = \sin \varphi$$

We have now merely to substitute the above values in the several integrals of art. 8, and to effect the integration according to φ between the limits 0 and π, regarding thereby p, p', λ, μ, ν, λ', μ', ν' as constants. This may be readily done; the nineteen results are deducible by appropriate permutations of λ, λ'; μ, μ'; and ν, ν', in accordance with the corresponding suffixes $_1$, $_2$, and $_3$, from the following six expressions:—

$$A_1 = \frac{\pi}{8} \int (3\lambda p^2 + 2\lambda' p p' + \lambda p'^2) d\theta.$$

$$A_{11} = \frac{\pi}{8} \int [(3\lambda^2 + \lambda'^2) p + 2\lambda\lambda' p'] d\theta.$$

$$A_{23} = \frac{\pi}{8} \int [(3\mu\nu + \mu'\nu') p + (\mu\nu' + \mu'\nu) p'] d\theta.$$

$$A_{111} = \frac{3\pi}{8} \int \lambda (\lambda^2 + \lambda'^2) d\theta.$$

$$A_{112} = \frac{\pi}{8} \int [(3\lambda^2 + \lambda'^2) \mu + 2\lambda\lambda' \mu'] d\theta.$$

$$A_{123} = \frac{\pi}{8} \int (3\lambda\mu\nu + \lambda'\mu'\nu + \lambda\mu'\nu' + \lambda'\mu\nu') d\theta.$$

By means of the equations to the curve the nine quantities involved in these integrals are readily expressible as functions of a single variable. This done, the integration in each case is to be extended to all points of the primitive curve (C).

13. I do not enter into the several interesting questions which here suggest them-

selves—as to the nature of the cubic locus of the origins of pedals of the present kind which have a constant volume, the conditions under which this locus degenerates to a quadric, and the position of the origin of the pedal of least volume—but pass at once to the case of a plane primitive curve, every pedal of which will be a surface generated by a circle, through two fixed points, whose magnitude varies at the same time that its plane rotates around the line joining those points. Taking the plane of the primitive as the coordinate plane of xy, we have clearly

$$p'=0,\ \lambda'=\mu'=\nu=0,\ \nu'=1,$$

and consequently

$$A_3=A_{23}=A_{31}=A_{113}=A_{223}=A_{333}=A_{123}=0,$$

$$A_1\ =\frac{3\pi}{8}\int\lambda p^2d\theta,\qquad A_2\ =\frac{3\pi}{8}\int\mu p^2d\theta,$$

$$A_{11}=\frac{3\pi}{8}\int\lambda^2pd\theta,\qquad A_{22}=\frac{3\pi}{8}\int\mu^2pd\theta,$$

$$A_{12}=\frac{3\pi}{8}\int\lambda\mu pd\theta,\qquad A_{33}=\frac{\pi}{8}\int pd\theta,$$

$$A_{111}=\frac{3\pi}{8}\int\lambda^3d\theta,\qquad A_{222}=\frac{3\pi}{8}\int\mu^3d\theta,$$

$$A_{112}=\frac{3\pi}{8}\int\lambda^2\mu d\theta,\qquad A_{221}=\frac{3\pi}{8}\int\lambda\mu^2d\theta,$$

$$A_{331}=\frac{\pi}{8}\int\lambda d\theta,\qquad A_{332}=\frac{\pi}{8}\int\mu d\theta.$$

When the primitive is a plane *closed* curve, the last six integrals, in general, vanish, and the locus of origins of equal pedals again degenerates to a quadric surface. The origin of the least pedal does not generally coincide with the *Krümmungs-Schwerpunct*, since A_1, A_2 have no longer the same values as in art. 4; it coincides with the centre of the primitive, however, whenever the latter possesses such a point. For instance, for a primitive circle (a) it will be found, on taking its centre for origin, that, with the exception of three, all the foregoing integrals vanish, and that these three acquire the values

$$A_{11}=A_{22}=\tfrac{3}{8}\pi^2a,\quad A_{33}=\tfrac{1}{4}\pi^2a.$$

The volume of the central or least pedal, (P_0), which is here a surface generated by the rotation of a circle with radius $\frac{a}{2}$ about one of its tangents, is easily found to be $\frac{1}{4}\pi^2a^3$, so that the volume of any other pedal will, by art. 8, be

$$P=\frac{\pi^2}{8}a(3x^2+3y^2+2z^2+2a^2),$$

and the locus of origins of pedals of the same volume a prolate spheroid.

14. To return to the case of surfaces: I propose to consider next the pedals of the ellipsoid, which, ever since the publication of FRESNEL'S researches on Light, have been

regarded with especial interest. The application to them of the foregoing principles will lead us to several new results.

With a view to this application, and in continuation of the subject of art. 10, I may add that when the primitive surface is symmetrical with respect to three rectangular planes, the integrals A_{12}, A_{23}, A_{31} likewise vanish, on taking these planes of symmetry for coordinate planes. In virtue of this property, which is evident from an inspection of the values in art. 8, the expression for the volume of any pedal assumes the simple form

$$P=P_0+A_{11}x^2+A_{22}y^2+A_{33}z^2.$$

If, further, as in the case of the ellipsoid, *the primitive be a closed convex surface*, the coefficients

$$A_{11}=\int p_0 d\sigma \cos^2\alpha, \quad A_{22}=\int p_0 d\sigma \cos^2\beta, \quad A_{33}=\int p_0 d\sigma \cos^2\gamma$$

will manifestly be sums of elements of the same sign, so that *the locus* (A) *of equal pedal origins will be an ellipsoid* whose axes coincide with the axes of symmetry of the primitive.

15. For the primitive ellipsoid

$$\frac{x^2}{a_1}+\frac{y^2}{a_2}+\frac{z^2}{a_3}=1,$$

the squares of whose semiaxes, written in descending order of magnitude, we will suppose to be a_1, a_2, a_3, we have the well-known formulæ

$$\cos\alpha=\frac{x}{a_1}p_0, \quad \cos\beta=\frac{y}{a_2}p_0, \quad \cos\gamma=\frac{z}{a_3}p_0,$$

$$\frac{1}{p_0^2}=\frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}+\frac{z^2}{a_3^2}=\frac{1}{a_1\cos^2\alpha+a_2\cos^2\beta+a_3\cos^2\gamma},$$

$$3P_0=\int p_0^3 d\sigma=\frac{1}{a_1a_2a_3}\int p_0^7 ds.$$

Both these equivalent expressions for the volume of the central or least pedal have their advantages. In the second the integration is supposed to be extended to all points of the ellipsoid; in the first, after expressing α, β, γ and thence p_0 by means of two suitable independent variables, to all points of the unit sphere. The limits in the latter case will not involve the axes, and by partial differentiation we shall clearly have

$$\frac{\partial P_0}{da_1}=\int p_0^2\frac{\partial p_0}{da_1}d\sigma=\tfrac{1}{2}\int d\sigma\, p_0\cos^2\alpha=\frac{A_{11}}{2},$$

with similar formulæ for A_{22} and A_{33}; so that the volume of any pedal whatever will be given by the formula

$$P=P_0+2\frac{\partial P_0}{da_1}x^2+2\frac{\partial P_0}{da_2}y^2+2\frac{\partial P_0}{da_3}z^2;$$

that is to say, it will be obtained by simple differentiation of the expression for P_0. At the same time it will be observed that P_0, being a homogeneous function of a_1, a_2, a_3 of the degree $\frac{3}{2}$, satisfies, identically, the relation

$$3P_0=2a_1\frac{\partial P_0}{da_1}+2a_2\frac{\partial P_0}{da_2}+2a_3\frac{\partial P_0}{da_3},$$

or, retaining the more convenient symbols A_{11}, A_{22}, A_{33},

$$3P_0 = a_1A_{11} + a_2A_{22} + a_3A_{33}.$$

16. From this, and the general formula for P in art. 14, a very simple relation may be at once established between the volume of the central pedal and that of any other whose origin is on one of the diagonals of the rectangular parallelopiped circumscribed to the ellipsoid. For the coordinates of any point on such a diagonal are given by the equations

$$\frac{x^2}{a_1} = \frac{y^2}{a_2} = \frac{z^2}{a_3} = \frac{r^2}{a},$$

where r is the radius vector to the point, and $a = a_1 + a_2 + a_3$ the square of the semi-diagonal in question. On substituting these values the two formulæ for P and P_0 give

$$P = \frac{a+3r^2}{a} P_0.$$

When $r^2 = a$, the origin of (P) coincides with a corner of the parallelopiped; and when $3r^2 = a$, it is a point on the ellipsoid; so that we may say, *the volume of the pedal whose origin is at any corner of the rectangular parallelopiped circumscribed to the primitive ellipsoid is four times that of the central pedal, and double that of the pedal at any one of the eight points wherein the ellipsoid is pierced by the diagonals of the parallelopiped.*

17. In order to establish further relations we will represent, generally, by x_i, y_i, z_i and r_i the coordinates and radius vector of any point (i) in space, and consider, first, the pedals (P_1), (P_2), (P_3) whose origins are at the extremities $({}_1)$, $({}_2)$, $({}_3)$ of any three conjugate diameters of a quadric (S′) concentric and co-axal with the primitive ellipsoid (S). The squared semiaxes of (S′) being a'_1, a'_2, a'_3 we have, of course,

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= a'_1, \\ y_1^2 + y_2^2 + y_3^2 &= a'_2, \\ z_1^2 + z_2^2 + z_3^2 &= a'_3; \end{aligned}$$

so that by substituting successively, in the general formula for (P), art. 14, the coordinates of the three points under consideration, and adding together the resulting equations, we have

$$P_1 + P_2 + P_3 = 3P_0 + a'_1A_{11} + a'_2A_{22} + a'_3A_{33} = 3\overline{P}.$$

The pedal $(\overline{P})$, whose volume is here put equal to one-third of the *constant* sum of the other three volumes, is easily seen, by the general formula for P, art. 14, to be that whose origin is at one of the points $\left(\sqrt{\frac{a'_1}{3}}, \sqrt{\frac{a'_2}{3}}, \sqrt{\frac{a'_3}{3}}\right)$, where the quadric (S′) is pierced by the diagonals of its circumscribed parallelopiped. If, then, we agree to take the volume of a pedal positively or negatively according as the diameter upon which its origin lies meets the quadric (S′) in real or imaginary points, we may say that *the algebraical sum of the volumes of three ellipsoid-pedals, whose origins are at the extremities of any conjugate diameters of a concentric and co-axal quadric, is constant, and equal to three times the volume of the pedal at the point where this quadric is pierced by a diagonal of its circumscribed rectangular parallelopiped.*

We may add, too, that the sum of the three pedal-volumes corresponding to origins situated at the extremities of conjugate diameters is not only invariable for one and the same quadric (S′), but for all concentric and co-axal quadrics which are inscribed in rectangular parallelopipeds, themselves inscribed in one and the same locus (A) of equal pedal origins. For the axes of all such quadrics clearly satisfy the condition

$$a'_1A_{11}+a'_2A_{22}+a'_3A_{33}=\text{const.}$$

18. When the quadric (S′) is not only concentric and co-axal with the primitive ellipsoid, but also similar to it, the diagonals of their circumscribed rectangular parallelopipeds coincide in direction; so that by art. 16, and putting

$$3r^2=a'_1+a'_2+a'_3=a',$$

the last relation becomes

$$P_1+P_2+P_3=3\overline{P}=3\frac{a+a'}{a}P_0.$$

When $a'=a$, that is to say, when the quadric (S′) coincides with the primitive ellipsoid, we learn that *the sum of the volumes of the three pedals whose origins are at the extremities of any conjugate diameters of the primitive ellipsoid is constant, and equal to six times the volume of the central or least pedal.*

The three pedals whose origins are the vertices of the primitive ellipsoid are, of course, included in this theorem.

19. When the quadric (S′) is a sphere, the conjugate diameters are at right angles to each other, and the diagonals of the circumscribed parallelopiped (cube) are equally inclined to the axes of the ellipsoid; hence *the sum of the volumes of the ellipsoid-pedals whose origins are the three vertices of any tri-rectangular triangle on a concentric sphere is constant, and equal to three times the volume of the pedal at a point on the sphere equidistant from the axes of the ellipsoid.* The value of this constant sum is

$$3P_0+r^2(A_{11}+A_{22}+A_{33}).$$

20. Lastly, when the quadric (S′) is an ellipsoid confocal with the primitive, we may put

$$a'_1-a_1=a'_2-a_2=a'_3-a_3=k^2,$$

and substitute the values of a'_1, a'_2, a'_3 in the general equation of art. 17. By so doing we find

$$P_1+P_2+P_3=6P_0+k^2(A_{11}+A_{22}+A_{33}).$$

Comparing this, therefore, with the expression at the end of the last article, we learn that *the sum of the volumes of the three pedals whose origins are at the extremities of any conjugate diameters of an ellipsoid confocal with the primitive is equal to double the sum of the volumes of the three pedals at the extremities of any three orthogonal diameters of a concentric sphere the square on whose radius is half the difference of the squares on the like-directed semiaxes of the confocals.* Of this general theorem the one at the end of art. 18 is a particular instance, corresponding to the case where the confocal ellipsoids coincide, and consequently $k=0$.

21. From the fundamental formula, written thus,

$$\text{H}=\frac{\text{P}-\text{P}_0}{r^2}=\text{A}_{11}\cos^2\lambda+\text{A}_{22}\cos^2\mu+\text{A}_{33}\cos^2\nu,$$

we may deduce further relations, as well as a construction for the volume of the pedal at any point. In the first place we learn that the *linear* magnitude H is constant at all points of the same radius vector; and secondly, that it is the limit to which $\frac{\text{P}}{r^2}=h$ approaches as the origin of the pedal recedes from the centre. This line h, being the altitude of a parallelopiped (of the same volume as the pedal) having for its base the square on the radius vector, we propose, for convenience of enunciation, to call the *pedal-altitude* at the point under consideration. Thus H will be the pedal-altitude at infinity on the line (λ, μ, ν); A_{11}, A_{22}, A_{33}, respectively, the pedal-altitudes at infinity on the three axes, and $(\text{A}_{11}+\text{A}_{22}+\text{A}_{33})$ that on the line equally inclined to the three axes.

Imagine now a central ellipsoid-pedal $(\dot{\text{P}})$, concentric and co-axal with the primitive, and such that the squares on its semiaxes are respectively proportional to the altitudes A_{11}, A_{22}, A_{33}. It is plain from the last equation that the squares on its radii vectores will be proportional to the pedal-altitudes at infinity on those vectores. The pedal-altitude at infinity on any line being thus determined by the auxiliary pedal $(\dot{\text{P}})$, that at any other point on the same line is easily found, and thence also the parallelopiped, equal in volume to the pedal which has that point for origin.

22. Between the pedal-altitudes at different points in space numerous relations might be established; we shall limit ourselves to one or two. Let $(_1)$, $(_2)$, $(_3)$ now denote the extremities of any three diameters, at right angles to each other, of the concentric and co-axal quadric (S′) before considered. Then the addition of the three formulæ (similar to the one last written) which refer to these extremities gives

$$h_1+h_2+h_3=\text{P}_0\left(\frac{1}{a'_1}+\frac{1}{a'_2}+\frac{1}{a'_3}\right)+\text{A}_{11}+\text{A}_{22}+\text{A}_{33}=3\bar{h},$$

since by a well-known theorem

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{1}{a'_1}+\frac{1}{a'_2}+\frac{1}{a'_3}.$$

The pedal-altitude $\bar{h}$, which is here put equal to the *constant* arithmetic mean of the other three, corresponds to the point on the quadric (S′) which is equidistant from its three axes, as may be easily seen by putting, in the formula of art. 21,

$$\cos^2\lambda=\cos^2\mu=\cos^2\nu=\tfrac{1}{3},$$

and observing that for such a point

$$\frac{3}{r^2}=\frac{1}{a'_1}+\frac{1}{a'_2}+\frac{1}{a'_3}.$$

Hence *the algebraical sum of the three pedal-altitudes at the extremities of any three orthogonal diameters of a quadric, concentric and co-axal with the primitive ellipsoid, is constant, and equal to three times the pedal-altitude at the extremity of a diameter of the quadric equally inclined to its axes.*

We may add, too, that *this sum is not only invariable for one and the same quadric* (S′), *but for all concentric and co-axal quadrics which pass through one and the same point, equidistant from the principal diametral planes of the primitive ellipsoid.* The quadric (S′) being a sphere, the pedal-altitudes at its several points are, of course, proportional to the pedal-volumes; so that we obtain again the theorem of art. 19.

23. Before proceeding to the actual calculation of the volume of an ellipsoid-pedal, we may remark, lastly, that for any four origins situated on a concentric and co-axal quadric the corresponding pedal-volumes satisfy the relation

$$\begin{vmatrix} P_1, & x_1^2, & y_1^2, & z_1^2 \\ P_2, & x_2^2, & y_2^2, & z_2^2 \\ P_3, & x_3^2, & y_3^2, & z_3^2 \\ P_4, & x_4^2, & y_4^2, & z_4^2 \end{vmatrix} = 0,$$

into the geometrical meaning of which, however, we will, at present, not inquire further.

24. I propose to show, in the next place, that the volume of any pedal may be expressed, symmetrically, by means of the first partial differential coefficients of the definite integral

$$V = \int_0^\infty \frac{dv}{\sqrt{(v+a_1)(v+a_2)(v+a_3)}}.$$

It is well known that when the coordinates x, y, z of any point of a surface are regarded as functions of two independent variables ϕ and v, we have the following equivalent expressions for three times the volume of the pyramid whose vertex is the coordinate origin, and base the surface-element ds enclosed between the curves $\phi=const.$, $v=const.$ and their respective consecutives:

$$p_0 ds = \begin{vmatrix} x, & y, & z \\ \frac{\partial x}{d\phi}, & \frac{\partial y}{d\phi}, & \frac{\partial z}{d\phi} \\ \frac{\partial x}{dv}, & \frac{\partial y}{dv}, & \frac{\partial z}{dv} \end{vmatrix} dv\, d\phi.$$

25. Now the equation of the primitive ellipsoid will obviously be satisfied, identically, by the assumptions

$$x^2 = a_1 \frac{v}{v+a_3} \cos^2\phi,$$

$$y^2 = a_2 \frac{v}{v+a_3} \sin^2\phi,$$

$$z^2 = a_3 \frac{a_3}{v+a_3};$$

which, when substituted in the above determinant and in the expression for p_0 given in art. 15, lead at once to the expressions

$$p_0 ds = -\frac{a_3\sqrt{a_1 a_2}}{2} \cdot \frac{dv\, d\phi}{(v+a_3)^{\frac{3}{2}}}.$$

$$\frac{1}{p_0^2} = \left[\frac{v+a_1}{a_1}\cos^2\phi + \frac{v+a_2}{a_2}\sin^2\phi\right]\frac{1}{v+a_3}.$$

Substituting these values in the second expression for P_0, given in art. 15, extending the integration over the ellipsoid-octant,—whereby the limits of ϕ will clearly be 0 and $\frac{\pi}{2}$, whilst those of v will be 0 and ∞,—and taking eight times the result, we have

$$3P_0 = -\frac{4}{\sqrt{a_1a_2}}\int_0^\infty\int_0^{\frac{\pi}{2}} \frac{(v+a_3)^{\frac{3}{2}}dv.d\phi}{\left[\frac{v+a_1}{a_1}\cos^2\phi+\frac{v+a_2}{a_2}\sin^2\phi\right]^3},$$

whence, by differentiation, we deduce

$$A_{33} = 2\frac{\partial P_0}{da_3} = -\frac{4}{\sqrt{a_1a_2}}\int_0^\infty\int_0^{\frac{\pi}{2}} \frac{(v+a_3)^{\frac{1}{2}}dv\, d\phi}{\left[\frac{v+a_1}{a_1}\cos^2\phi+\frac{v+a_2}{a_2}\sin^2\phi\right]^3}.$$

The integration according to ϕ presents no difficulty, and when effected gives the result

$$A_{33} = \frac{\pi}{4}\int_0^\infty \left[\frac{3a_1^2}{(v+a_1)^2}+\frac{2a_1a_2}{(v+a_1)(v+a_2)}+\frac{3a_2^2}{(v+a_2)^2}\right]\frac{(v+a_3)dv}{\sqrt{R}},$$

where, for brevity, we have put

$$R = (v+a_1)(v+a_2)(v+a_3).$$

26. A more convenient form can be given to the above expression for A_{33} by introducing the partial differential coefficients of the two symmetrical integrals

$$V = \int_0^\infty \frac{dv}{\sqrt{R}}, \quad W = \int_0^\infty \frac{vdv}{\sqrt{R}}.$$

In fact if, for brevity, we indicate the results of the operations

$$\frac{\partial}{da_1}, \ \frac{\partial}{da_2} \ldots\ldots \frac{\partial^2}{da_1^2}, \ \frac{\partial^2}{da_1da_2}, \ldots\ldots$$

performed on any subject, by giving to the symbol of that subject the suffixes $_1$, $_2$, ... $_{11}$, $_{12}$, ..., we shall have

$$\frac{1}{\pi}A_{33} = a_3(a_1^2V_{11}+2a_1a_2V_{12}+a_2^2V_{22})+a_1^2W_{11}+2a_1a_2W_{12}+a_2^2W_{22},$$

as may be easily verified.

27. This expression, however, may itself be resolved into a simpler one involving V_1, V_2, V_3 alone. To effect this resolution we may observe that, in virtue of the identity

$$\frac{dR}{dv} = R_1+R_2+R_3,$$

we have

$$W_1+W_2+W_3 = -\tfrac{1}{2}\int_0^\infty (R_1+R_2+R_3)\frac{vdv}{\sqrt{R^3}} = \int_0^\infty v d\left(\frac{1}{\sqrt{R}}\right);$$

from which, by partial integration, we deduce

$$W_1+W_2+W_3 = -V,$$

since $\frac{v}{\sqrt{R}}$ clearly vanishes at both limits. From this expression, again, we obtain by differentiation the relations

$$W_{11}+W_{12}+W_{13}=-V_1,$$
$$W_{12}+W_{22}+W_{23}=-V_2,$$
$$W_{13}+W_{23}+W_{33}=-V_3.$$

By subjecting the integral V to a precisely similar treatment, it will be found that

$$\begin{aligned} V_1+V_2+V_3&=-\frac{1}{\sqrt{a_1a_2a_3}}, \\ &=-2a_1(V_{11}+V_{12}+V_{13}), \\ &=-2a_2(V_{12}+V_{22}+V_{23}), \\ &=-2a_3(V_{13}+V_{23}+V_{33}). \end{aligned}$$

Further, since

$$V_1=-\tfrac{1}{2}\int_0^\infty \frac{1}{v+a_1}\cdot\frac{dv}{\sqrt{R}}, \text{ \&c.} \ldots$$

$$V_{12}=\tfrac{1}{4}\int_0^\infty \frac{1}{(v+a_1)(v+a_2)}\,\frac{dv}{\sqrt{R}}, \text{ \&c.} \ldots$$

we have, on resolving the coefficients of $\frac{dv}{\sqrt{R}}$ in V_{12}, V_{23}, V_{31} into partial fractions,

$$2(a_1-a_2)V_{12}=V_1-V_2,$$
$$2(a_2-a_3)V_{23}=V_2-V_3,$$
$$2(a_3-a_1)V_{31}=V_3-V_1;$$

and in like manner we also find that

$$2(a_1-a_2)W_{12}=a_2V_2-a_1V_1,$$
$$2(a_2-a_3)W_{23}=a_3V_3-a_2V_2,$$
$$2(a_3-a_1)W_{31}=a_1V_1-a_3V_3.$$

28. Now the last four groups of equations clearly suffice for the expression of A_{33} (art. 26) in terms of V_1, V_2, V_3, and thence, by mere permutations of suffixes, we may obtain the values of A_{11}, A_{22}. The results, after due simplification, may be thus written:

$$A_{11}=-\frac{\pi}{2}\Big[(\ a_2+\ a_3)a_1V_1+(\ a_3+2a_1)a_2V_2+(2a_1+\ a_2)a_3V_3\Big],$$

$$A_{22}=-\frac{\pi}{2}\Big[(2a_2+\ a_3)a_1V_1+(\ a_3+\ a_1)a_2V_2+(\ a_1+2a_2)a_3V_3\Big],$$

$$A_{33}=-\frac{\pi}{2}\Big[(\ a_2+2a_3)a_1V_1+(2a_3+\ a_1)a_2V_2+(\ a_1+\ a_2)a_3V_3\Big].$$

From these values of the pedal-altitudes at infinity on each of the axes (art. 21) we obtain, by addition, the following value of the pedal-altitude at infinity on a line equally inclined to these axes:

$$A_{11}+A_{22}+A_{33}=-2\pi[(a_2+a_3)a_1V_1+(a_3+a_1)a_2V_2+(a_1+a_2)a_3V_3].$$

Again, in virtue of the relation at the end of art. 15, we at once deduce the following expression for the volume of the central pedal of the ellipsoid,

$$P_0=-\frac{\pi}{2}\Big[m_1a_1V_1+m_2a_2V_2+m_3a_3V_3\Big];$$

if, for brevity, we put

$$3m_1=(a_1+a_2+a_3)(a_2+a_3)+a_2^2+a_3^2,$$

$$3m_2=(a_1+a_2+a_3)(a_3+a_1)+a_3^2+a_1^2,$$

$$3m_3=(a_1+a_2+a_3)(a_1+a_2)+a_1^2+a_2^2.$$

Lastly, for the volume of any pedal (P) whose origin A is at x, y, z, we have the expression

$$P=-\frac{\pi}{2}\Big[M_1.a_1V_1+M_2.a_2V_2+M_3.a_3V_3\Big];$$

where again, for brevity, we put

$$3M_1=(3r^2+a)(a_2+a_3)+3(a_2y^2+a_3z^2)+a_2^2+a_3^2,$$

$$3M_2=(3r^2+a)(a_3+a_1)+3(a_3z^2+a_1x^2)+a_3^2+a_1^2,$$

$$3M_3=(3r^2+a)(a_1+a_2)+3(a_1x^2+a_2y^2)+a_1^2+a_2^2,$$

r^2 and a being, as usual, abbreviations for $x^2+y^2+z^2$ and $a_1+a_2+a_3$. The volume of the primitive ellipsoid, when expressed by means of V_1, V_2, V_3, is

$$S=-\frac{4\pi}{3}\Big[a_2a_3.a_1V_1+a_3a_1.a_2V_2+a_1a_2.a_3V_3\Big],$$

as is at once evident from one of the relations in art. 27. The integral V itself, when thus expressed, has the value

$$V=-2[a_1V_1+a_2V_2+a_3V_3];$$

for it may readily be shown to be a homogeneous function of a_1, a_2, a_3 of the degree $-\frac{1}{2}$.

I do not dwell upon the many interesting expressions of S and V by means of pedal-volumes, but proceed at once to the expression of the foregoing results by means of elliptic integrals.

29. The integral V, by means of whose partial differential coefficients the volumes of all pedals have been expressed, is at once converted into an elliptic integral of the first kind by the substitution

$$\sin^2\varphi=\frac{a_1-a_3}{v+a_1},$$

whereby the limits 0 and ∞ of v will correspond, respectively, to the limits θ and 0 of ϕ, provided

$$\theta=\cos^{-1}\sqrt{\frac{a_3}{a_1}}=\sin^{-1}\sqrt{\frac{a_1-a_3}{a_1}}.$$

The result of this substitution is easily found to be

$$V=\frac{2}{\sqrt{a_1-a_3}}\int_0^\theta\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}=2\,\frac{F}{\sqrt{a_1-a_3}},$$

where

$$k^2=\frac{a_1-a_2}{a_1-a_3}$$

is clearly positive and less than unity.

Representing also, with LEGENDRE, by E the elliptic integral, of the second kind,

$$E(\theta,k)=\int_0^\theta d\phi\sqrt{1-k^2\sin^2\phi},$$

and differentiating the preceding value of V, it will be found that

$$\begin{aligned}
V_1&=-\frac{1}{a_1-a_2}\cdot\frac{F}{\sqrt{a_1-a_3}}+\frac{1}{a_1-a_2}\cdot\frac{E}{\sqrt{a_1-a_3}},\\
V_2&=\frac{a_3}{a_2-a_3}\,\frac{1}{\sqrt{a_1a_2a_3}}+\frac{1}{a_1-a_2}\cdot\frac{F}{\sqrt{a_1-a_3}}-\frac{a_1-a_3}{(a_1-a_2)(a_2-a_3)}\cdot\frac{E}{\sqrt{a_1-a_3}},\\
V_3&=\frac{-a_2}{a_2-a_3}\,\frac{1}{\sqrt{a_1a_2a_3}}+\frac{1}{a_2-a_3}\cdot\frac{E}{\sqrt{a_1-a_3}}.
\end{aligned}$$

By substituting these values in the formulæ of art. 20, we might at once obtain the values of A_{11}, A_{22}, A_{33}, P_0, and P expressed in elliptic integrals. Since the volume of any pedal (P), however, may be deduced from that of the central pedal (P_0) by mere differentiation (art. 15), the following complete expression for P_0 will here suffice:—

$$P_0=\frac{\pi}{6}\left[(2a-a_1)\sqrt{\frac{a_2a_3}{a_1}}+(aa_3+a_3^2-a_1a_2)\,\frac{F}{\sqrt{a_1-a_3}}+2(a_1-a_3)a\,\frac{E}{\sqrt{a_1-a_3}}\right].$$

This expression, I may add, agrees precisely with the one first obtained by Professor TORTOLINI in 1844*.

30. If we allow a_3 to diminish indefinitely, the amplitude θ approaches the limit $\frac{\pi}{2}$, and the modulus k acquires the value

$$k_1=\sqrt{\frac{a_1-a_2}{a_1}}.$$

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* CRELLE's Journal, vol. xxxi. p. 28. At the time the present paper was communicated to the Royal Society I was under the impression that the central pedal of the ellipsoid was the only one whose volume had hitherto been calculated. I have since found that Dr. MAGENER, in a paper "On the Cubature of Ellipsoid-pedals" (GRUNERT's Journal, t. xxxiv. 1860), first gave the complete expression for P in elliptic integrals. Although the simple relation between P_0 and P, above referred to, appears to have escaped Dr. MAGENER's notice, it is due to him to state that he not only determined the loci (A) of the origins of ellipsoid-pedals of equal volume, but also succeeded in giving to P a very interesting and symmetrical form, by introducing the partial differential coefficients of the well-known double integral to which JACOBI, in 1833 (CRELLE's Journal, vol. x.), reduced the quadrature of the reciprocal of the primitive ellipsoid.

The elliptic functions E and F thus become transformed into the complete ones $E\left(\frac{\pi}{2}, k_1\right)$ and $F\left(\frac{\pi}{2}, k_1\right)$, or more simply, E_1 and F_1.

Representing generally by [U] the limit to which any function U approaches when a_3 diminishes indefinitely, we deduce from the expressions in art. 29 the limiting values

$$[V_1]=\frac{1}{a_1-a_2}\cdot\frac{E_1}{\sqrt{a_1}}-\frac{1}{a_1-a_2}\cdot\frac{F_1}{\sqrt{a_1}},$$

$$[V_2]=-\frac{a_1}{a_2}\cdot\frac{1}{a_1-a_2}\cdot\frac{E_1}{\sqrt{a_1}}+\frac{1}{a_1-a_2}\cdot\frac{F_1}{\sqrt{a_1}},$$

$$[V_3]=\infty, \quad [V_3\sqrt{a_3}]=-\frac{1}{\sqrt{a_1a_2}}, \quad [a_3V_3]=0.$$

$$[P_0]=\frac{\pi}{6}\sqrt{a_1}\{2(a_1+a_2)E_1-a_2F_1\}.$$

This last is the volume of the central pedal surface of an ellipse (art. 13). By substitution in art. 28, it will be found that the volume of any other pedal of this curve is given by the formula

$$[P]-[P_0]=\frac{\pi}{2}\frac{\sqrt{a_1}}{a_1-a_2}\left\{[(2a_1-a_2)E_1-a_2F_1]x^2+[(a_1-2a_2)E_1+a_2F_1]y^2+(a_1-a_2)E_1z^2\right\},$$

to which expression we should have been led at once had we sought, directly, the values of A_{11}, A_{22}, A_{33} as exhibited in art. 13. In fact when $a_1=a_2$, the above formula may be easily reduced to the one already found in art. 13 for the volume of the pedal surface of a circle.

31. I give, lastly, the modifications of the preceding formulæ which correspond to the special cases of ellipsoids of rotation.

For the prolate spheroid $a_2=a_3$, and

$$V=\frac{2}{\sqrt{a_1-a_3}}\log\left[\sqrt{\frac{a_1-a_3}{a_3}}+\sqrt{\frac{a_1}{a_3}}\right],$$

$$P_0=\frac{\pi}{6}\left\{(2a_1+3a_3)\sqrt{a_1}+\frac{3a_3^2}{\sqrt{a_1-a_3}}\log\left[\sqrt{\frac{a_1-a_3}{a_3}}+\sqrt{\frac{a_1}{a_3}}\right]\right\},$$

$$P=P_0+2\frac{\partial P_0}{da_1}x^2+\frac{\partial P_0}{da_3}(y^2+z^2).$$

At either focus $x^2=a_1-a_3$, $y=z=0$, and the volume of (P) becomes

$$P=P_0+2\frac{\partial P_0}{da_1}(a_1-a_3)=\frac{4\pi}{3}a_1\sqrt{a_1},$$

which is, of course, the volume of the sphere whose diameter is the major axis of the generating ellipse.

For the oblate spheroid $a_1=a_2$, and hence

$$V=\frac{2}{\sqrt{a_1-a_3}}\cos^{-1}\sqrt{\frac{a_3}{a_1}},$$

$$P_0=\frac{\pi}{6}\left[(3a_1+2a_3)\sqrt{a_3}+\frac{3a_1^2}{\sqrt{a_1-a_3}}\cos^{-1}\sqrt{\frac{a_3}{a_1}}\right],$$

$$P=P_0+\frac{\partial P_0}{da_1}(x^2+y^2)+2\frac{\partial P_0}{da_3}z^2;$$

which last formula, when $a_3=0$, is also reducible to the last formula in art. 13 for the volume of any pedal (P) of a circle, regarded as the limit of a surface, one of whose dimensions has been allowed to diminish indefinitely.

III. *On the* Archeopteryx *of* von Meyer, *with a description of the Fossil Remains of a Long-tailed species, from the Lithographic Stone of Solenhofen. By Professor* Owen, *F.R.S. &c.*

Received November 6,—Read November 20, 1862.

The first evidence of a Bird in strata of the Oxfordian or Corallian stage of the Oolitic series was afforded by the impression of a single feather, in a slab of the lithographic calcareous laminated stone, or slate, of Solenhofen; it is described and figured with characteristic minuteness and care by M. Hermann von Meyer, in the fifth part of the 'Jahrbuch für Mineralogie*.' He applies to this fossil impression the term *Archeopteryx lithographica*; and although the probability is great that the class of Birds was represented by more than one genus at the period of the deposit of the lithographic slate, and generic identity cannot be predicated from a solitary feather, I shall assume it in the present instance, and retain for the genus, which can now be established on adequate characters, the name originally proposed by the distinguished German palæontologist †.

At the Meeting of the Mathematico-Physical Class of the Royal Academy of Sciences of Munich, on the 9th of November, 1861, Professor Andreas Wagner communicated the discovery, in the lithographic slate of Solenhofen, of a considerable portion of the skeleton of an animal with impressions of feathers radiating fanwise from each anterior limb, and diverging obliquely in a single series from each side of a long tail.

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* 1861, p. 561.

† A specific diagnosis deduced from the characters of a single feather presupposes that such characters are common to every feather of the bird so defined, and the impression of a second feather differing greatly in its shape and proportions, as in Plate IV. fig. 8, would represent a distinct species in Palæontology; otherwise the characters afforded by a feather cannot be held to be distinctive of a species.

From the number of species of *Pterodactylus*, some having short, some having long tails, in the lithographic slate of Bavaria, it is probable that there may have been different species of *Archeopteryx* so characterized: the future possible discovery of a short-tailed *Archeopteryx* with impressions of feathers corresponding with that of the *Archeopteryx lithographica*, v. Meyer, would impose upon its describer the duty of applying a new specific name to the long-tailed *Archeopteryx* with the differently-shaped feathers, to which the name *lithographica* would thus prove to have been wrongly applied. Moreover, as winged reptiles are not peculiar to the lithographic modification of oolitic deposits, the term *lithographica* may prove as little distinctive of an *Archeopteryx* as of a *Pterodactylus*.

On these grounds the author distinguished in his original communication, as in the Catalogue of the Fossils in the British Museum, the species of *Archeopteryx*, indicated by the specimen which, for the first time, has yielded any knowledge of the specific characters of one of the genus, by the term expressive of the best-marked of those characters, and by which *Archeopteryx macrura*, Ow., differs most conspicuously rom every other known species of bird.

These and other particulars of the fossil Professor Wagner gave on the authority of M. Witte, Law-Councillor (Oberjustiz-Rath) in Hanover, who had seen the fossil in the possession of M. Häberlein, District Medical Officer (Landarzt) of Pappenheim.

Upon the report thus furnished to him, Professor Wagner proposed for the remarkable fossil the generic name *Griphosaurus*, conceiving it to be a long-tailed Pterodactyle with feathers. His state of health prevented his visiting Pappenheim for a personal inspection of the fossil; and, unfortunately for palæontological science, which is indebted to him for many valuable contributions, Professor Wagner shortly after expired.

I thereupon communicated with Dr. Häberlein, and reported on the nature and desirability of the fossils in his possession to the Trustees of the British Museum: they were accordingly inspected by my colleague Mr. Waterhouse, F.Z.S.; and an interesting and instructive selection, including the subject of the present paper, has been purchased for the Museum.

The specimen is divided between the counterpart halves of a split slab of lithographic stone: the moiety (Plate I.) containing the greater number of the petrified bones exhibits such proportion of the skeleton from the inferior or ventral aspect.

The lower half of an arched furculum (merry-thought, 58) marks, by its relative position to the wings, the fore part of the trunk. From this portion of the furculum to the root of the tail measures 4½ inches; the length of the caudal series of vertebræ (*Cd*, *Cd′*) is 8 inches; but the terminal tail-feathers extend 3 inches further, making the length of the tail 11 inches. From the end of the tail to the anterior border of the wing-feather impressions is 1 foot 8½ inches. From the outer border of the impression of the left wing (*d*) to that of the right wing measures 1 foot 4 inches. The front margin of the slab of stone has been broken away short of the anterior border of the impression of the outspread left wing, and the head or skull of the specimen may have been included in that part of the quarry or stone from which the present slab has been detached. The preserved parts of the feathered creature indicate its size to have been about that of a Rook or Peregrine Falcon. The exposed bones on one moiety of the split slab (Plate I.) are—

The lower portion of the furculum (58) above mentioned.

Portion of the left os innominatum, showing part of the ilium (62) and ischium (63), with the acetabulum (*a*).

Twenty caudal vertebræ (*Cd*) in a consecutive and naturally articulated series.

Several slender curved ribs (*pl*), most of them sternal (*h*), irregularly scattered about the region of the trunk.

Left scapula (51′).

Proximal half of left humerus (53′), entire, and part of the distal half.

Left radius (54′) and ulna (55′).

Left carpus (56′) and portion of a metacarpal bone (57′).
Right scapula (51).
Right humerus (53), wanting part of the bony wall and the proximal end.
Right radius (54) and ulna (55).
Two metacarpal bones (57).
Two unguiculate phalanges (I and II).
Right femur (65), tibia (66), and bones of the foot (68, *i*, *ii*, *iii*, *iv*).
Left femur (65′) and tibia (66′).

Impressions of the quill-feathers of the wings and tail. Impressions of parts of finer feathers and down at the side of the body.

The opposite moiety of the split slab contains only one claw-bone (Plate I. fig. I′), belonging to the impression of the unguiculate digit (I) of the right wing, and a few slender curved rib-like bones, in addition to those shown on the lower moiety; of which bones the counterpart displays the impressions, and in some instances, as in the femora, the thin outer crust of the shaft.

The furculum, pelvis, and bones of the tail are in their natural undisturbed position, as in the skeleton of the animal. The left scapula has been displaced backward, and lies outside of, and nearly parallel with, the left os innominatum. The left humerus extends outward and a little forward from its scapular articulation, from which it has not been dislocated. The antibrachium is bent directly inward towards the trunk; and the wing-feathers, of which twelve primaries may be counted, diverge about an inch or less in advance of the carpus.

The right scapula retains almost its natural relative position to the trunk, and is imbedded in the matrix, exposing its lower sharp margin. The right humerus extends backward; the right antibrachium is bent forward, outside of and close upon the humerus; the two metacarpal or proximo-phalangeal bones (57) extend forward in the same direction, but have been dislocated inward. Impressions of about fourteen long quill-feathers, from 6 to 7 inches in length, like those of the left wing, diverge from an extent of about 3 inches, parallel with and outside of the metacarpo-phalangeal bones.

The right femur extends from its acetabular articulation backward and a little outward, reaching as far as the eighth caudal vertebra. The tibia extends directly outward and backward from the knee-joint; the metatarse is bent upon the tibia obliquely forward and inward; and the toes extend in nearly the same direction, the foot being contracted. The left femur is dislocated from the pelvis; its head is opposite the eighth caudal vertebra; the shaft extends forward and a little outward; the tibia extends from the knee-joint more directly outward and a little forward.

The best-preserved impressions of the quill-feathers of the wing measure 6 inches in length, with a breadth of vane of nearly 1 inch; the anterior series of barbs being the shorter, or the anterior part of the vane being less broad than the posterior part; and the end of the vane is obtusely rounded, as at *d*, Plate I., and in fig. 7, Plate IV. The area covered by the diverging quill-feathers of the left wing measures 6 inches across its widest part, near the ends of the feathers; that of the right wing occupies a space of 11 inches from before backward; but this difference is due to the three posterior primaries being dislocated from the rest and directed backward. The under part of the wing being exposed, a few shorter feathers, 'under-coverts,' are seen crossing rather obliquely the 'primaries': one of these (Plate IV. fig. 7 *a*) is exquisitely preserved.

The impressions of the tail-feathers may be discerned from the third to the last caudal vertebræ (Plate I. *Cd*), the right series being complete; the anterior fifth of the left series being wanting. Twenty feathers succeed each other, from before backward, on the right side; and the last thirteen feathers of the left side are preserved. The principal tail-feathers correspond in number with the tail-vertebræ, and diverge, outward and backward (a pair from each vertebra), at an angle of 45° with the line of the tail, becoming more acute towards the end (Plate I. *Cd'*), where the two feathers forming the pair from the sides of the last caudal vertebra extend nearly parallel with each other, and in the axis of the tail, about $3\frac{1}{2}$ inches beyond the end of that vertebra. The length of the anterior tail-feathers is about an inch, and they gradually increase to a length of about 5 inches in the 15th, 16th, and 17th pairs (Plate IV. fig. 8); and gradually decrease, with a more backward direction, to the last pair, which have a length of 3 inches 8 lines. Thus the tail, which is about 11 inches in length, gradually expands to a breadth of $3\frac{1}{2}$ inches opposite the last two vertebræ, and terminates by an obtusely rounded or almost squared or truncate end.

In general shape and proportions it resembles rather the tail of a Petaurus or Squirrel than of a modern bird; while the wings, in their present state of preservation, agree in form and proportion with those of the Gallinaceous or 'round-winged' birds.

The scapula (Plate I. 51', and Plate II. fig. 1, 51) is 1 inch 10 lines in length, 4 lines across the articular end, $2\frac{1}{2}$ lines across the neck, and very gradually expanding, towards the base, to a breadth of 3 lines. In its slightly bent, lamelliform or sabre-shaped figure, and in the concavity between the glenoid articulation and the short acromial projection on the outer side, it closely resembles the scapula of a bird.

In the *Pterodactylus suevicus* (Plate II. fig. 3, 51), a species which accords in general size with the *Archeopteryx*, the scapula is broader in proportion to its length, and exhibits a slight double or sigmoid flexure lengthwise.

The extent of the furcular arch (Plate I. 58, and Plate IV. fig. 1), or connate clavicles, which is preserved, measures from end to end, following the curve, about 2 inches; the breadth at the apex of the curve is 2 lines; but this is obtuse, and the piers diverge at a right angle, but curving from each other; so that the arch is an open or rounded one, not contracted and pointed as in the true Gallinaceous birds; the furcular bone, moreover, is as thick as the slender part of the shaft of the humerus.

No skeleton of the Pterodactyle has shown a furculum. The best-preserved specimens, such as the *Pterodactylus suevicus*, figured by QUENSTEDT [*], exhibit the scapula and coracoid entire, without a trace of clavicle, separate or confluent.

The prominence beyond the left scapula (Plate I. 51′) suggested at first view the humeral end of the coracoid, but I believe it to be part of the humerus corresponding with the tuberosity on the ulnar side of the sessile semioval head, overarching the pneumatic foramen in the bird. The humerus of *Archeopteryx* (Plate I. 53, 53′, and Plate II. fig. 1, 53) is nearly 3 inches in length, with the same slight sigmoid flexure as in the bird. The pectoral ridge (ib. *b*) has a basal extent of 1 inch: the breadth of the humerus at this part is 6 lines, one-half of which breadth appears to be due to the pectoral ridge. In contour it most resembles that in the *Corvidæ* (Plate II. fig. 4), the border being continued almost straight down from the low upper angle; but there is a better-marked lower angle in *Archeopteryx*, where the border of the process curves with a slight concavity to subside in the shaft.

The Pterodactyle (Plate II. fig. 3, 53) presents a well-marked difference from the bird in the greater extent to which the pectoral ridge projects from the shaft of the humerus, and in the minor relative extent of its base. The humerus, moreover, is straight, shorter in proportion to the antibrachium, and thicker in proportion to its length, with a different character of the distal articulation. In *Archeopteryx* the humerus closely resembles that of the bird, and presents about the same proportion, in length, to the trunk as in the Peregrine Falcon (Plate II. fig. 2, 53), the Touraco, and most *Gallinæ*.

The radius (Plate I. 54, and Plate II. fig. 1, 54) is slender and straight. The ulna (*ib.* 55) is thicker, rather longer, and slightly bent, leaving a well-marked interosseous space between the two bones: it expands at both ends to contribute the chief share in both the elbow- and wrist-joints. The right ulna (Plate I. 55) shows the convexity at the part of the proximal end next the radius, as in modern birds. Both ulna and radius closely resemble the antibrachial bones of the bird. The length of the ulna is 2 inches 8 lines—bearing nearly the same proportion to the humerus as in some *Scansores* and *Gallinaceæ*.

In Pterodactyles (Plate II. fig. 3) the radius (54) and ulna (55) are of equal thickness, are straight, leave no interosseous space, take equal shares in the formation of the elbow- and wrist-joints, and the antibrachium is always much longer than the humerus.

A single carpal, of large size, wedged between the end of the radius and the base of a metacarpal, is shown on the left side of *Archeopteryx* (Plate I. 56), indicating a structure of the wrist like that in the bird.

On the right side an irregular mass of spar occupies the position of a thick carpus or metacarpus, twisted inward at right angles to the antibrachium; but this is a doubtful indication. An inch from the antibrachium, nearer the medial line, but, like the antibrachium, directed forward, are two longish bones, with expanded proximal articulations and straight shafts, growing slender to their distal ends, which come in contact (Plate I. 57, and Plate II. fig. 1, 57). The proximal articular surfaces are convex, indented by grooves; that of the shorter bone is 4 lines in advance of that of the longer. The latter is 1 inch 5 lines in length, or about half the length of the ulna; its small distal end is obtuse, and may have been articular. The contiguous shorter bone extends beyond the end of the other, and seems to terminate by a small convex condyle. These appear to be metacarpals; they bear the same relation of length to the antibrachium as do the two terminally coalesced metacarpals in the bird (Plate II. fig. 2, 57). If they be the homologues of these, they retain their original individuality or distinctness, and they are more equal in thickness. If they be proximal phalanges of the two digits answering to those which constitute the penultimate joint of the pinion of the bird (Plate II. fig. 2, IV), they differ in being relatively longer and more equal in length and thickness.

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[*] Ueber *Pterodactylus suevicus*, 4to, Tübingen, 1855.

Half an inch from the outer of the two bones of the pinion, and external to, but on the same transverse parallel as, its distal articulation, is the impression of a slender bone, about 11 lines long, extending forward in the same line or direction as the above pinion-bones. At the distal end of the slender bone is the impression of part of a compressed curved bone, grooved along the side, 4 lines in length, 1 line in breadth; this dimension slightly decreasing as the bone recedes, curving from the longer slender supporting bone: it is most like the basal half of an ungual phalanx, supported by a long and slender penultimate phalanx (Plate I. II).

In advance and external to the foregoing is the bone itself, of a corresponding penultimate phalanx, 11 lines in length, half a line in thickness of shaft; expanded at both ends, but most so at the distal one, which supports a beautifully perfect claw-phalanx, preserved in the opposite slab (Plate I. I'), and indicated by its impression (*ib.* I) in the moiety which retains most of the bones of *Archeopteryx*. The claw-phalanx is 8 lines in a straight line, 2½ lines broad at the base, with a degree of curvature equal to that of the claw-phalanx of a Raptorial bird; grooved along the side; with the base produced, at the under or concave side, for the insertion of the flexor tendon, and with a sharp apex.

This claw resembles that of the mid-claw of the hind foot (Plate I. *iii*); but the bone, which plainly appears to be in penultimate phalangeal relation with it, is twice as long and only half as thick as the penultimate phalanx in the foot, and the repetition of the same character of penultimate phalanx in the less definite or less perfect indication of the other claw (Plate I. II) indicates that the hand of *Archeopteryx*, besides being concerned in supporting the remiges or quill-feathers of a wing, also supported two moderately long and slender free digits, each terminated by a strong, curved, sharp-pointed claw (as in the restoration, Plate II. fig. 1, 57, I, II, III, IV).

It is true that the parts of the present skeleton show a certain amount of dislocation, and one of the claw-bearing digits might have belonged to the left wing; but this is less probable than that they are on their right side. So much of the skeleton of the hand as is exposed to view in the present specimen unquestionably accords in its proportions with that of the bird (compare fig. 1, 57, with fig. 2, 57, Plate II.).

The anterior of the three digits which are developed in the bird's pinion (*ib.* fig. 2, II) remains free, and in some species supports a claw or spur *. The digit answering to the middle one in the pinion of birds of flight, supports, in *Apteryx*, a terminal curved claw. But if my interpretation of the appearances above described in the present fossil be correct, *Archeopteryx* differs markedly from all known birds in having two free unguiculate digits in the hand; and these digits, in the slenderness of the penultimate phalanx, do resemble the unguiculate digits in the hand of the Pterodactyle (Plate II. fig. 3, II). But the claw has not the characteristic depth or breadth of that of the Pterodactyle; and there is no trace of the much-lengthened metacarpal and phalangeal bones of the fifth digit, or peculiar wing-finger, of the flying Reptile (*ib.* V).

Had the manus of *Archeopteryx* been constructed for the support of a membranous wing, the extent to which the skeleton is preserved, and the ordinary condition of the fossil *Pterosauria* in lithographic slate, render it almost certain that some of these most characteristic elongated slender bones of the wing-finger (Plate II. fig. 3, $\text{V}_{1,2,3,4}$) would have been preserved if they had existed in the present specimen. But, besides the negative evidence, the positive proof of the ornithic proportions of the hand or pinion, of the existence of quill-feathers, and the manifest attachment of the principal ones, or 'primaries,' to the carpal and metacarpal parts of a short terminal segment of the limb, sufficiently evince the true class-affinity of the *Archeopteryx*.

The pelvis is chiefly represented by a bone on the left side (Plate I. $_{62}$), bearing the nearest resemblance to the iliac bone of a bird. A circular acetabulum, 3 lines in diameter (ib. *a*), is defined by a sharp border backed by matrix, not by bone. An oblong plate of bone extends in advance of the acetabulum 11 lines, with a breadth at the acetabulum of 7 lines, diminishing to a breadth of 4 lines, and then expanding to one of 5 lines. The margin of the bone next to the sacrum is nearly straight; the opposite or outer border is sinuous, being concave as it leaves the acetabulum, and then convex with an obtusely rounded anterior end. The exposed surface is smooth and polished. Transversely this surface is concave at the medial, convex at the lateral half. The bone is continued backward along the medial side of the acetabulum, of a breadth equal to that of the cavity; and behind it for the same extent, with a breadth of 7 lines, where it is interrupted by the well-defined curve of the anterior border of a large oval vacuity, one boundary of which is broken away at 6 lines' distance from the acetabulum.

I conclude that here is shown the left os innominatum, including the anterior two-thirds of the ilium, and the anterior half, or more, of the coalesced ischium. The anterior iliac border of the acetabulum ends abruptly and obtusely, precisely at the part where the acetabular end of the os pubis articulates with the ilium in the young bird; the ischium, however, appearing to meet that part of the ilium at a lower level (in the exposed surface of the fossil), and sending a very short process towards the acetabulum. The ischium (Plate I. $_{63}$), behind the acetabulum and external (as it lies) to the oval interspace between it and the ilium, shows the anterior curved boundary of a smaller or narrower vacuity, which I take to have intervened between the ischium and pubis.

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* *E. g.* Syrian Blackbird (*Merula dactyloptera*), Spur-winged Goose (*Anser gambensis*), Jacana (*Parra jacana*). The Screamer (*Palamedea cornuta*) has two spurs; the Megapode (*Megapodius*) has a tubercular rudiment of a pinion-claw.

We have here, therefore, plain indications of a large ischio-iliac interspace, answering to that called 'great ischiatic foramen or notch' (ib. *i'*), and the smaller ischio-pubic vacuity called 'obturator foramen' (ib. *o*), under conditions of size, formation, and relative position to the acetabulum, known only in the class of birds. The acetabulum itself, moreover, instead of being a bony cup, is a direct circular perforation of the os innominatum, as in birds.

Sufficient is known of the pelvis of the Pterodactyle to show that the ilium is relatively shorter and narrower than in the present fossil; that the pubic and ischial bones are distinct, short, broad, subtriangular plates, and that they contribute to form, with the ilium, a bony cup for the head of the femur.

Whether the pubis has retained its individuality in *Archeopteryx*, or has been broken away from the part of the ilium indicative of the place of its original attachment and relations to the acetabulum, I cannot determine. So far as the appearance of the pelvis can be discerned and, by me, interpreted, they give no evidence of a reptilian structure.

A confused mass of coalesced vertebræ, much shorter and broader than those of the tail, covers the proximal end of the right femur, and extends forward between it and the left innominatum. The sparry material which has crystallized in the vacuities of all the widely and apparently pneumatically excavated bones of the *Archeopteryx* chiefly represents the sacral portion of the spine, in which a series of six or seven short and broad transverse processes, in close contact on the right side, can alone be distinguished. From this indication, the sacrum would seem to have been at least 2 inches in length, and nearly 1 inch in breadth. The inferior or central surface, as in the case of the slightly dislocated left innominatum, is towards the observer, but is much mutilated.

The broad, subquadrate, short, compressed spines of one or two lumbar vertebræ are dimly discernible in front of the sacrum. No trace of the vertebral column in advance of these is visible, nor any part of the sternum; trunk, neck, and head are all wanting. The remains of *Archeopteryx*, as preserved in the present split slab of lithographic stone, recalled to mind the condition in which I have seen the carcase of a Gull or other seabird left on estuary sand after having been a prey to some carnivorous assailant. The viscera and chief masses of flesh, with the cavity containing and giving attachment to them, are gone, with the muscular neck and perhaps the head, while the indigestible quill-feathers of the wings and tail, with more or less of the limbs, held together by parts of the skin, and with such an amount of dislocation as the bones of the present specimen exhibit, remain to indicate what once had been a bird.

Perhaps the most decisive mark of the class-relationship of the *Archeopteryx* is afforded by the bones of the pelvic appendage or extremity, especially of the foot.

The mark of reptilian nature on which CUVIER mainly relied in his masterly analysis of the Pterodactyle's skeleton, was the separate state of the tarsals, and of the metatarsals supporting the digits, with the different number of joints in each digit. In the present specimen, a single coalesced tarso-metatarsal bone (Plates I. & III. fig. 1, 68) articulates at one end with the tibia; at the other, by a trifid trochlear end, with three toes (*ii*, *iii*, *iv*) directed forward: a shorter opposing toe (*i*) is connected with the metatarse a little above and behind the inner trochlea.

The femur (Plates I. & III. fig. 1, 65) is 2 inches 4½ lines long, and 2 lines in diameter at the middle of the shaft, which is slightly bent, with the concavity backward. In the Pterodactyle (Plate III. fig. 4, 65) the femur is straight. In some birds (*Corythaix*, Plate III. fig. 2, 65) it shows the same bend as in *Archeopteryx*.

The tibia of *Archeopteryx* (*ib.* fig. 1, 66) is 3 inches 2 lines long, with a shaft of 1½ line in diameter; it is straight. On the left side (Plate I. 66′), where its back surface appears, it shows the division of the hinder border of the upper articular surface into two lobes; but these are thicker, more rounded or convex, and with a deeper mid-cleft than in those birds that best show this division. In the fossil, however, the sharper contour of this part of the bone is indicated by the thin layer imbedded in the depression on the counterpart slab.

The right tibia (*ib.* 66) exposes its inner or tibial side, and neither the bone nor the impression exhibits a procnemial ridge. The head of the tibia is produced obtusely below the fore part of the knee-joint. The procnemial production varies much in different birds; in some *Raptores* (*Falco trivirgatus*, Plate III. fig. 3, 66), and in most *Volitores*, it would not leave a more marked indication than in *Archeopteryx*. The distal end of the tibia expands anteriorly, and the contour shown by the inner surface of the right tibia, and the hinder and inner part of the left one, agrees with the peculiar structure of that part in birds.

In the proportion of the tibia to the femur, exceeding as it does the latter bone by rather more than one-fourth of its own length, *Archeopteryx* (Plate III. fig. 1) resembles some birds (Grouse, Touracos (*ib.* fig. 2), many *Insessores*); but the thigh is proportionally longer in *Archeopteryx* than in the majority of birds, especially those (e. g. *Cursores*, *Grallatores*) which are remarkable for the length of leg. In the Pterodactyle (*ib.* fig. 4) the tibia (66) is more nearly of equal length with the femur (65). Whatever trace or proportion of the fibula may have existed in *Archeopteryx*, if preserved, is buried in the matrix beneath the exposed parts of the tibia.

There is no indication, in either the fossil bones or their impressions, of a separate or distinct tarsus. The upper end of the coalesced metatarsals (Plates I. & III. fig. 1, 68) shows the calcaneal process and the tendinal groove on its inner side. The thin bony crust of the inner side of this single composite bone adheres to the impression on the counterpart slab; the cast of the medullary cavity in the usual clear, light-coloured spar represents the major part of the shaft; but the innermost and the middle of the three distal condyles, or trochlear joints, are well preserved. The length of the tarso-metatarsal to the end of the mid-trochlea is 1 inch 10½ lines, to the end of the inner trochlea 1 inch 8½ lines: this characteristic bird-bone in *Archeopteryx* thus resembles the same in *Gallinaceæ* and some other groups in which the inner trochlea is least produced, and differs from the *Raptores* (Plate III. fig. 3) and others in which the trochleæ terminate on the same or nearly the same level.

The short metatarsal of the innermost or back toe (Plate III. fig. 1, *i*) begins at the lower third of the metatarsus (68); has an extent of attachment, shown to be ligamentous by a linear tract of matrix, of 2½ lines; and its convex articular end is about the same distance above the inner trochlea as that is above the middle trochlea of the connate metatarsals. Thus the proportion of the metatarsus to the tibia resembles the average or common proportion in birds (*ib.* fig. 2), having neither the extreme length of the Grallatorial, the extreme shortness of the Volitorial, nor the robustness of the Raptorial modifications (*ib.* fig. 3) of this characteristic bone.

The difference from the Reptilian structure, and especially from the Pterosaurian modification thereof (Plate III. fig. 4, 68), is here most striking. The tarsus (ib. *a l*) is a distinct segment in the volant reptiles, and the metatarsals (*ib.* 68) equally retain their distinctness, and correspond in number with the toes. The entire tarso-metatarsal segment of the limb in the Pterodactyle is much shorter in proportion to the tibia than in *Archeopteryx* and most birds.

The innermost or back toe of *Archeopteryx* (Plate III. fig. 1, *i*) consists of two phalanges, each 4 lines in length: the second phalanx is curved, slender, pointed, with an obtuse process on the under or plantar side of the articulation, closely resembling the claw-phalanx of the bird: the toe is shorter and more slender than in the *Raptores* (*ib.* fig. 3, *i*, 1, 2), longer and more slender than in the *Rasores*, more curved than in the *Grallatores*, corresponding in its proportions, as in the relative length of the proximal phalanx, with the same toe in perching birds. The second toe (*ib.* fig. 1, *ii*), the innermost of the three directed forward, consists of three phalanges (1, 2, 3) of nearly equal length, that of the entire toe being 1 inch 3 lines. The third (ib. *iii*), or mid-toe of the three front ones, is 1 inch 9 lines in length, and consists of four phalanges, the second (2) and penultimate (3) being rather the shortest. These toes, with their claw-phalanges, equally accord in structure and proportions with the Insessorial type of foot. The termination of the claw-phalanx of the outermost (fourth) toe (ib. *iv*) projects beyond and from beneath that of the second toe, indicating a length intermediate between that of the second and third toes, but more nearly that of the second toe: traces of the other joints of the fourth toe are sufficiently plain to determine that it was not bent back, but that it accorded in position and direction with the Insessorial, not the Scansorial, type of foot. All the claw-bones correspond in the proportions of breadth to length with the bird-type of those bones, and not with the compressed deep form which they present in Pterodactyles.

The structure of the foot, and the proportion which its metatarsal bone bears to the tibia, lead me to restrict the account of the closer comparisons of the bones of *Archeopteryx* with those of other birds to the species of *Insessores* and *Raptores* which best accord with the fossil in general size. The furculum of *Archeopteryx* (Plate IV. fig. 1) presents the proportional strength, thickness, and span of the arch which characterize the diurnal *Raptores* (*ib.* fig. 3): but the piers or crura do not arch into one another below by so open a curve; they have converged in a form more angular, more like that in the Owls (*ib.* fig. 4, *Nyctea nivea*), and still more like that in some *Grallæ*, with a strong furculum, as, *e. g.*, in the Spoonbill (*Platalea leucorodia*) and Argala (*ib.* fig. 5); only, as before remarked, the type of pelvic limb precludes any useful comparison with birds of the Wading order. The furculum in *Columbidæ* and *Cracidæ* (*ib.* fig. 6) is feeble in comparison with that of *Archeopteryx*: in the more typical *Gallinaceæ*, the still more slender piers of the furculum meet at an acute angle, and develope a compressed plate of bone from the apex. The furculum of *Archeopteryx* is that of a bird of a more powerful flight than in the true *Gallinaceæ*. In the *Corvidæ* (*ib.* fig. 2), in which the furculum is narrower in proportion to its length than in *Falconidæ*, the piers unite by a wider curve than in *Archeopteryx*.

The scapula of *Archeopteryx* (Plate II. fig. 1, $_{51}$) bears nearly the same proportion in length to the humerus and femur as in some of the more slender-limbed *Falconidæ* (*Falco trivirgatus*, Plate II. fig. 2, $_{51}$). But the humerus seems to have been more slender than in the Falcon (*Falco trivirgatus*), which comes nearest to *Archeopteryx* in this respect. The form of the pectoral ridge presents the difference previously pointed out.

In the Kites (*Milvus*) and Perns (*Pernis*) the humerus is proportionally longer than in *Archeopteryx*: in the *Corvidæ* it is proportionally thicker (Plate II. fig. 4, *Corvus corax*). It is by the proportion of the antibrachium (*ib.* fig. 1, $_{54}$, $_{55}$) to the humerus that *Archeopteryx* departs furthest from the Raptorial and Insessorial types, whilst it closely resembles the true *Gallinaceæ*, the antibrachium being rather shorter than the humerus; and this condition of the wing-bones accords with the indication of the proportions of the primary quill-feathers, as in the short rounded wing of Grouse and Pheasants. The bones of the segment of the hand giving attachment to the primaries are not preserved in the left wing of *Archeopteryx*; two of those on the right side are preserved, and the manus shows, apparently, in the two distinct sets of phalanges, terminated each by a compressed, curved, sharp-pointed claw, the departure, next in importance after the tail, from the structures of modern and known tertiary birds.

Few of the bones, even the best-preserved ones of *Archeopteryx*, permit a close or minute comparison of superficial features and markings with their recent homologues in birds or reptiles.

The osseous remains of *Archeopteryx* being included between the halves of a split slab, it might be supposed that the configuration of the outer surface of the fossilized bone must be demonstrable on one or other of the moieties: it is not so. The long contact of the phosphate with the carbonate of lime has resulted in a certain degree of disintegration or partial decomposition of the former, which has baffled every attempt to detach the matrix from the bone, or the bone from the matrix, where they have come to hand in their original contact. Only in the instances of the bones with the thickest osseous walls, as those of the feet, and especially the claw-bones, is the surface entire; and this has been exposed by the splitting of the slab, and needs no working out by tool.

Were it not for the large proportional size of their cavities, the general configuration of the long bones of the limbs could not have been so well preserved and presented for the requisite comparison. When these bones sank in the soft fine calcareous mud which has hardened into the peculiar stone which the progress of lithographic art has rendered so valuable, the sparry matter in solution, percolating the matrix and entering the cavities of the bones, has slowly crystallized there, and ultimately filled them by a compact body of spar. The degree to which this represents the original bone gives the measure of the pneumatic cavities and cancelli in the skeleton of *Archeopteryx*, and shows that the proportion of the original osseous matter must have been that which we observe in the present day in birds of flight.

The great and striking difference, and that which gives its enigmatical character to this fossil bird's skeleton, is the number, or rather the proportions and distinctness, of the caudal vertebræ; their under surface is exposed, or rather the sparry casts of the cavities of their bodies, the thin crust of the bone adhering to the impressions of the counterpart. The best view of the under surface of the caudal centrum, thus obtainable, shows a slight expansion of the two articular ends, which join those of the contiguous vertebræ by simple flattened surfaces, having the margin obtuse. The mid-line of the under surface is slightly canaliculate, the impression probably of the caudal artery (Plate IV. fig. 8). There is no trace of hæmal arch, or spine, or articular surface for such, in any part of the caudal series; nor is there any appearance of the ossified thread-like ligaments which are so conspicuous in the tail of the Pterodactyle. The first five of these vertebræ show transverse processes progressively diminishing in breadth and length to the fifth caudal: no trace of such processes is visible in the succeeding vertebræ. The length of the first caudal vertebra is 3½ lines; this dimension gradually increases to the eighth caudal, the centrum or body of which is 6 lines in length, and that dimension is retained to the sixteenth caudal, when it gradually diminishes to the last caudal, which is 5 lines in length, and terminates in a point.

The impressions of the quills of the anterior shorter tail-feathers show that they were attached, ligamentously, to the end of the transverse processes in the anterior ones, and in the succeeding caudals to the sides of the vertebræ, each of these vertebræ supporting a pair of plumes. The under surface of the tail-feathers being exposed, the median groove of the shaft of the vane is clearly shown. The barbs of the vane are as distinctly and inimitably preserved in this delicate and fine-grained lithographic matrix

(Plate IV. fig. 8), as in the impression of the single shorter and broader feather from the same formation described by M. HERMANN VON MEYER*. The narrower series at the fore part of one feather overlaps the margin of the broader series of barbs of the preceding feather.]

With the exception of the caudal vertebræ, and possibly of the bi-unguiculate and less confluent condition of the manus, the parts of the skeleton preserved in this rare fossil feathered animal accord with the strictly ornithic modifications of the vertebrate skeleton.

The main departure therefrom is in a part of that skeleton most subject to variety. In Bats there are short-tailed and long-tailed species, as in Rodents, Pterodactyles, and many other natural groups of air-breathing vertebrates; and it now is manifest that, at the period of the deposition of the lithographic slate, a like variety obtained in the feathered class. Its unexpected and almost startling character is due to the constancy with which all birds of the neozoic and modern periods present the short bony tail, accompanied in most of them with that further departure from type exemplified by the coalescence and special modification of the terminal vertebræ, to form the peculiar 'ploughshare bone' supporting the coccygeal glands, and giving attachment to the limited number of fanwise radiating *rectrices*, constituting the outward and visible tail in existing birds. All birds, however, in their embryonic state exhibit the caudal vertebræ distinct, and, in part of the series, gradually decreasing in size to the pointed terminal one.

In the embryo Rook (Pl. III. fig. 6), the proper extent of the caudal vertebræ is shown by the divergence of the parts of the ilia (62) to form the acetabula (*a*); and as many as ten free, but short, vertebræ are indicated beyond this part (*Cd*). Five or six of the anterior of these subsequently coalesce with each other and with the hinder halves of the ilia, lengthening out the sacrum to that extent. The tail is further shortened by the welding together of three terminal vertebræ to form the ploughshare bone.

In the young Ostrich from eighteen to twenty such vertebræ may be counted, freely exposed, between the parts of the iliac bones behind the acetabula; of which vertebræ seven or eight are afterwards annexed to the enormously prolonged sacrum, by coalescing with the backwardly produced ilia; while two or three vertebræ are welded together to form the terminal slender styliform bone of the tail, without undergoing the 'ploughshare' modification. In *Archeopteryx* the embryonal separation persists with such a continued growth of the individual vertebræ as is commonly seen in tailed Vertebrates, whether reptilian or mammalian.

The modification and specialization of the terminal bones of the spinal column in modern birds is closely analogous to that which converts the long, slender, symmetrical, many-jointed tail of the modern embryo-fish into that short and deep symmetrical shape, with coalescence of terminal vertebræ into a compressed lamelliform bone, to which the term 'homocercal' applies; such extreme development or transformation passing through the protocercal and usually the heterocercal stages, at which latter stage, in palæozoic and many mesozoic fishes, it was in different degrees arrested.

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* Jahrbuch für Mineralogie, &c., 1861, p. 561.

Thus we discern, in the main differential character of the by-fossil-remains-oldest-known feathered Vertebrate, a retention of a structure embryonal and transitory in the modern representatives of the class, and a closer adhesion to the general vertebrate type. The same evidence is afforded by the minor extent to which the anchylosing process has been carried on in the pinion, and by the apparent retention of two unguiculate digits on the radial side of the metacarpo-phalangeal bones, modified for the attachment of the primary quill-feathers. But when we recall the single unguiculate digit in the wing of *Pteropus*, and the number of such digits, equalling that in *Pterodactylus*, in the fore foot of the Flying Lemur (*Galeopithecus*), the tendency to see only a reptilian character in what may have been the structure of the manus in *Archeopteryx* receives a due check.

The best-determinable parts of its preserved structure declare it unequivocally to be a Bird, with rare peculiarities indicative of a distinct order in that class. By the law of correlation we infer that the mouth was devoid of lips, and was a beak-like instrument fitted for preening the plumage of *Archeopteryx*. A broad and keeled breast-bone was doubtless associated in the living bird with the great pectoral ridge of the humerus, with the furculum, and with the other evidences of feathered instruments of flight.

Explanation of the Plates.

PLATE I.

The moiety of the split slab of Lithographic Slate, containing, with the impressions of the feathers, the major part of the fossilized skeleton of *Archeopteryx*:—nat. size.

n. Concretionary nodules: the larger one consists of matrix, which filled a cavity, *n'*, formed by a thin layer of brownish and crystalline matter; which may be, as suggested by Mr. John Evans, F.G.S., part of the cranium with the cast of the brain of the *Archeopteryx*.

n'. Cavity with a layer of brown matter, in the counterpart slab, which was applied to the nodule, *n*.

Fig. 2. Fore part of the brain of a Magpie (*Corvus pica*, L.).

Fig. 3, *p'*. Premaxillary bone and, fig. 1, *p*, its impression, resembling that of a fossil fish. The other letters and figures are explained in the text.

PLATE II.

Wing-bones—of the *Archeopteryx* (restored, fig. 1), of a Bird (*Falco trivirgatus*, fig. 2), and of a Pterodactyle (*Pterodactylus suevicus*, Quenst., fig. 3), and the humerus of a Raven (*Corvus corax*, fig. 4).

PLATE III.

Fig. 1. Bones of the leg of *Archeopteryx*.

Fig. 2. Bones of the leg of a Touraco (*Corythaix*).

Fig. 3. Bones of the leg of a Falcon (*Falco trivirgatus*).

Fig. 4. Bones of the leg of *Pterodactylus suevicus*, Quenst.

Fig. 5. Pelvis and caudal vertebræ of a newly-hatched Ostrich:—nat. size.

Fig. 6. Pelvis and caudal vertebræ of an embryo Rook (magnified 6 diameters).

In both figures, 62 ilium; 63 ischium; 64 pubis; *a*, acetabulum; *Cd*, caudal vertebræ.

PLATE IV.

Fig. 1. Portion of the furculum of *Archeopteryx*.

Fig. 2. Furculum of a Raven (*Corvus corax*).

Fig. 3. Furculum of a Falcon (*Falco peregrinus*).

Fig. 4. Furculum of an Owl (*Nyctea nivea*).

Fig. 5. Furculum of a Stork (*Ciconia argala*).

Fig. 6. Furculum of a Curassow (*Crax alector*).

Fig. 7. Impressions of the basal part of two 'primaries' and of four entire 'under-coverts' of the left wing of *Archeopteryx*.

Fig. 8. Impressions of the caudal plumes of the 15th and 16th caudal vertebræ of *Archeopteryx*.

Fig. 9. Two bone-cells or lacunæ, femur of *Dinornis*.

Fig. 10. Two bone-cells or lacunæ, wing-bone of *Pterodactylus*. (From Quekett's 'Catalogue of the Histological Series, Museum of the Royal College of Surgeons,' 4to, vol. ii. plate 9. fig. 29, and plate 10. fig. 16, showing identity of character.)

IV. *On the Strains in the Interior of Beams.*

By George Biddell Airy, *F.R.S., Astronomer Royal.*

Received November 6,—Read December 11, 1862.

I HAVE long desired to possess a theory which should enable me to express and to compute numerically the actual strain or strains upon every point in the interior of a beam or girder, under circumstances analogous to those which occur in ordinary engineering applications,—partly for information on the amount of force actually sustained by the different particles of the cast or wrought iron in a solid beam, partly as a guide in the construction of lattice-bridges. The memoirs and treatises on the theories of elasticity and strains, to which I have referred, have given me no assistance*. I have therefore constructed a theory, in a form which (I believe) is new, which solves completely the problems that I had proposed to myself, and which, as I think, may, with due attention to details, be applied to all the cases that are likely to present themselves as interesting. This theory, with some of its first applications, I ask leave to place before the Royal Society.

1. It is supposed, in the following investigations, that the beam consists of one lamina in a vertical plane,—the idea of a solid beam being supplied by the conception of a multitude of such laminæ side by side, all subject to similar strains, and therefore exerting no force one upon another. It is also supposed that the thickness of the lamina is uniform, and that its form is rectangular, the depth of the beam being equal throughout: these suppositions are made only for the sake of simplicity, as there does not appear to be any difficulty of principle in applying the theory to cases not restricted by these conditions, although the complexity would be much increased. It also appears necessary to suppose that the material of the beam yields equally, with equal forces, in different directions. Another physical supposition, which appears to be necessary for complete solution of the problem, will be stated when we reach the discussion of the first instance.

2. It is to be remarked that our theory is not intended to take account of all the strains possible in a beam, but only of those which are introduced by the weight of the beam or its load in the position in which it is used. A beam, whether of cast iron or of wrought iron, is, by the process of its manufacture, in most instances affected by permanent strains; so that, while the lamina is lying on its flat side, some parts are ready to

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* Since completing this essay, I have found that considerable progress had been made in the case of figure 5, by Professor W. J. M. Rankine.

burst asunder, while others are severely compressed. When the lamina is placed in a vertical plane, these accidental strains will be combined with the strains which are produced by the weight of the beam, &c.; nevertheless our attention will be confined strictly to the latter. The algebraical expression of this idea is, that we do not want complete solutions of our differential equations; we only want solutions which will satisfy those equations; and among solutions which possess this property, we may have respect to the laws of pressure antecedently known from simpler investigations.

3. For the unit of force we shall use the weight of a unit of surface of the lamina; but in writing the expressions, we shall omit the word "weight," as no ambiguity can be produced by its absence. For the unit of the force of compression, or of tension (which is merely compression with changed sign, or negative compression), we must refer to such considerations as the following. A force of tension is not a force acting in a single line; it is a force acting in parallel or nearly parallel lines, with nearly constant magnitude over a considerable extent of surface. In a large structure, like the Britannia Bridge for instance, on any space one inch broad there is a certain force of tension; but on the neighbouring space of one inch broad there is the same force of tension, and so for each inch in a long succession there is sensibly the same force of tension. The force of tension, acting on a certain breadth measured perpendicularly to the direction of tension, will therefore be proportional to that breadth, or will be equal to the weight of a surface or ribbon whose breadth is the breadth which sustains the action, and whose length varies with the magnitude of the tension. That length is the proper measure of tension. When the breadth subject to the action $=1$ (the unit of linear measure), the amount of action is expressed simply by that length; when the breadth has another value, the amount of action is the product of the value of breadth by the length which measures the tension. The same remarks apply to the measure of compression.

4. We must now consider the effect of tension estimated in a direction inclined at an angle φ to the direction of tension. Suppose that a cut is made through the lamina, at right angles to the direction of tension, and that the effect of tension is to separate the sides of the cut. And suppose the direction of tension to rotate in the plane of the lamina. As the rotation proceeds, the tendency to open the cut diminishes, till, when $\varphi=90^\circ$, the tendency vanishes entirely. But when φ becomes greater than 90°, the tendency to open the cut is restored, and when $\varphi=180^\circ$, it is exactly as great as when $\varphi=0$. As φ is further increased, the tendency diminishes by the same degrees, and vanishes for $\varphi=270^\circ$; then increases till $\varphi=360^\circ$. It is never converted into a force of compression, and its changes are the same for positive and for negative changes of φ. These considerations show that the effect must be represented by a formula containing only even powers of $\cos\varphi$. And the following consideration will show that there will be only one term, multiplying $\cos^2\varphi$. When the tension acts at right angles to the cut, if t be the length which measures the tension, and if l be the length of a portion of the cut, the force which acts is the weight of the ribbon whose length is t and breadth l; and is

therefore $=lt$. But when the direction has rotated through ϕ, the force acting obliquely on l is the weight of the ribbon whose length is t and breadth $l.\cos\phi$, and is therefore $=lt.\cos\phi$. And this force is not normal to the cut, but makes the angle ϕ with the normal; and therefore the force which is normal to the cut, acting on the length l, is $lt.\cos\phi\times\cos\phi=lt.\cos^2\phi=l\times t.\cos^2\phi$. Consequently the measure of the tension, at the angle ϕ to the original tension, is $t.\cos^2\phi$. The same theorem applies to compression.

5. We must now proceed to consider the coexistence of two or more forces of compression or extension. There is no difficulty in conceiving that a plate of metal may at the same time be extended in one direction and compressed in another direction transversal to the former. But on consideration it will be found equally easy to conceive that a plate of metal may sustain at the same time several forces of compression, or of extension, or of both. It is easy to devise an apparatus which will produce these effects. Such forces may exist in the strains of a beam; and it is important to show that they can be included in a simple investigation. The following theorem is now to be proved. "Whatever be the number and directions of the forces of compression and extension, their combination may in all cases be represented by the combination of two forces at right angles,—these forces being sometimes both of compression, sometimes both of extension, sometimes one a force of extension and the other a force of compression, and generally unequal in magnitude." The following is the demonstration. Suppose that there are forces of compression (forces of extension being represented as negative forces of compression) of magnitudes A_1, A_2, &c., acting in directions which make angles α_1, α_2, &c. with a fixed line. Let us estimate the effect of their combination in a direction making any angle ψ with the same line. The angles between the directions of the several forces and this direction are respectively $\alpha_1-\psi$, $\alpha_2-\psi$, &c.; and therefore, by the last article, their effects in the direction ψ are $A_1.\cos^2(\alpha_1-\psi)$, $A_2.\cos^2(\alpha_2-\psi)$, &c.; or

$$A_1.\cos^2\alpha_1.\cos^2\psi\ +2A_1.\cos\alpha_1.\sin\alpha_1.\cos\psi.\sin\psi\ +A_1.\sin^2\alpha_1.\sin^2\psi,$$
$$A_2.\cos^2\alpha_2.\cos^2\psi\ +2A_2.\cos\alpha_2.\sin\alpha_2.\cos\psi.\sin\psi\ +A_2.\sin^2\alpha_2.\sin^2\psi,$$
$$\&c.;$$

the sum of which may be represented by

$$\Sigma(A.\cos^2\alpha).\cos^2\psi+\Sigma(2A.\cos\alpha.\sin\alpha).\cos\psi.\sin\psi+\Sigma(A.\sin^2\alpha).\sin^2\psi,$$

or

$$a.\cos^2\psi\ +b.\cos\psi.\sin\psi\ +c.\sin^2\psi;$$

where a, b, c may have any magnitude and either sign. And it is to be shown that we can find a force B acting at the angle β, and a force C acting at the angle $\beta+90°$, whose combination will produce the same effect.

Now the effect of these forces, by the theorem of last article, is

$$B.\cos^2(\beta-\psi)\ +C.\cos^2(\beta+90°-\psi),$$

or

$$B.\cos^2\beta.\cos^2\psi\ +2B.\cos\beta.\sin\beta.\cos\psi.\sin\psi\ +B.\sin^2\beta.\sin^2\psi$$
$$+C.\sin^2\beta.\cos^2\psi\ -2C.\sin\beta.\cos\beta.\cos\psi.\sin\psi\ +C.\cos^2\beta.\sin^2\psi.$$

Comparing this, term by term, with the former,

$$B\,.\cos^2\beta+C\,.\sin^2\beta=a;$$
$$B\,.\sin^2\beta+C\,.\cos^2\beta=c;$$
$$(B-C)\,.\sin 2\beta \qquad =b.$$

The difference of the first and second equations gives

$$(B-C)\,.\cos 2\beta=a-c;$$

and the quotient of the third by this gives

$$\tan 2\beta=\frac{b}{a-c};$$

which always gives a possible value for β.

Then $B-C=\frac{b}{\sin 2\beta}$ or $=\frac{a-c}{\cos 2\beta}$, which is always possible. And, by adding the first and second equations,

$$B+C=a+c.$$

By the combination of B+C and B−C, B and C are found. Thus all the elements may be found, for representing the effect of any number of forces of compression or extension, by the effect of two forces of compression or extension acting at right angles to each other. Our succeeding investigations therefore will be confined to the consideration of two such forces acting at each point.

We are now in a state to proceed with the consideration of the strains in a beam.

6. In fig. 1, Plate V., let the parallelogram represent a beam, supported in any way, as for instance by having one end fixed into a wall, and subject to any force, as for instance the vertical reaction R of a support at distance h. If R is negative, it will represent a weight hanging on the beam. Conceive a line to pass in any curved or crooked direction, from the lower to the upper edge, dividing the beam into two parts, a near part and a distant part. This division is to be understood merely as a line visible to the eye; it is not to be contemplated as a mechanical separation; for if it were such, the metal on one side could be considered as acting upon the metal on the other side only in the direction perpendicular to the separating line; which action, in many cases (as when the separating line is vertical), would obviously be incompetent to support the distant part of the beam. The compressions and tensions, which we can suppose to exist while the continuity is mechanically uninterrupted, will suffice (with or without other forces) to support the distant part. Now if the upper end of the curve terminates in the upper edge of the beam, conceive the curve to continue along that edge till it meets the upper angle at the end of the beam; if it terminates in the vertical end of the beam, conceive it carried upwards till it meets the upper angle; thus the special actions which sometimes operate in the limiting lines will be separated from those in the dividing curve. Let r and s be the length and depth of the beam; x the horizontal abscissa (measured from o), and y the vertical ordinate (measured from the lower edge)

of any point of the curve. At the first limit of the curve, the coordinates are z, 0; at the last, the coordinates are r, s.

7. The distant part of the beam is supported by the forces of compression (this term, with negative values, including tensions) across every part of the curve, combined with the reaction R. At the point whose coordinates are x, y, conceive that there is one force of compression B whose direction makes the angle β to the left side of y produced, and another force of compression C whose direction makes the angle $\beta+90°$ to the left side of y produced. And, in figure 2, consider the actions of these on the small element δs of the curve, or rather the actions on a portion of the lamina, including δs. Let θ be the angle made by δs with y. The direction of the action of B makes with δs the angle $\beta+\theta$; and therefore the breadth of the ribbon representing its action is $\delta s\times\sin(\beta+\theta)$, and its whole force is $\mathrm{B}.\delta s\times\sin(\beta+\theta)$. Resolving this in the directions of x and y, we have for the effects of B on the distant part of the beam,

$$\text{In the direction } x,\quad \mathrm{B}.\delta s\times\sin(\beta+\theta)\times\sin\beta,$$

$$\text{In the direction } y,\quad -\mathrm{B}.\delta s\times\sin(\beta+\theta)\times\cos\beta.$$

In like manner, the effects of C on the distant part of the beam are,

$$\text{In the direction } x,\quad \mathrm{C}.\delta s\times\sin(\beta+90°+\theta)\times\sin(\beta+90°),$$

$$\text{In the direction } y,\quad -\mathrm{C}.\delta s\times\sin(\beta+90°+\theta)\times\cos(\beta+90°).$$

Expanding the sine, we have, for the whole force in the direction x,

$$\{\mathrm{B}.\sin^2\beta+\mathrm{C}.\cos^2\beta\}.\cos\theta.\delta s+\{\mathrm{B}.\cos\beta.\sin\beta-\mathrm{C}.\sin\beta.\cos\beta\}.\sin\theta.\delta s,$$

and for the whole force in the direction y,

$$\{-\mathrm{B}.\sin\beta.\cos\beta+\mathrm{C}.\cos\beta.\sin\beta\}.\cos\theta.\delta s+\{-\mathrm{B}.\cos^2\beta-\mathrm{C}.\sin^2\beta\}.\sin\theta.\delta s.$$

But $\cos\theta.\delta s=\delta y$, $\sin\theta.\delta s=\delta x$. And using for convenience the following letters,

$$\mathrm{L}=\mathrm{B}.\sin^2\beta+\mathrm{C}.\cos^2\beta,$$

$$\mathrm{M}=(\mathrm{B}-\mathrm{C}).\sin\beta.\cos\beta,$$

$$\mathrm{Q}=-\mathrm{B}.\cos^2\beta-\mathrm{C}.\sin^2\beta,$$

we have for the whole forces on the element δs,

$$\text{In the direction } x,\quad \mathrm{L}.\delta y+\mathrm{M}.\delta x,$$

$$\text{In the direction } y,\quad -\mathrm{M}.\delta y+\mathrm{Q}.\delta x.$$

It must be borne in mind that the force in direction x acts in a line whose vertical ordinate is y, and that the force in direction y acts in a line whose horizontal ordinate is x.

8. There is another force acting on this portion of the distant part, namely, the weight of the lamina included between the ordinates corresponding to x and $x+\delta x$; which, estimated in the direction y, is $-y.\delta x$, acting in a line whose horizontal ordinate

And, besides these forces which act at every point of the curve, there is the reaction $+\mathrm{R}$ in the direction y, acting in a line whose horizontal ordinate is h.

9. We have now collected all the elements for the equations of equilibrium of the distant part of the beam, and we proceed to form those equations. For δy we shall put $p\,.\,\delta x$. The equations are as follows:

First, equation for forces in x:

$$\int dx\,.\,(\mathrm{L}p+\mathrm{M})=0. \qquad (1.)$$

Second, equation for forces in y:

$$\int dx\,.\,(-\mathrm{M}p+\mathrm{Q}-y)+\mathrm{R}=0. \qquad (2.)$$

Third, equation of moments:

$$\int dx\,.\,(\mathrm{L}yp+\mathrm{M}y+\mathrm{M}xp-\mathrm{Q}x+xy)-\mathrm{R}h=0. \qquad (3.)$$

It will be convenient at once to make $y-\mathrm{Q}=\mathrm{O}$; and the equations become

$$\int dx\,.\,(\mathrm{L}p\;+\mathrm{M})=0. \qquad (4.)$$

$$\int dx\,.\,(\mathrm{M}p\;+\mathrm{O})-\mathrm{R}=0. \qquad (5.)$$

$$\int dx\,.\,(\mathrm{L}yp+\mathrm{M}y+\mathrm{M}xp+\mathrm{O}x)-\mathrm{R}h=0. \qquad (6.)$$

10. We shall now introduce a consideration which will prove singularly advantageous for the solution of these equations. Referring to figure 3, the equations which we have obtained apply to the curve *a b c d e f*. The same equations, *mutatis mutandis*, apply to the curve *a b g d e f*. Hence the variations in those equations produced by passing from one of these curves to the other will $=0$. Now these variations are clearly such as are treated in the Calculus of Variations. We may therefore form the variations of the equations according to the rules of the Calculus of Variations, and equate those variations to zero. R and Rh will disappear.

11. The left side of equations (4.), (5.), (6.), is in each case a function of x, y, p (L, M, and O depending on the position of the point in the lamina, and therefore being functions of x and y), and of no other differential coefficients. Therefore the equation of variations in each case, in the usual language of the Calculus of Variations, will have the form $\mathrm{N}-\frac{d(\mathrm{P})}{dx}=0$. Applying this in each instance we have;—

For $\delta\,.\int dx\,.\,(\mathrm{L}p+\mathrm{M})$:

$$\mathrm{N}=\frac{d\mathrm{L}}{dy}p+\frac{d\mathrm{M}}{dy}; \quad \mathrm{P}=\mathrm{L}; \quad \frac{d(\mathrm{P})}{dx}=\frac{d\mathrm{L}}{dx}+\frac{d\mathrm{L}}{dy}p;$$

therefore

$$\frac{d\mathrm{L}}{dy}p+\frac{d\mathrm{M}}{dy}-\frac{d\mathrm{L}}{dx}-\frac{d\mathrm{L}}{dy}p=0,$$

or

$$\frac{d\mathrm{M}}{dy}-\frac{d\mathrm{L}}{dx}=0. \qquad (7.)$$

For $\delta\,.\int dx(\mathrm{M}p+\mathrm{O})$: in the same manner,

$$\frac{d\mathrm{O}}{dy}-\frac{d\mathrm{M}}{dx}=0. \qquad (8.)$$

For $\delta.\int dx(\mathrm{L}yp+\mathrm{M}y+\mathrm{M}xp+\mathrm{O}x)$:

$$\mathrm{N}=\frac{d\mathrm{L}}{dy}yp+\mathrm{L}p+\frac{d\mathrm{M}}{dy}y+\mathrm{M}+\frac{\ }{dy}xp+\frac{d\mathrm{O}}{dy}x;\qquad \mathrm{P}=\mathrm{L}y+\mathrm{M}x;$$

$$\frac{d(\mathrm{P})}{dx}=\frac{d\mathrm{L}}{dx}y+\frac{d\mathrm{L}}{dy}py+\mathrm{L}p+\frac{d\mathrm{M}}{dx}x+\frac{d\mathrm{M}}{dy}px+\mathrm{M};$$

therefore

$$\frac{d\mathrm{L}}{dy}yp+\mathrm{L}p+\frac{d\mathrm{M}}{dy}y+\mathrm{M}+\frac{d\mathrm{M}}{dy}xp+\frac{d\mathrm{O}}{dy}x-\frac{d\mathrm{L}}{dx}y-\frac{d\mathrm{L}}{dy}py-\mathrm{L}p-\frac{d\mathrm{M}}{dx}x-\frac{d\mathrm{M}}{dy}px-\mathrm{M}=0,$$

or

$$y\left(\frac{d\mathrm{M}}{dy}-\frac{d\mathrm{L}}{dx}\right)+x\left(\frac{d\mathrm{O}}{dy}-\frac{d\mathrm{M}}{dx}\right)=0.$$

This equation, by virtue of equations (7.) and (8.), is identically true, and therefore adds nothing to our knowledge. The information, then, that we have obtained from our process is comprised in the two equations

$$\frac{d\mathrm{M}}{dy}=\frac{d\mathrm{L}}{dx}; \quad \dots\dots\dots \quad (7.)$$

$$\frac{d\mathrm{O}}{dy}=\frac{d\mathrm{M}}{dx}. \quad \dots\dots\dots \quad (8.)$$

From this it follows that L, M, O are the three partial differential equations of the second order of a function F of x and y, such that

$$\mathrm{L}=\frac{d^2\mathrm{F}}{dy^2},\quad \mathrm{M}=\frac{d^2\mathrm{F}}{dxdy},\quad \mathrm{O}=\frac{d^2\mathrm{F}}{dx^2};$$

and we may substitute these symbols for L, M, O, in the equations of equilibrium of the distant part of the beam.

12. If it had been necessary to use expressions of the utmost possible generality, we must have said

$$\mathrm{L}=\frac{d^2\mathrm{F}}{dy^2}+\phi(y),\quad \mathrm{M}=\frac{d^2\mathrm{F}}{dxdy},\quad \mathrm{O}=\frac{d^2\mathrm{F}}{dx^2}+\psi(x),$$

where the forms of the functions ϕ and ψ are arbitrary. Suppose now that F is so determined that the substitution of $\frac{d^2\mathrm{F}}{dy^2}$, $\frac{d^2\mathrm{F}}{dxdy}$, and $\frac{d^2\mathrm{F}}{dx^2}$ for L, M, O will satisfy the equations (4.), (5.), (6.) in their entirety. Then the substitution of $\phi(y)$ and $\psi(x)$ alone must satisfy those equations deprived of their constant terms; and therefore $\phi(y)$ and $\psi(x)$ may be multiplied to any degree, or different functions of the same character may be added to them. These remarks clearly indicate that these functions represent accidental strains such as we have spoken of in article 2, and they are therefore to be neglected. We confine ourselves therefore to the terms

$$\mathrm{L}=\frac{d^2\mathrm{F}}{dy^2},\quad \mathrm{M}=\frac{d^2\mathrm{F}}{dxdy},\quad \mathrm{O}=\frac{d^2\mathrm{F}}{dx^2}.$$

13. Making these substitutions, and restoring for pdx its original expression dy, the equations become the following:

$$\int\left(dy\,.\,\frac{d^2F}{dy^2}+dx\,.\,\frac{d^2F}{dxdy}\right)=0\,; \qquad \ldots\ldots\ldots \qquad (9.)$$

$$\int\left(dy\,.\,\frac{d^2F}{dxdy}+dx\,.\,\frac{d^2F}{dx^2}\right)-R=0\,; \qquad \ldots\ldots\ldots \qquad (10.)$$

$$\int\left(dy\,.\,\frac{d^2F}{dy^2}\,.\,y+dx\,.\,\frac{d^2F}{dxdy}y+dy\,.\,\frac{d^2F}{dxdy}x+dx\,.\,\frac{d^2F}{dx^2}x\right)-Rh=0\,; \ldots\ldots \qquad (11.)$$

the integrals being taken from x, $y=z$, 0, to x, $y=r$, s. Now

$$\frac{d^2F}{dy^2}dy+\frac{d^2F}{dxdy}dx=d\left(\frac{dF}{dy}\right); \qquad \frac{d^2F}{dxdy}dy+\frac{d^2F}{dx^2}dx=d\left(\frac{dF}{dx}\right);$$

and the same symbols appear in the bracket of equation (11.). Hence the equations become

$$\int.\,d\left(\frac{dF}{dy}\right)=0\,; \qquad \ldots\ldots\ldots \qquad (12.)$$

$$\int.\,d\left(\frac{dF}{dx}\right)-R=0\,; \qquad \ldots\ldots\ldots \qquad (13.)$$

and

$$\int\left\{y\,.\,d\left(\frac{dF}{dy}\right)+x\,.\,d\left(\frac{dF}{dx}\right)\right\}-Rh=0.$$

Integrating the quantities under the bracket by parts, the bracket becomes

$$y\frac{dF}{dy}+x\frac{dF}{dx}-\int\left(dy\,.\,\frac{dF}{dy}+dx\,.\,\frac{dF}{dx}\right).$$

But $\frac{dF}{dy}dy+\frac{dF}{dx}dx=d(F)$. The value of the bracket, therefore, is $y\frac{dF}{dy}+x\frac{dF}{dx}-F$; and equation (11.) becomes

$$\int.\,d\left\{y\frac{dF}{dy}+x\frac{dF}{dx}-F\right\}-Rh=0. \qquad \ldots\ldots\ldots \qquad (14.)$$

Attaching the subscripts z, 0 and r, s to the symbols or brackets, to denote the values which the expressions assume when z, 0 or r, s are substituted for x, y, the equations finally become

$$\left(\frac{dF}{dy}\right)_{r,s}-\left(\frac{dF}{dy}\right)_{z,0}=0. \qquad \ldots\ldots\ldots \qquad (15.)$$

$$\left(\frac{dF}{dx}\right)_{r,s}-\left(\frac{dF}{dx}\right)_{z,0}-R=0. \qquad \ldots\ldots\ldots \qquad (16.)$$

$$\left\{y\frac{dF}{dy}+x\frac{dF}{dx}-F\right\}_{r,s}-\left\{y\frac{dF}{dy}+x\frac{dF}{dx}-F\right\}_{z,0}-Rh=0. \qquad \ldots\ldots \qquad (17.)$$

From these, by very simple treatment, the form of F may be found; and from that form every required expression will be deduced with great facility.

14. A slight familiarity with the expressions for strains, as given by simple theory in some ordinary cases, is sufficient to convince us that F will contain only integer

powers of x and y. Assume, therefore,

$$F=S+Ty+Uy^2+Vy^3+Wy^4+\&c.,$$

where S, T, U, V, W, &c. are functions of x; then

$$\frac{dF}{dy}=\quad T+2Uy+3Vy^2+4Wy^3+\&c.$$

For r, s, the value of this is

$$T_r+2U_r.s+3V_r.s^2+4W_r.s^3+\&c.$$

For z, 0, its value is

$$T_z.$$

The expression $\left(\frac{dF}{dy}\right)_{r,s}-\left(\frac{dF}{dy}\right)_{z,0}$ will therefore contain the function T_z, where z is absolutely arbitrary. It is impossible that equation (15.) can subsist, except by making $T_z=0$, and therefore $T_r=0$, and generally $T=0$.

Again, omitting T, we find (using the accents to indicate differential coefficients)

$$\frac{dF}{dx}=S'+U'y^2+V'y^3+W'y^4+\&c.$$

For r, s, the value of this is

$$S'_r+U'_r.s^2+V'_r.s^3+W'_r.s^4+\&c.$$

For z, 0, its value is

$$S'_z.$$

For the same reason as before, S' generally $=0$. Therefore if S have any value, it is a mere numerical constant; and this will disappear in each of the equations (15.), (16.), (17.); and therefore it may be entirely omitted. The expression for F will therefore be reduced to $Uy^2+Vy^3+Wy^4+\&c.$ We shall hereafter show that ordinary investigations entitle us to assume that the expression for F will really be limited to the first two terms of this series, and that the powers of x will not be higher than the second; and therefore we shall suppose

$$F=(ax^2+bx+c)y^2+(ex^2+fx+g)y^3.$$

We can now proceed with instances.

15. *Example* 1. Suppose the beam to project from a wall, and to sustain no load except its own weight.

Here $R=0$; and the three equations (15.), (16.), (17.), with the last assumption for F, become

$$\begin{aligned}&(2ar^2+2br+2c)s+(3er^2+3fr+3g)s^2=0,\\&(2ar\ +b)s^2\qquad +(2er\ +f)s^3\qquad =0,\\&(3ar^2+2br+c)s^2\ +(4er^2+3fr+2g)s^3=0.\end{aligned}$$

Determining from these the values of b, c, g, we change the expression for F to the following:

$$F=\left\{\begin{array}{l}\{ax^2+(-2ar-2ers-fs)x+ar^2+2er^2s+frs\}y^2\\+\{ex^2+fx-er^2-fr\}y^3\end{array}\right\}.$$

To determine the constants a, e, f, which remain, we must have recourse to other considerations.

16. If we suppose the beam cut through in a vertical line corresponding to abscissa x, and if we make the usual assumptions in regard to the horizontal forces acting between the two parts and thus sustaining the moment of the distant part, namely, that there is a neutral point in the centre of the depth—that on the upper side of this neutral point the forces are forces of tension, and on the lower side are forces of compression—and that these forces are proportional to the distances from the neutral point, with equal coefficients on both sides,—then we can ascertain the horizontal force at every point. But I remark that it appears to me that these suppositions involve a distinct hypothesis as to the physical structure of the material. They seem to imply that the actual extensions or compressions correspond exactly to the curvature of the edge of the lamina, and that the forces of elasticity so put into play correspond to the amount of extension or compression. The experiments of Mr. W. H. BARLOW appear to modify this theory; and it seems probable that, when duly followed into their mathematical consequences, they may require the introduction into the formula for F of other powers of y. Leaving this question open, I shall now proceed, on the usual assumptions, to compute the horizontal force at every point of the vertical division.

17. Let the horizontal force at elevation y, estimated as compression, be represented by $c.\left(\frac{s}{2}-\quad\right)$; the force on the element δy is the ribbon $\delta y\times c.\left(\frac{s}{2}-y\right)$; its moment is $y\times\delta y\times c.\left(\frac{s}{2}-y\right)=c\left(\frac{sy}{2}-y^2\right)\delta y$; and the entire moment is $c\int dy\left(\frac{sy}{2}-y^2\right)=c\left(\frac{sy^2}{4}-\frac{y^3}{3}\right)$; which, from $y=0$ to $y=s$, is $-\frac{cs^3}{12}$. The moment produced by the weight of the distant part of the bar is the product of its weight by the horizontal distance of its centre of gravity, or is $(r-x)\times s\times\frac{r-x}{2}=\frac{s(r-x)^2}{2}$. The equation of moments is therefore, $-\frac{cs^3}{12}+\frac{s(r-x)^2}{2}=0$. From this, $c=\frac{6(r-x)^2}{s^2}$; and the horizontal compression-force at elevation $y=\frac{6}{s^2}.(r-x)^2.\left(\frac{s}{2}-y\right)$; or the horizontal compression-force on the element $\delta y=\frac{6}{s^2}.(r-x)^2.\left(\frac{s}{2}-y\right)\delta y$.

18. But we have the means of expressing the same horizontal force in terms of F. For, in the last expressions of art. 7, conceive the dividing line to be vertical; that is, conceive $\delta x=0$, and $\delta s=\delta y$; then we have for the compression-force on the element δy in direction x, the expression $\mathrm{L}.\delta y$; which, giving to L its value from the end of art. 11, becomes $\frac{d^2\mathrm{F}}{dy^2}\delta y$.

Comparing these two expressions, $\frac{d^2\mathrm{F}}{dy^2}=\frac{6}{s^2}(r-x)^2.\left(\frac{s}{2}-y\right)$. And, using the last formula of art. 14,

$$\left.\begin{cases}\{2ax^2+(-4ar-4ers-2fs)x+2ar^2+4er^2s+2frs\} \\ +\{6ex^2+6fx-6er^2-6fr\}y\end{cases}\right\}=\frac{6}{s^2}(r-x)^2.\left(\frac{s}{2}-y\right);$$

Comparing the coefficients of y, $ex^2+fx+(-er^2-fr)=-\frac{1}{s^2}x^2+\frac{2r}{s^2}x-\frac{r^2}{s^2}$. The first term gives $e=-\frac{1}{s^2}$; the second gives $f=\frac{2r}{s^2}$; the third gives $\frac{r^2}{s^2}-\frac{2r^2}{s^2}=-\frac{r^2}{s^2}$, which is identical.

Then substituting these in the term independent of y, and comparing,

$$2ax^2+\left(-4ar+\frac{4r}{s}-\frac{4r}{s}\right)x+2ar^2-\frac{4r^2}{s}+\frac{4r^2}{s}=\frac{3}{s}x^2-\frac{6r}{s}x+\frac{3}{s}r^2.$$

The first term gives $a=\frac{3}{2s}$; and this makes the second and third comparisons to become identical equations. The circumstance, that the determination of the constants from some terms causes the other terms to agree, gives evidence of the agreement of the two lines of theory, inasmuch as those remaining terms are obtained in the two theories by totally different operations, each peculiar to its own theory.

We may now therefore use $\frac{d^2\mathrm{F}}{dy^2}=\frac{6}{s^2}(r-x)^2.\left(\frac{s}{2}-y\right)$.

19. From this we find

$$\mathrm{F}=\frac{6}{s^2}.(r-x)^2.\left(\frac{sy^2}{4}-\frac{y^3}{6}\right);$$

from which

$$\mathrm{L}=\frac{d^2\mathrm{F}}{dy^2}=\frac{3}{s^2}.(r-x)^2.(s-2y) = s.\frac{3r^2}{s^2}.\left(1-\frac{x}{r}\right)^2.\left(1-\frac{2y}{s}\right);$$

$$\mathrm{M}=\frac{d^2\mathrm{F}}{dxdy}=\frac{-12}{s^2}.(r-x).\left(\frac{sy}{2}-\frac{y^2}{2}\right)=-s.\frac{6r}{s}.\left(1-\frac{x}{r}\right).\frac{y}{s}.\left(1-\frac{y}{s}\right);$$

$$\mathrm{N}=-2\mathrm{M} = s.\frac{12r}{s}.\left(1-\frac{x}{r}\right).\frac{y}{s}.\left(1-\frac{y}{s}\right);$$

$$\mathrm{O}=\frac{d^2\mathrm{F}}{dx^2}=\frac{12}{s^2}.\left(\frac{sy^2}{4}-\frac{y^3}{6}\right) = s.\left(3\frac{y^2}{s^2}-2\frac{y^3}{s^3}\right);$$

$$\mathrm{Q}=y-\mathrm{O}=s\frac{y}{s}-\mathrm{O}=s\left(\frac{y}{s}-3\frac{y^2}{s^2}+2\frac{y^3}{s^3}\right)=s.\frac{y}{s}.\left(1-\frac{y}{s}\right).\left(1-\frac{2y}{s}\right).$$

Put $v=\frac{x}{r}$, $w=\frac{y}{r}$, and omit the general multiplier s. And as the succeeding operations, while kept in the symbolical form, become rather cumbrous, assume for $\frac{r}{s}$ a numerical value, as 5. Then

$$\mathrm{L}=75.(1-v)^2.(1-2w);$$

$$\mathrm{N}=60.(1-v).w.(1-w);$$

$$\mathrm{Q}=w.(1-w).(1-2w).$$

From these (see art. 7),

$$\tan 2\beta=\frac{\mathrm{N}}{\mathrm{L}+\mathrm{Q}};$$

$$\mathrm{C}-\mathrm{B}=\frac{\mathrm{N}}{\sin 2\beta}=\frac{\mathrm{L}+\mathrm{Q}}{\cos 2\beta};$$

$$\mathrm{C}+\mathrm{B}=\mathrm{L}-\mathrm{Q};$$

which give the numerical values of the three elements B, C, β of the strains at every point.

By means of these formulæ, the numbers of Table I. (end of the Memoir) have been computed and the lines of pressure traced in Plate V. fig. 4. They give complete information on the nature and magnitude of the strains to which such a beam is subject.

20. *Example* 2. A beam of length $2r$ and depth s rests, at its two ends, freely on piers, and sustains no load except its own weight.

Assume, as before,

$$\mathrm{F}=(ax^2+bx+c)y^2+(ex^2+fx+g)y^3,$$

and remark that the distant pier exerts a reaction vertically upwards, of magnitude rs at distance $2r$. The three equations (15.), (16.), (17.), taking the integrals from $z, 0$ to $2r, s$, become

$$\begin{aligned}
&(\ 8ar^2+4br+2c)s+(12er^2+6fr+3g)s^2 &&=0;\\
&(\ 4ar\ +b)s^2 \qquad +(\ 4er\ +f\,)s^3 \qquad -rs &&=0;\\
&(12ar^3+4br+c)s^2\ +(16er^2+6fr+2g)s^3-2r^2s&&=0.
\end{aligned}$$

When from these we determine the values of b, c, g, and substitute them in the expression for F, it becomes

$$\mathrm{F}=\left\{\begin{array}{l}\left\{ax^2+\left(-4ar-4ers-fs+\frac{r}{s}\right)x+\left(4ar^2+8er^2s+2frs-\frac{2r^2}{s}\right)\right\}y^2\\ +\{ex^2+fx+(-4er^2-2fr)\}y^3\end{array}\right\}.$$

21. The horizontal pressure at any point of any vertical line across the beam at distance x will be found on the usual theory as follows. The compression at any elevation y being represented, as in article 17, by $c.\left(\frac{s}{2}-y\right)$, the entire moment is, as in that article, $-\frac{cs^3}{12}$. The moment produced by the weight of the distant part of the beam, whose length is $2r-x$, is $\frac{s(2r-x)^2}{2}$; and the moment produced by the reaction at the distant pier is $-rs\times(2r-x)$. The equation of moment is therefore

$$-\frac{cs^3}{12}+\frac{s(2r-x)^2}{2}-rs\times(2r-x)=0,$$

or

$$-\frac{cs^2}{12}+\frac{(2r-x)^2}{2}-\frac{2r(2r-x)}{2}=0,$$

or

$$-\frac{cs^2}{12}-\frac{x(2r-x)}{2}=0.$$

From this, $c=\frac{6x^2-12rx}{s^2}$; and the horizontal compression-force at elevation y

$$=\frac{6x^2-12rx}{s^2}\cdot\left(\frac{s}{2}-y\right).$$

Therefore, as in article 18,

$$\frac{d^2\text{F}}{dy^2}=\frac{6x^2-12rx}{s^2}\left(\frac{s}{2}-y\right).$$

And, using the last formula of article 20,

$$\left.\begin{array}{l}\left\{2ax^2+\left(-8ar-8ers-2fs+\frac{2r}{s}\right)x+\left(8ar^2+16er^2s+4frs-\frac{4r^2}{s}\right)\right\}\\ +\{6ex^2+6fx+(-24er^2-12fr)\}y\end{array}\right\}=\frac{6x^2-12rx}{s^2}\left(\frac{s}{2}-y\right).$$

Comparing the coefficients of y, $6ex^2+6fx+(-24er^2-12fr)=-\frac{6}{s^2}x^2+\frac{12r}{s^2}x$. The first term gives $e=-\frac{1}{s^2}$; the second gives $f=\frac{2r}{s^2}$; the third gives $\frac{24r^2}{s^2}-\frac{24r^2}{s^2}=0$, which is identical.

Substituting these in the term independent of y, and comparing,

$$2ax^2+\left(-8ar+\frac{8r}{s}-\frac{4r}{s}+\frac{2r}{s}\right)x+\left(8ar^2-\frac{16r^2}{s}+\frac{8r^2}{s}-\frac{4r^2}{s}\right)=\frac{3}{s}x^2-\frac{6r}{s}x.$$

The first term gives $a=\frac{3}{2s}$; and on substituting this, the second and third comparisons become identical equations. The evidence of correctness of theory is therefore satisfactory; and we may use $\frac{d^2\text{F}}{dy^2}=\frac{6x^2-12rx}{s^2}\left(\frac{s}{2}-y\right)$.

22. From this we find

$$\text{F}=\frac{6x^2-12rx}{s^2}\left(\frac{sy^2}{4}-\frac{y^3}{6}\right);$$

from which

$$\text{L}=\frac{d^2\text{F}}{dy^2}=\frac{6x^2-12rx}{s^2}\cdot\left(\frac{s}{2}-y\right)\qquad=-s\cdot\frac{6r^2}{s^2}\cdot\frac{x}{r}\cdot\left(1-\frac{x}{2r}\right)\cdot\left(1-\frac{2y}{s}\right);$$

$$\text{M}=\frac{d^2\text{F}}{dxdy}=\frac{12x-12r}{s^2}\cdot\left(\frac{sy}{2}-\frac{y^2}{2}\right)\qquad=-s\cdot\frac{6r}{s}\cdot\left(1-\frac{x}{r}\right)\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right);$$

$$\text{N}=\quad-2\text{M}\qquad=\quad s\cdot\frac{12r}{s}\cdot\left(1-\frac{x}{r}\right)\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right);$$

$$\text{O}=\frac{d^2\text{F}}{dx^2}=\frac{12}{s^2}\left(\frac{sy^2}{4}-\frac{y^3}{6}\right)\qquad=\quad s\cdot\left(3\frac{y^2}{s^2}-2\frac{y^3}{s^3}\right);$$

$$\text{Q}=y-\text{O}=s\frac{y}{s}-\text{O}=s\left(\frac{y}{s}-3\frac{y^2}{s^2}+2\frac{y^3}{s^3}\right)=\quad s\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right)\cdot\left(1-\frac{2y}{s}\right).$$

As before, put $v=\frac{x}{r}$, $w=\frac{y}{s}$; and suppose $\frac{r}{s}=5$. Then, omitting s,

$$\text{L}=-75\,.\,v\,.\,(2-v)\,.\,(1-2w);$$

$$\text{N}=\quad 60\,.\,(1-v)\,.\,w\,.\,(1-w);$$

$$\text{Q}=\quad w\,.\,(1-w)\,.\,(1-2w);$$

after which we may use the same formulæ as before, namely,

$$\tan 2\beta = \frac{N}{L+Q},$$

$$C-B = \frac{N}{\sin 2\beta} = \frac{L+Q}{\cos 2\beta},$$

$$C+B = L-Q;$$

by means of which the numbers have been computed for Table II. (end of the Memoir), and the lines have been traced that are exhibited in Plate VI. fig. 5.

23. There is one part of the pressures which it is matter of great interest to compute, namely, the pressures exerted on different parts of the end portion of the beam which rests on the pier. It will be seen in figure 7 that this part is not free from pressure; there are at every point a large force of compression in one direction, and a large force of tension in another direction. And the circumstances of this part differ from those of any other vertical section of the beam in this respect, that there is no opposing force. In all other sections, a thrust of compression on one side is met by a thrust of compression on the other side, and so for tension; and though there may be a tendency to crush or to disrupt the particles of the metal, yet there is no great tendency to force a small sectional portion horizontally or vertically. But on the end portion, where the forces of compression and tension are not so met, there are or may be great tendencies to force that end portion horizontally or vertically. We proceed now to investigate these tendencies.

24. First, for the horizontal pressure. The force B (which is estimated as a compression), acting in a direction which makes the angle β with the vertical, upon the element δy (as measured in the vertical direction) or $\sin\beta \, . \, \delta y$ (as measured in the direction transverse to B), does really exert the pressure $B \sin\beta \, . \, \delta y$ in the direction of B, or the pressure $B \, . \sin\beta \, . \, \delta y \times \sin \beta$, or $B \, . \sin^2\beta \, . \, \delta y$, in the horizontal direction. Similarly, the force C exerts the pressure $C \, . \sin^2(\beta+90^\circ) \, . \, \delta y$, or $C \, . \cos^2\beta \, . \, \delta y$, in the horizontal direction. The entire horizontal force upon the element δy is therefore

$$(B \, . \sin^2\beta + C \, . \cos^2\beta) \, . \, \delta y = L \, . \, \delta y = \frac{d^2F}{dy^2}\delta y.$$

In the instance before us, of a beam resting on two piers,

$$\frac{d^2F}{dy^2} = \frac{6x^2-12rx}{s^2} \, . \left(\frac{s}{2}-y\right);$$

and at the end of the beam, where $x=2r$, this quantity $=0$ whatever be the value of y. The same applies where $x=0$. There is no tendency therefore to bend or distort the end portion.

25. Secondly, for the vertical pressure. The pressure $B \, . \sin\beta \, . \, \delta y$ in the direction of B, found in last article, will produce the pressure $B \, . \sin\beta \, . \cos\beta \, . \, \delta y$ in the direction vertically downwards. Similarly, the force C will produce the pressure

$$C \, . \sin(\beta+90^\circ) \, . \cos(\beta+90^\circ) \, . \, \delta y$$

vertically downwards. The whole downwards pressure therefore on the element δy is $(\mathrm{B}.\sin\beta.\cos\beta+\mathrm{C}.\sin\overline{90^\circ+\beta}.\cos\overline{90^\circ+\beta}).\delta y$, or $(\mathrm{B}-\mathrm{C}).\sin\beta.\cos\beta.\delta y$, or $\mathrm{M}.\delta y$; which in the present instance $=\frac{12x-12r}{s^2}\cdot\left(\frac{sy}{2}-\frac{y^2}{2}\right)\cdot\delta y$. At the end of the beam, where $x=2r$, this $=\frac{6r}{s^2}.(sy-y^2).\delta y$. Let $y'=s-y$ (that is, let the ordinate be measured from the upper edge downwards); then the downwards pressure on the element $\delta y'$ of the end portion $=\frac{6r}{s^2}\cdot(sy'-y'^2).\delta y'$. Integrating this from the top downwards, we find for the pressure which a horizontal section of the end portion must sustain,

$$\frac{6r}{s^2}\cdot\left(\frac{sy'^2}{2}-\frac{y'^3}{3}\right)=rs\cdot\frac{y'^2}{s^2}\left(3-\frac{2y'}{s}\right).$$

At the middle of the depth this $=\frac{rs}{2}$; at the base it $=rs$. It appears therefore that every part of the end portion which rests upon the pier is subject to a very heavy pressure (such as affects no other part of the beam), increasing from the top to the bottom, where it is equal to the weight of half the beam.

It was undoubtedly from a clear perception of the magnitude of this pressure (though not reduced to the formulæ of mathematical investigation) that Mr. ROBERT STEPHENSON, in the construction of the Britannia Bridge, was induced to insert the strong end-frames in each of the tubes, at the places where they rest on their piers.

26. *Example* 3. A beam of length $2r$ and depth s rests, at its two ends, freely on piers, and carries a weight W at the distance a from the left-hand extremity.

For convenience, we will suppose a to be not greater than r. This will include every case, as the supposition a' greater than r is the same as the supposition a less than r measured from the right-hand extremity, if $a+a'=2r$.

In examples 1 and 2, we have selected a form for F which satisfied the equations (15.), (16.), (17.), applying to F, and we have then shown that this form represents properly the horizontal pressure determined from the ordinary theory. In the present example, which is unsymmetrical and complicated, we shall find the form for F (a discontinuous form) which represents the horizontal pressure as determined from the ordinary theory, and shall show that this form satisfies in all parts the equations (15.), (16.), (17.).

27. The pressure upon the left-hand pier is $rs+\mathrm{W}\cdot\frac{2r-a}{2r}$; and that upon the right-hand pier is $rs+\mathrm{W}\cdot\frac{a}{2r}$. The reactions of the piers have the same values, but in the opposite direction. For a transverse section at the ordinate x, where x is less than a, the forces which produce moments are the following: the weight $s\times(2r-x)$ acting at distance $\frac{2r-x}{2}$; the weight W at distance $a-x$; and the reaction $rs+\mathrm{W}\cdot\frac{a}{2r}$ at distance $2r-x$. The sum of their moments, estimated as compressing the upper part, is

$$(2r-x).\left(rs+\mathrm{W}\cdot\frac{a}{2r}-s\cdot\frac{2r-x}{2}\right)-\mathrm{W}(a-x),\ =\left(rs+\mathrm{W}\cdot\frac{2r-a}{2r}\right)x-\frac{s}{2}x^2.$$

(The same value will be found if we consider the moment as produced by the weight of bar and the reaction on the left side of x.) Treating this as in article 17, we find the horizontal compression-force at elevation y

$$=\frac{6}{s^2}\left\{\left(2r+\text{W}\frac{2r-a}{rs}\right)x-x^2\right\}\cdot\left(y-\frac{s}{2}\right).$$

This, as in preceding instances, ought to equal $\frac{d^2\text{F}}{dy^2}$; and therefore F ought to equal

$$\frac{6}{s^2}\left\{\left(2r+\text{W}\cdot\frac{2r-a}{rs}\right)x-x^2\right\}\cdot\left(\frac{y^3}{6}-\frac{sy^2}{4}\right).$$

This formula applies to any point of the part of the bar included between $x=0$ and $x=a$, which we shall call the "first part." For any point of the "second part," or the part included between $x=a$ and $x=2r$, there is no weight W on the right hand; the forces producing moments are the weight $s\times(2r-x)$ acting at distance $\frac{2r-x}{2}$, and the reaction $rs+\text{W}\frac{a}{2r}$ at distance $2r-x$; the sum of their moments, estimated as compressing the upper part, is

$$(2r-x)\left(rs+\text{W}\frac{a}{2r}-s\frac{2r-x}{2}\right)=\text{W}a+\left(rs-\text{W}\frac{a}{2r}\right)x-\frac{s}{2}x^2;$$

whence, as in article 17, the horizontal compression-force at elevation y

$$=\frac{6}{s^2}\left\{\frac{2\text{W}a}{s}+\left(2r-\text{W}\frac{a}{rs}\right)x-x^2\right\}\cdot\left(y-\frac{s}{2}\right),$$

which ought to equal $\frac{d^2\text{F}}{dy^2}$; and therefore F ought to equal

$$\frac{6}{s^2}\left\{\frac{2\text{W}a}{s}+\left(2r-\text{W}\cdot\frac{a}{rs}\right)x-x^2\right\}\cdot\left(\frac{y^3}{6}-\frac{sy^2}{4}\right).$$

This formula applies to any point of the part of the bar included between $x=a$ and $x=2r$, or to any point of the "second part." The function changes its form, or is discontinuous, when x passes the value a,—the two formulæ, however, giving the same value for F when $x=a$. We have now to ascertain whether the discontinuous function does in all parts satisfy the equations (15.), (16.), (17.).

28. First, suppose the integrals to begin from a point z in the "first part." It is unnecessary to make an elaborate trial of equation (15.), because, as our assumed value for F contains the multiplier $\frac{y^3}{6}-\frac{sy^2}{4}$, and $\frac{d\text{F}}{dy}$ therefore contains the multiplier $\frac{y^2}{2}-\frac{sy}{2}$, $\frac{d\text{F}}{dy}$ will necessarily vanish at both the limits for y (namely $y=0$, $y=s$) which enter into the formulæ of (15.). In regard to the other equations, the integrals must be taken by the formulæ of the "first part" from z, 0, to a, s; and by the formulæ of the "second part" from a, s, to $2r$, s; and the constant forces are $+\text{W}$ at abscissa a and $-\left(rs+\text{W}\frac{a}{2r}\right)$ at abscissa $2r$.

For equation (16.), $\frac{d\mathrm{F}}{dx}$ in the "first part" $=\frac{6}{s^2}\left\{2r+\mathrm{W}\,.\,\frac{2r-a}{rs}-2x\right\}\left(\frac{y^3}{6}-\frac{sy^2}{4}\right)$; which for z, 0, $=0$, and for a, s, $=\frac{-6}{s^2}\left\{2r-2a+\mathrm{W}\,.\,\frac{2r-a}{rs}\right\}.\frac{s^3}{12}$. And $\frac{d\mathrm{F}}{dx}$ in the "second part"

$$=\frac{6}{s^2}\left\{2r-\mathrm{W}\frac{a}{rs}-2x\right\}.\left(\frac{y^3}{6}-\frac{sy^2}{4}\right);$$

which for a, s,

$$=\frac{-6}{s^2}\left\{2r-2a-\mathrm{W}\frac{a}{rs}\right\}.\frac{s^3}{12},$$

and for $2r$, s,

$$=-\frac{6}{s^2}\left\{-2r-\mathrm{W}\frac{a}{rs}\right\}\frac{s^3}{12}.$$

The sum of the two portions of the integral will therefore be

$$-\frac{s}{2}\left\{0+2r-2a+\mathrm{W}\,.\,\frac{2r-a}{rs}-2r+2a+\mathrm{W}\frac{a}{rs}-2r-\mathrm{W}\frac{a}{rs}\right\}=-\frac{s}{2}\left\{-2r+\mathrm{W}\frac{2r-a}{rs}\right\}$$
$$=rs-\mathrm{W}\,.\,\frac{2r-a}{2r}.$$

To this are to be added $+\mathrm{W}$ and $-\left(rs+\mathrm{W}\frac{a}{2r}\right)$, or $-rs+\mathrm{W}\frac{2r-a}{2r}$; the sum is 0. Equation (16.) therefore is satisfied when z is in the "first part."

For equation (17.): omitting $y\,\frac{d\mathrm{F}}{dy}$ (because, as is explained above, it cannot produce any term), it will be found that in the "first part"

$$x\frac{d\mathrm{F}}{dx}-\mathrm{F}=\frac{6}{s^2}\cdot\left\{-x^2\right\}\cdot\left(\frac{y}{6}-\frac{sy^2}{4}\right);$$

which for z, 0, $=0$, and for a, s, $=\frac{6}{s^2}\cdot a^2\cdot\frac{s^3}{12}$. And in the "second part,"

$$x\frac{d\mathrm{F}}{dx}-\mathrm{F}=\frac{6}{s^2}\left\{-\frac{2\mathrm{W}a}{s}-x^2\right\}\cdot\left(\frac{y^3}{6}-\frac{sy^2}{4}\right);$$

which for a, s,

$$=\frac{6}{s^2}\left\{\frac{2\mathrm{W}a}{s}+a^2\right\}\frac{s^3}{12},$$

and for $2r$, s,

$$=\frac{6}{s^2}\left\{\frac{2\mathrm{W}a}{s}+4r^2\right\}\cdot\frac{s^3}{12}.$$

The sum of the two portions of the integral will therefore be

$$\frac{s}{2}\left\{0+a^2-\frac{2\mathrm{W}a}{s}-a^2+\frac{2\mathrm{W}a}{s}+4r^2\right\}=2r^2s.$$

To this are to be added $+\mathrm{W}a$ and $-\left(rs+\mathrm{W}\frac{a}{2r}\right)2r$, or $-2r^2s$; the sum is 0. Equation (17.) therefore is satisfied when z is in the "first part."

29. Second, suppose the integrals to begin from a point z in the "second part." As before, it is unnecessary to make a trial of equation (15.), which is necessarily satisfied. In regard to equations (16.) and (17.), the integrals are only to be taken by the formulæ

of the "second part" from z, 0, to $2r$, s; and the only constant force is $-\left(rs+\mathrm{W}\frac{a}{2r}\right)$ at abscissa $2r$.

For equation (16.), $\frac{d\mathrm{F}}{dx}$ in the "second part" $=\frac{6}{s^2}\left\{2r-\mathrm{W}\frac{a}{rs}-2x\right\}\cdot\left(\frac{y^3}{6}-\frac{sy^2}{4}\right)$, which for z, 0, $=0$, and for $2r$, s, $=-\frac{s}{2}\left\{-2r-\mathrm{W}\frac{a}{rs}\right\}$, or $rs+\mathrm{W}\frac{a}{2r}$. To this is to be added $-\left(rs+\mathrm{W}\frac{a}{2r}\right)$; the sum is 0. Equation (16.) therefore is satisfied when z is in the "second part."

For equation (17.), $x\frac{d\mathrm{F}}{dx}-\mathrm{F}$ in the "second part" $=\frac{6}{s^2}\left\{-\frac{2\mathrm{W}a}{s}-x^2\right\}\cdot\left(\frac{y^3}{6}-\frac{sy^2}{4}\right)$, which for z, 0, $=0$, and for $2r$, s, $=\frac{s}{2}\left\{\frac{2\mathrm{W}a}{s}+4r^2\right\}=+\mathrm{W}a+2r^2s$. To this is to be added $-\left(rs+\mathrm{W}\frac{a}{2r}\right)2r$, or $-2r^2s-\mathrm{W}a$; the sum is 0. Equation (17.) therefore is satisfied when z is in the "second part."

30. It appears therefore that our equations (15.), (16.), (17.) are in all parts of this loaded bar satisfied by the discontinuous formula which we found for F; and therefore that formula is to be adopted in the further calculations. But different calculations must be made for the "first part" and the "second part."

First Part, from $x=0$ to $x=a$.	Second Part, from $x=a$ to $x=2r$.
$\mathrm{F}=\frac{6}{s^2}\cdot\left\{\left(2r+\mathrm{W}\cdot\ \ \frac{-a}{\ }\right)x-x^2\right\}\cdot\left(\frac{y^3}{6}-\frac{sy^2}{4}\right)$;	$\mathrm{F}=\frac{6}{s^2}\cdot\left\{\frac{2\mathrm{W}a}{s}+\left(2r-\mathrm{W}\cdot\frac{a}{rs}\right)x-x^2\right\}\cdot\left(\frac{y}{6}-\frac{sy^2}{4}\right)$;
$\mathrm{L}=\frac{6}{s^2}\cdot\left\{\left(2r+\mathrm{W}\cdot\ \ \frac{-a}{\ }\right)x-x^2\right\}\cdot\left(y-\frac{s}{2}\right)$;	$\mathrm{L}=\frac{6}{s^2}\cdot\left\{\frac{2\mathrm{W}a}{s}+\left(2r-\mathrm{W}\cdot\frac{a}{rs}\right)x-x^2\right\}\cdot\left(y-\frac{s}{2}\right)$;
$\mathrm{M}=\frac{6}{s^2}\cdot\left\{\ 2r+\mathrm{W}\cdot\ \ \frac{-a}{\ }-2x\right\}\cdot\left(\frac{y^2}{2}-\frac{sy}{2}\right)$;	$\mathrm{M}=\frac{6}{s^2}\left\{2r-\mathrm{W}\cdot\frac{a}{rs}-2x\right\}\cdot\left(\frac{y^2}{2}-\frac{sy}{2}\right)$;
$\mathrm{N}=\frac{6}{s^2}\cdot\left\{2r+\mathrm{W}\cdot\frac{2r-a}{rs}-2x\right\}\cdot(sy-y^2)$;	$\mathrm{N}=\frac{6}{s^2}\left\{2r-\mathrm{W}\frac{a}{rs}-2x\right\}\cdot(sy-y^2)$;
$\mathrm{O}=s\cdot\left(3\frac{y^2}{s^2}-2\frac{y^3}{s^3}\right)$;	$\mathrm{O}=s\cdot\left(3\frac{y^2}{s^2}-2\frac{y^3}{s^3}\right)$;
$\mathrm{Q}=s\cdot\left(\frac{y}{s}-3\frac{y^2}{s^2}+2\frac{y^3}{s^3}\right)$.	$\mathrm{Q}=s\cdot\left(\frac{y}{s}-3\frac{y^2}{\ }+2\frac{y^3}{\ }\right)$

To diminish the number of symbols, we will at once assume that W = weight of half the bar $=rs$. Then we have

$\mathrm{L}=\frac{6}{s^2}\cdot\left\{(4r-a)x-x^2\right\}\cdot\left(y-\frac{s}{2}\right)$;	$\mathrm{L}=\frac{6}{s^2}\left\{2ra+(2r-a)x-x^2\right\}\cdot\left(y-\frac{s}{2}\right)$;
$\mathrm{N}=\frac{6}{s^2}\cdot\left\{4r-a-2x\right\}\cdot(sy-y^2)$;	$\mathrm{N}=\frac{6}{s^2}\left\{2r-a-2x\right\}\cdot(sy-y^2)$;
$\mathrm{Q}=s\cdot\left(\frac{y}{s}-\ \frac{y^2}{\ }+2\frac{y^3}{\ }\right.$	$\mathrm{Q}=s\left(\frac{y}{s}-3\frac{y^2}{s^2}+2\frac{y^3}{s^3}\right)$.

And we will now select the cases which it appears desirable to compute numerically.

31. The strains upon the beam are not at all affected by placing a weight upon its end (supposed strong enough to resist distortion of form). It appears probable, therefore, that the extreme changes of opposite character will be given, on the one hand, by placing the weight upon the centre of the beam's length, or making $a=r$; on the other hand by placing the weight upon the centre of one half of the beam, or making $a=\frac{r}{2}$. We will proceed first with the formulæ for the case when the weight is upon the centre, or $a=r$. It is unnecessary here to make calculations for the two segments of the beam, as the strains will be symmetrical with respect to the two extremities. As before, $\frac{r}{s}$ is taken $=5$.

Weight *rs* placed on the centre of the beam's length.

$$\mathrm{L}=\frac{-6}{s^2}\left\{3rx-x^2\right\}.\left(\frac{s}{2}-y\right)=-s.\frac{3r^2}{s^2}.\left\{\frac{3x}{r}-\frac{x^2}{r^2}\right\}.\left(1-\frac{2y}{s}\right)=-s.75.v.(3-v).(1-2w);$$

$$\mathrm{N}=\frac{6}{s^2}\{3r-2x\}.y.(s-y)\;=\;s.\frac{12r}{s}.\left\{\frac{3}{2}-\frac{x}{r}\right\}.\frac{y}{s}\left(1-\frac{y}{s}\right)\;=\;s.60.\left\{\frac{3}{2}-v\right\}.w.(1-w);$$

$$\mathrm{Q}\qquad\qquad=\;s.w.(1-w).(1-2w);$$

from all which, as before, the general factor *s* may be omitted.

Proceeding now with the other case, or

Weight *rs* placed on the centre of the first half of the beam's length,

the formulæ for the "first part," from $x=0$ to $x=a=\frac{r}{2}$, or from $v=0$ to $v=0{\cdot}5$, will be

$$\mathrm{L}=\frac{-6}{s^2}.\left\{\frac{7r}{2}x-x^2\right\}.\left(\frac{s}{2}-y\right)=-s.\frac{3r^2}{s^2}.\left\{\frac{7}{2}.\frac{x}{r}-\frac{x^2}{r^2}\right\}.\left(1-\frac{2y}{s}\right)=-s.75.v\left(\frac{7}{2}-v\right).(1-2w);$$

$$\mathrm{N}=\frac{6}{s^2}.\left\{\frac{7r}{2}-2x\right\}y.(s-y)\;=\;s.\frac{12r}{s}.\left\{\frac{7}{4}-\frac{x}{r}\right\}.\frac{y}{s}.\left(1-\frac{y}{s}\right)\;=\;s.60.\left\{\frac{7}{4}-v\right\}.w.(1-w);$$

$$\mathrm{Q}\qquad\qquad=\;s.w.(1-w).(1-2w);$$

and those for the "second part," from $x=\frac{r}{2}$ to $x=2r$, or from $v=0{\cdot}5$ to $v=2{\cdot}0$, will be

$$\mathrm{L}=-\frac{6}{s^2}.\left\{r^2+\frac{3r}{2}x-x^2\right\}.\left\{\frac{s}{2}-y\right\}=-s.\frac{3r^2}{s^2}.\left(2-\frac{x}{r}\right).\left(\frac{1}{2}+\frac{x}{r}\right).\left(1-\frac{2y}{s}\right)=-s.75.(2-v).(\frac{1}{2}+v$$

$$\mathrm{N}=\;\frac{6}{s^2}.\left\{\frac{3}{2}-2x\right\}.(sy-y^2)\;=\;s.\frac{12r}{s}.\left\{\frac{3}{4}-\frac{x}{r}\right\}.\frac{y}{s}.\left(1-\frac{y}{s}\right)\;=\;s.60.\left(\frac{3}{4}-v\right).w.(1$$

$$\mathrm{Q}\qquad\qquad=\;s.w.(1-w).(1-2$$

from all which the factor *s* may be omitted.

32. For all these cases, the same formulæ as before are to be used in the ultimate

calculations of the magnitudes and directions of the strains, namely,

$$\tan 2\beta=\frac{N}{L+Q},$$

$$C-B=\frac{N}{\sin 2\beta}=\frac{L+Q}{\cos 2\beta},$$

$$C+B=L-Q.$$

By means of these, the numbers have been computed for Table III., and Table IV. parts 1 and 2 (end of the Memoir), and the lines of figs. 6 and 7, Plate VI. have been traced.

33. It is worthy of remark that, in figures 4 and 5, the lines representing the direction of thrust, and also those representing the direction of pull, are continuous; but in figures 6 and 7 they are discontinuous, the two segments of each curve, at their meeting in the ordinate vertically below the weight, having different tangential directions. This follows as an inevitable consequence of the assumption in art. 16; I think it probable that a hypothesis like that of Mr. W. H. Barlow would remove the discontinuity. An investigation similar to that of art. 25 would show that, at these points, the transverse section of the beam must be sufficiently strong to support the weight by thrust (if the weight is on the top of the beam), or by tension (if the weight is carried by or attached to the bottom of the beam).

34. There are cases somewhat different from those already considered, whose importance and singularity of principle are such as to make them worthy of special notice. In Mr. Robert Stephenson's construction of the Britannia Bridge, the strength of the tubes was nearly doubled by the following admirable arrangement. The junction of the ends of successive tubes, at their meeting on the piers, was effected, not while the two successive tubes rested on the bearings which they were finally to take, but while the distant end of one of the tubes was considerably elevated. It is a problem of no great difficulty to ascertain what elevation ought to be given in order to reduce the maximum strains on the bridge to their smallest value; when the best arrangement is made, the strains are reduced to one-half of their original value. The singularity of the mathematical principle consists in this, that there is impressed on the end-frame of the tube or beam a strain of the nature of a couple, or (as it is called in the preceding articles) a moment. Where there are three or more connected tubes, the middle tube, or each of the middle tubes, has such a moment-strain at each end; but each of the external tubes has a moment-strain at one end only (inasmuch as, at the land termination of the bridge, there are no means of applying such a strain). There are therefore two different cases, requiring different investigations.

35. Take, first, the case of a middle tube in which a moment-strain is impressed on each end, the directions of the two strains (supposed equal) being opposed, so that both tend to raise the middle of the tube. The pressures upon the two piers will not be disturbed, because the effects of the two strains upon the entire beam balance. If now we consider the forces which act on the distant part of the beam (using the lan-

uage of art. 7), we shall have to combine, with forces formerly recognized, the moment which acts on the distant end. By the known laws of translation of the place of application of a moment, we may suppose this moment applied at the imaginary division of the bar. Thus, at every vertical section of the bar, there is combined with the ordinary moment of strains a moment equal to that impressed on each end. The most advantageous magnitude for this moment is evidently half the magnitude of moment at the beam's centre, with opposite sign; for if we use a smaller value we leave too much moment at the centre, and if we use a larger value we impress too great a straining moment at the junction above the pier.

36. Now in art. 21 we found, for the horizontal thrust in a point of any vertical section, $\frac{6x^2-12rx}{s^2}\cdot\left(\frac{s}{2}-y\right)$. As regards the variations of x, this is greatest when $x=r$, and its value is then $-\frac{6r^2}{s^2}\left(\frac{s}{2}-y\right)$. One half of this with changed sign, or $+\frac{3r^2}{s^2}\left(\frac{s}{2}-y\right)$, is now to be applied to the expression for horizontal thrust in every part of the beam's length. Hence the expression to be used for horizontal thrust or compression is

$$\frac{6x^2-12rx+3r^2}{s^2}\cdot\left(\frac{s}{2}-y\right),$$

and therefore

$$\mathrm{F}=\frac{6x^2-12rx+3r^2}{s^2}\cdot\left(\frac{sy^2}{4}-\frac{y^3}{6}\right).$$

It will be seen immediately that this quantity satisfies the equations (15.) and (16.), the integrals being taken from z, 0 to $2r$, s. But with regard to equation (17.), we must consider that in the instance before us a moment is to be introduced which has not presented itself before, namely, the moment impressed on the distant end. The value of that moment, which (with the sign contemplated in forming equation (17.)) is $-\int dy\cdot\frac{3r^2}{s^2}\left(\frac{sy}{2}-y^2\right)$, becomes $+\frac{r^2s}{4}$. Hence equation (17.) becomes in this case

$$\left\{y\frac{d\mathrm{F}}{dy}+x\frac{d\mathrm{F}}{dx}-\mathrm{F}\right\}_{2r,\,s}-\left\{y\frac{d\mathrm{F}}{dy}+x\frac{d\mathrm{F}}{dx}-\mathrm{F}\right\}_{z,\,0}-2r^2s+\frac{r^2s}{4}=0.$$

And, on making the substitutions, this equation is satisfied.

37. Therefore we are to adopt

$$\mathrm{F}=\frac{6x^2-12rx+3r^2}{s^2}\cdot\left(\frac{sy^2}{4}-\frac{y^3}{6}\right),$$

from which

$$\mathrm{L}=\frac{d^2\mathrm{F}}{dy^2}=\frac{6x^2-12rx+3r^2}{s^2}\cdot\left(\frac{s}{2}-y\right)=\quad s\cdot\frac{3r^2}{s^2}\cdot\left(\frac{1}{2}-\frac{2x}{r}+\frac{x^2}{r^2}\right)\cdot\left(1-\frac{2y}{s}\right);$$

$$\mathrm{M}=\frac{d^2\mathrm{F}}{dxdy}=\frac{12x-12r}{s^2}\cdot\left(\frac{sy}{2}-\frac{y^2}{2}\right)\qquad=-s\cdot\frac{6r}{s}\cdot\left(1-\frac{x}{r}\right)\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right);$$

$$\mathrm{N}=-2\mathrm{M}\qquad\qquad=\quad s\cdot\frac{12r}{s}\cdot\left(1-\frac{x}{r}\right)\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right);$$

$$\text{O}=\frac{d^2\text{F}}{dx^2}=\frac{12}{s^2}\cdot\left(\frac{sy^2}{4}-\frac{y^3}{6}\right)\qquad=s\cdot\left(3\frac{y^2}{s^2}-2\frac{y^3}{s^3}\right);$$

$$\text{Q}=y-\text{O}=s\frac{y}{s}-\text{O}=s\left(\frac{y}{s}-3\frac{y^2}{s^2}+2\frac{y^3}{s^3}\right)=s\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right)\cdot\left(1-\frac{2y}{s}\right).$$

Or, if $v=\frac{x}{r}$, $w=\frac{y}{s}$, $\frac{r}{s}=5$, and the multiplier s be omitted,

$$\text{L}=75\,.\left\{(1-v)^2-\frac{1}{2}\right\}.(1-2w);$$

$$\text{N}=60\,.(1-v)\,.\,w\,.(1-w);$$

$$\text{Q}=w\,.(1-w).(1-2w).$$

Then

$$\tan 2\beta=\frac{\text{N}}{\text{L}+\text{Q}};\quad \text{C}-\text{B}=\frac{\text{N}}{\sin 2\beta}=\frac{\text{L}+\text{Q}}{\cos 2\beta};\quad \text{C}+\text{B}=\text{L}-\text{Q},$$

by which the numbers for Table V. have been computed, and the curves of figure 8, Plate VII. have been drawn.

38. Take, secondly, the case of an end tube, on which a moment is impressed only at one end. In this case, the effect of that moment is not balanced by a moment impressed at the other end, and must be balanced by an increase of pressure on the near pier (at which the moment is impressed), and a decrease of pressure on the distant pier. The value $\frac{r^2s}{4}$ of moment will be balanced by an increase of pressure $\frac{rs}{8}$ on the near pier, and a decrease of pressure $\frac{rs}{8}$ on the distant pier. Hence the pressure on the distant pier will be $rs-\frac{rs}{8}=\frac{7rs}{8}$. From this (as in art. 21) the moment produced by the weight of the distant part of the beam $=\frac{s(2r-x)^2}{2}$, and the moment produced by the reaction of the distant pier is $=-\frac{7rs}{8}\times(2r-x)$. The equation of moments is now

$$-\frac{cs^3}{12}+\frac{s(2r-x)^2}{2}-\frac{7rs}{8}\times(2r-x)=0;$$

or

$$-\frac{cs^3}{12}+s(2r-x).\left\{r-\frac{x}{2}-\frac{7r}{8}\right\}=0;$$

or

$$-\frac{cs^3}{12}-s(2r-x).\left(\frac{x}{2}-\frac{r}{8}\right)=0$$

From this,

$$c=\frac{(6x-12r)\left(x-\frac{r}{4}\right)}{s^2};$$

and the horizontal compression-force at elevation y

$$=\frac{(6x-12r)\left(x-\frac{r}{4}\right)}{s^2}\left(\frac{s}{2}-y\right)$$

Therefore we are to take for trial

$$F=\frac{(6x-12r)\left(x-\frac{r}{4}\right)}{s^2}\left(\frac{sy^2}{4}-\frac{y^3}{6}\right).$$

Remarking that the reaction of the distant pier $=\frac{7rs}{8}$, and that its moment upwards $=\frac{7rs}{8}\times 2r$, it will be found that this function satisfies equations (15.), (16.), (17.).

39. Adopting therefore

$$F=\frac{(6x-12r)\cdot\left(x-\frac{r}{4}\right)}{s^2}\cdot\left(\frac{sy^2}{4}-\frac{y^3}{6}\right),$$

we have

$$L=\frac{d^2F}{dy^2}=\frac{(6x-12r)\cdot\left(x-\frac{r}{4}\right)}{s^2}\cdot\left(\frac{s}{2}-y\right)=\quad s\cdot\frac{3r^2}{s^2}\cdot\left(\frac{x}{r}-2\right)\cdot\left(\frac{x}{r}-\frac{1}{4}\right)\cdot\left(1-\frac{2y}{s}\right);$$

$$M=\frac{d^2F}{dxdy}=\frac{12x-\frac{27}{2}r}{s^2}\cdot\left(\frac{sy}{2}-\frac{y^2}{2}\right)\qquad=-s\cdot\frac{6r}{s}\cdot\left(\frac{9}{8}-\frac{x}{r}\right)\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right);$$

$$N=-2M\qquad=\quad s\cdot\frac{12r}{s}\cdot\left(\frac{9}{8}-\frac{x}{r}\right)\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right);$$

$$O=\frac{d^2F}{dx^2}=\frac{12}{s^2}\cdot\left(\frac{sy^2}{4}-\frac{y^3}{6}\right)\qquad=\quad s\cdot\left(3\frac{y^2}{s^2}-2\frac{y^3}{s^3}\right);$$

$$Q=y-O\qquad=\quad s\cdot\frac{y}{s}\cdot\left(1-\frac{y}{s}\right)\cdot\left(1-\frac{2y}{s}\right).$$

And, with $v=\frac{x}{r}$, $w=\frac{y}{s}$, $\frac{r}{s}=5$, $s=1$,

$$L=-75.(2-v).\left(v-\frac{1}{4}\right)\cdot(1-2w);$$

$$N=\ 60\cdot\left(\frac{9}{8}-v\right).w.(1-w);$$

$$Q=\ w.(1-w).(1-2w);$$

$$\tan 2\beta=\frac{N}{L+Q};\quad C-B=\frac{N}{\sin 2\beta}=\frac{L+Q}{\cos 2\beta};\quad C+B=L-Q;$$

by which the numbers for Table VI. have been computed, and the curves of figure 9 have been drawn.

40. These instances will probably suffice as applications of the theory to the most important cases of practice, and as examples of the modifications on subordinate points which may be required in investigating strains where the forms or other circumstances are different from those considered here.

41. Perhaps useful information may be derived from the diagrams and tables of numbers for guiding the construction of Latticed Bridges. Thus, in such cases as those of figures 5, 6, 7, the upper and lower edges require great longitudinal strength in the middle of the beam's length, but very little near the ends; on the contrary, powerful lattice-work is required near the ends, but very little near the middle. In the case of figure 8 these remarks require very considerable modification.

Table I.—Strains on the interior points of a beam which projects from a wall, supporting no other weight. The length of the beam is supposed to be five times its depth. The two numbers in each division of the Table are the values of the two principal strains, the unit being the depth of the beam. The positive sign denotes compression, and the negative sign tension. The angle is that by which the first-written strain is inclined to y, in the direction of diminishing x for increase of y. The direction of the second-written strain is at right angles to that of the first-written strain.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from the wall, to the whole length of the beam).										
	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
1·0	−75·00 0·00 90° 0′	−60·75 0·00 90° 0′	−48·00 0·00 90° 0′	−36·75 0·00 90° 0′	−27·00 0·00 90° 0′	−18·75 0·00 90° 0′	−12·00 0·00 90° 0′	− 6·75 0·00 90° 0′	− 3·00 0·00 90° 0′	− 0·75 0·00 90° 0′	0·00 0·00
0·9	−60·12 + 0·19 87 30	−48·72 + 0·19 87 10	−38·52 + 0·19 86 50	−29·52 + 0·19 86 20	−21·72 + 0·19 85 40	−15·12 + 0·19 84 50	− 9·72 + 0·19 83 40	− 5·52 + 0·19 81 40	− 2·51 + 0·18 78 10	− 0·70 + 0·17 70 40	0·00 0·00
0·8	−45 51 + 0·60 84 0	−36·95 + 0·60 83 20	−29·30 + 0·60 82 30	−22·55 + 0·59 81 30	−16·69 + 0·59 80 20	−11·74 + 0·58 78 30	− 7·68 + 0·57 76 10	− 4·50 + 0·55 72 40	− 2·20 + 0·50 67 20	− 0·73 + 0·38 59 50	0·00 0·00
0·7	−31·27 + 1·35 78 40	−25·56 + 1·34 77 30	−20·44 + 1·32 76 10	−15·92 + 1·30 74 30	−11·98 + 1·27 72 40	− 8·64 + 1·22 70 10	− 5·87 + 1·15 67 0	− 3·66 + 1·04 63 10	− 1·97 + 0·86 58 30	− 0·77 + 0·55 53 30	0·00 0·00
0·6	−17·89 + 2·94 68 10	−14·95 + 2·85 66 40	−12·29 + 2·74 65 0	− 9·90 + 2·60 63 10	− 7·78 + 2·43 61 10	− 5·92 + 2·22 58 50	− 4·31 + 1·95 56 30	− 2·92 + 1·62 54 0	− 1·75 + 1·20 51 20	− 0·78 + 0·68 48 50	0·00 0·00
0·5	− 7·50 + 7·50 45 0	− 6·75 + 6·75 45 0	− 6·00 + 6·00 45 0	− 5·25 + 5·25 45 0	− 4·50 + 4·50 45 0	− 3·75 + 3·75 45 0	− 3·00 + 3·00 45 0	− 2·25 + 2·25 45 0	− 1·50 + 1·50 45 0	− 0·75 + 0·75 45 0	0·00 0·00
0·4	− 2·94 +17·89 21 50	− 2·85 +14·95 23 20	− 2·74 +12·29 25 0	− 2·60 + 9·90 26 50	− 2·43 + 7·78 28 50	− 2·22 + 5·92 31 10	− 1·95 + 4·31 33 30	− 1·62 + 2·92 36 0	− 1·20 + 1·75 38 40	− 0·68 + 0·78 41 10	0·00 0·00
0·3	− 1·35 +31·27 11 20	− 1·34 +25·56 12 30	− 1·32 +20·44 13 50	− 1·30 +15·92 15 30	− 1·27 +11·98 17 20	− 1·22 + 8·64 19 50	− 1·15 + 5·87 23 0	− 1·04 + 3·66 26 50	− 0·86 + 1·97 31 30	− 0·55 + 0·77 36 30	0·00 0·00
0·2	− 0·60 +45·51 6 0	− 0·60 +36·95 6 40	− 0·60 +29·30 7 30	− 0·59 +22·55 8 30	− 0·59 +16·69 9 40	− 0·58 +11·74 11 30	− 0·57 + 7·68 13 50	− 0·55 + 4·50 17 20	− 0·50 + 2·20 22 40	− 0·38 + 0·73 30 10	0·00 0·00
0·1	− 0·19 +60·12 2 30	− 0·19 +48·72 2 50	− 0·19 +38·52 3 10	− 0·19 +29·52 3 40	− 0·19 +21·72 4 20	− 0·19 +15·12 5 10	− 0·19 + 9·72 6 20	− 0·19 + 5·52 8 20	− 0·18 + 2·51 11 50	− 0·17 + 0·70 19 20	0·00 0·00
0·0	0·00 +75·00 0 0	0·00 +60·75 0 0	0·00 +48·00 0 0	0·00 +36·75 0 0	0·00 +27·00 0 0	0·00 +18·75 0 0	0·00 +12·00 0 0	0·00 + 6·75 0 0	0·00 + 3·00 0 0	0·00 + 0·75 0 0	0·00 0·00

Table II.—Strains on the interior points of a beam whose two ends rest upon piers, and which supports no other weight. The length of the beam is supposed to be ten times its depth. The two numbers in each division of the Table are the values of the two principal strains, the unit being the depth of the beam. The positive sign denotes compression, and the negative sign tension. The angle is that by which the first-written strain is inclined to y, in the direction of diminishing x for increase of y. The direction of the second-written strain is at right angles to that of the first-written strain.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).		Values of v (the proportion of the horizontal ordinate of a point, measured from one pier, to half the length of the beam).										
		0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
	1 0	0·00 00·0 ° ′	0·00 +14·25 0° 0′	0·00 +27·00 0° 0′	0 00 +38·25 0° 0′	0·00 +48·00 0° 0′	0·00 +56·25 0° 0′	0·00 +63·00 0 0′	0·00 +68·25 0° 0′	0·00 +72·00 0° 0′	0·00 +74·25 0° 0′	0·00 +75·00 0° 0′
	0·9	− 2·66 + 2·74 45 20	− 0·43 +11·90 11 40	− 0·14 +21·82 5 40	− 0 05 +30·72 3 30	0·00 +38·47 2 30	+ 0·03 +45·04 1 40	+ 0·05 +50·42 1 10	+ 0·06 +54·61 0 50	+ 0·07 +57·61 0 30	+ 0 07 +59·40 0 20	+ 0 07 60·00 0 0
	0·8	− 4·75 + 4·85 45 20	− 1·72 +10·37 22 50	− 0·77 +17·07 12 50	− 0·39 +23·43 8 10	− 0·19 +29·09 5 40	− 0 08 +33·92 1 0	0·00 +37·90 2 50	+ 0·05 +41·00 2 0	+ 0·08 +43·22 1 20	+ 0·09 +44·56 0 40	+ 0·10 +45·00 0 0
	0·7	− 6·26 + 6·34 45 10	− 3·44 + 9·22 31 50	− 1·91 +12·80 21 40	− 1·10 +16·49 15 0	− 0·64 +19·92 10 50	− 0·35 +22·94 7 50	− 0·17 +25·45 5 40	− 0·05 +27·43 4 0	+ 0·03 +28·86 2 30	+ 0·07 +29·71 1 10	+ 0·08 +30·00 0 0
	0 6	− 7·18 + 7·22 45 10	− 5·18 + 8·08 38 50	− 3·63 + 9·08 32 30	− 2·46 +10·16 26 30	− 1·62 +11·26 21 0	− 1·01 +12·31 16 20	− 0 59 +13·23 12 20	− 0·29 +13·99 8 50	− 0·10 +14·54 5 40	+ 0·01 +14·89 2 50	+ 0·05 +15·00 0 0
	0·5	− 7·50 + 7·50 45 0	− 6·75 + 6·75 45 0	− 6·00 + 6·00 45 0	− 5·25 + 5·25 45 0	− 4·50 + 4·50 45 0	− 3·75 + 3·75 45 0	− 3·00 + 3·00 45 0	− 2·25 + 2·25 45 0	− 1·50 + 1·50 45 0	− 0·75 + 0·75 45 0	0·00 0·00
	0 4	− 7·22 + 7·18 44 50	− 8·08 + 5·18 51 10	− 9·08 + 3·63 57 30	−10·16 + 2·46 63 30	−11·26 + 1·62 69 0	−12·31 + 1·01 73 40	−13 23 + 0·58 77 40	−13·99 + 0·29 81 10	−14·54 + 0·10 84 20	−14·89 − 0·01 87 10	−15·00 − 0·05 90 0
	0·3	− 6·34 + 6·26 44 50	− 9·22 + 3·44 58 10	−12·80 + 1·91 68 20	−16·49 + 1·10 75 0	−19·92 + 0·64 79 10	−22·94 + 0·35 82 10	−25·45 + 0·17 84 20	−27·43 + 0·05 86 0	−28·86 − 0·03 87 30	−29·72 − 0·07 88 50	−30·00 − 0·08 90 0
	0·2	− 4·85 + 4·75 44 40	−10·37 + 1·72 67 10	−17·07 + 0·73 77 10	−23·43 + 0·39 81 50	−29·09 + 0·19 84 20	−33·92 + 0·08 86 0	−37·90 0·00 87 10	−41·00 − 0·05 88 0	−43·22 − 0·08 88 40	−44·56 − 0·09 89 20	−45·00 − 0·10 90 0
	0·1	− 2·74 + 2·66 44 40	−11·90 + 0·43 78 20	−21·82 + 0·14 84 20	−30·72 + 0·05 86 30	−38·47 0 00 87 30	−45·04 − 0 03 88 20	−50·42 − 0·05 88 50	−54·61 − 0 06 89 10	−57·61 − 0·07 89 30	−59·40 − 0·07 89 40	−60·00 − 0·07 90 0
	0·0	0·00 0·00	−14·25 0·00 90 0	−27·00 0·00 90 0	−38·25 0·00 90 0	−48·00 0·00 90 0	−56·25 0·00 90 0	−63·00 0·00 90 0	−68 25 0·00 90 0	−72·00 0·00 90 0	−74·25 0·00 90 0	−75·00 0·00 90 0

The numbers from $v=0·0$ to $v=1·0$ apply also from $v=2·0$ to $v=1·0$, changing the sign of the angle. In regard to the pressure on the piers, where $v=0·0$ and $w=0·0$, see article 24.

Table III.—Strains on the interior points of a beam whose two ends rest upon piers, and which supports at the middle of its length a weight equal to half the weight of the beam. The length of the beam is supposed to be ten times its depth. The two numbers in each division of the Table are the values of the two principal strains, the unit being the depth of the beam. The positive sign denotes compression, and the negative sign tension. The angle is that by which the first-written strain is inclined to y, in the direction of diminishing x for increase of y. The direction of the second-written strain is at right angles to that of the first-written strain.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from one pier, to half the length of the beam).										
	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
1·0	0·00 0·00 ° ′	0·00 +21·75 0° 0′	0·00 +42·00 0° 0′	0·00 +60·75 0° 0′	0·00 +78·00 0° 0′	0·00 +93·75 0° 0′	0·00 +108·00 0° 0′	0·00 +120·75 0° 0′	0·00 +132·00 0° 0′	0·00 +141·75 0° 0′	0·00 +150·00 0° 0′
0·9	– 4·01 + 4·09 45 10	– 0·72 +18·19 11 50	– 0·29 +33·96 5 50	– 0·14 +48·82 3 50	– 0·07 +62·54 2 40	– 0·03 +75·10 2 0	+ 0·01 + 86·47 1 40	+ 0·02 + 96·65 1 20	+ 0·01 +105·64 1 0	+ 0·05 +113·42 0 50	+ 0·06 +120·02 0 40
0·8	– 7·15 + 7·25 45 10	– 2·76 +15·91 23 0	– 1·37 +26·67 13 10	– 0·80 +37·34 8 50	– 0·49 +47·39 6 20	– 0·31 +56·66 4 50	– 0·19 + 65·09 3 50	– 0·11 + 72·65 3 0	– 0·05 + 79·34 2 30	0·00 + 85·15 2 0	+ 0·03 + 90·07 1 30
0·7	– 9·41 + 9·49 45 10	– 5·42 +14·21 32 0	– 3·26 +20·14 22 10	– 2·08 +26·47 16 0	– 1·42 +32·71 12 0	– 0·95 +38·53 9 20	– 0·65 + 43·93 7 20	– 0·44 + 48·82 5 50	– 0·28 + 53·17 4 40	– 0·17 + 56·95 3 50	– 0·08 + 60·17 3 0
0·6	–10·78 +10·82 45 0	– 8·11 +12·50 39 0	– 6·23 +14·47 33 0	– 4·45 +16·65 27 30	– 3·27 +18·92 22 50	– 2·40 +21·20 18 50	– 1·75 + 23·40 15 30	– 1·25 + 25·44 12 50	– 0·88 + 27·33 10 30	– 0·60 + 29·00 8 30	– 0·38 + 30·43 6 50
0·5	–11·25 +11·25 45 0	–10·50 +10·50 45 0	– 9·75 + 9·75 45 0	– 9·00 + 9·00 45 0	– 8·25 + 8·25 45 0	– 7·50 + 7·50 45 0	– 6·75 + 6·75 45 0	– 6·00 + 6·00 45 0	– 5·25 + 5·25 45 0	– 4·50 + 4·50 45 0	– 3·75 + 3·75 45 0
0·4	–10·82 +10·78 45 0	–12·51 + 8·11 51 0	–14·47 + 6·23 57 0	–16·65 + 4·45 62 30	–18·92 + 3·27 67 10	–21·20 + 2·40 71 10	– 23·40 + 1·75 74 30	– 25·44 + 1·25 77 10	– 27·33 + 0·88 79 30	– 29·00 + 0·60 81 30	– 30·42 + 0·38 83 10
0·3	– 9·49 + 9·41 44 50	–14·21 + 5·42 58 0	–20·14 + 3·26 67 50	–26·47 + 2·08 74 0	–32·71 + 1·42 78 0	–38·53 + 0·95 80 40	– 43·93 + 0·66 82 40	– 48·82 + 0·44 84 10	– 53·17 + 0·28 85 20	– 56·95 + 0·17 86 10	– 60·17 + 0·08 87 0
0·2	– 7·25 + 7·15 44 50	–15·91 + 2·76 67 0	–26·67 + 1·37 76 50	–37·34 + 0·80 81 10	–47·39 + 0·49 83 40	–56·66 + 0·31 85 10	– 65·09 + 0·19 86 10	– 72·65 + 0·11 87 0	– 79·34 + 0·05 87 30	– 85·15 0·00 88 0	– 90·07 – 0·04 88 30
0·1	– 4·09 + 4·01 44 50	–18·19 + 0·72 78 10	–33·96 + 0·29 84 10	–48·82 + 0·14 86 10	–62·54 + 0·07 87 20	–75·10 + 0·03 88 0	– 86·47 – 0·01 88 20	– 96·65 – 0·02 88 40	–105·64 – 0·04 89 0	–113·42 – 0·05 89 10	–120·02 – 0·06 89 20
0·0	0·00 0·00	–21·75 0·00 90 0	–42·00 0·00 90 0	–60·75 0·00 90 0	–78·00 0·00 90 0	–93·75 0·00 90 0	–108·00 0·00 90 0	–120·75 0·00 90 0	–132·00 0·00 90 0	–141·75 0·00 90 0	–150·00 0·00 90 0

The numbers from $v=0·0$ to $v=1·0$ apply also from $v=2·0$ to $v=1·0$, changing the sign of the angle.

TABLE IV. Part I.—Strains on the interior points of a beam whose two ends rest upon piers, and which supports at the middle of its first half-length a weight equal to half the weight of the beam. Continued in Table IV. Part II.

The explanations are the same as those of Tables II. and III.

Values of *w* (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of *v* (the proportion of the horizontal ordinate of a point, measured from the first pier, to half the length of the beam).										
	First part of the beam, to the supported weight.						Second part of the beam, beyond the supported weight (continued in next Table).				
	0·0	0·1	0·2	0·3	0·4	0·5	0·5	0·6	0·7	0·8	0·9
1·0	0·00 0·00 ° ′	0·00 +25·50 0° 0′	0·00 +49·50 0° 0′	0·00 +72·00 0° 0′	0·00 +93·00 0° 0′	0·00 +112·50 0° 0′	0·00 +112·50 0° 0′	0·00 +115·50 0° 0′	0·00 +117·00 0° 0′	0·00 +117·00 0° 0′	0·00 +115·50 0° 0′
0·9	− 4·69 + 4·76 45 10	− 0·86 +21·33 11 50	− 0·37 +40·04 6 0	− 0·19 +57·87 3 50	− 0·10 +74·58 2 50	− 0·05 + 90·13 2 10	+ 0·07 + 90·00 0 30	+ 0·07 + 92·40 0 20	+ 0·07 + 93·60 0 10	+ 0·07 + 93·60 − 0 10	+ 0·07 + 92·40 − 0 20
0·8	− 8·35 + 8·45 45 10	− 3·28 +18·68 23 10	− 1·67 +31·46 13 20	− 1·00 +44·30 9 0	− 0·65 +56·54 6 30	− 0·43 + 68·03 5 0	+ 0·08 + 67·52 1 0	+ 0·09 + 69·31 0 40	+ 0·10 + 70·20 0 10	+ 0·10 + 70·20 − 0 10	+ 0·09 + 69·31 − 0 40
0·7	−10·98 +11·07 45 10	− 6·42 +16·70 32 0	− 3·94 +23·82 22 20	− 2·58 +31·46 16 10	− 1·77 +39·06 12 20	− 1·26 + 46·34 9 40	+ 0·03 + 45·06 2 0	+ 0·06 + 46·22 1 10	+ 0·08 + 46·80 0 20	+ 0·08 + 46·80 − 0 20	+ 0·06 + 46·22 − 1 10
0·6	−12·58 +12·62 45 0	− 9·57 +14·72 39 0	− 7·23 +17·17 33 10	− 5·44 +19·89 27 40	− 4·11 +22·76 23 10	− 3·11 + 25·66 19 20	− 0·10 + 22·64 4 30	0·00 + 23·15 2 40	+ 0·04 + 23·41 0 50	+ 0·04 + 23·41 − 0 50	0·00 + 23·15 − 2 40
0·5	−13·12 +13·12 45 0	−12·38 +12·38 45 0	−11·63 +11·63 45 0	−10·88 +10·88 45 0	−10·12 +10·12 45 0	− 9·38 + 9·38 45 0	− 1·88 + 1·88 45 0	− 1·13 + 1·13 45 0	− 0·38 + 0·38 45 0	− 0·38 + 0·38 −45 0	− 1·13 + 1·13 −45 0
0·4	−12·62 +12·58 45 0	−14·72 + 9·57 51 0	−17·17 + 7·23 56 50	−19·89 + 5·44 62 20	−22·76 + 4·11 66 50	− 25·66 + 3·11 70 40	− 22·64 + 0·10 85 30	− 23·15 0·00 87 20	− 23·41 − 0·04 89 10	− 23·41 − 0·04 −89 10	− 23·15 0·00 −87 20
0·3	−11·07 +10·98 44 50	−16·70 + 6·42 58 0	−23·82 + 3·94 67 40	−31·46 + 2·58 73 50	−39·06 + 1·77 77 40	− 46·34 + 1·26 80 20	− 45·06 − 0·03 88 0	− 46·22 − 0·06 88 50	− 46·80 − 0·08 89 40	− 46·80 − 0·08 −89 40	− 46·22 − 0·06 −88 50
0·2	− 8·45 +8·35 44 50	−18·67 + 3·28 66 50	−31·46 + 1·67 76 40	−44·30 + 1·00 81 0	−56·54 + 0·65 83 30	− 68·03 + 0·43 85 0	− 67·52 − 0·08 89 0	− 69·31 − 0·09 89 20	− 70·20 − 0·10 89 50	− 70·20 − 0·10 −89 50	− 69·31 − 0·09 −89 20
0·1	− 4·76 + 4·69 44 50	−21·33 + 0·86 78 10	−40·01 +0·37 84 0	−57·87 + 0·19 86 10	−74·58 + 0·11 87 10	− 90·13 + 0·05 87 50	− 90·01 − 0·07 89 30	− 92·40 − 0·07 89 40	− 93·60 − 0·07 89 50	− 93·60 − 0·07 −89 50	− 92·40 − 0·07 −89 40
0·0	0·00 0·00	−25·50 0·00 90 0	−49·50 0·00 90 0	−72·00 0·00 90 0	−93·00 0·00 90 0	−112·50 0·00 90 0	−112·50 0·00 90 0	−115·50 0·00 90 0	−117·00 0·00 90 0	−117·00 0·00 −90 0	−115·50 0·00 −90 0

Table IV. Part II. (Continuation of Table IV. Part I.).—Strains on the interior points of a beam whose two ends rest upon piers, and which supports at the middle of its first half-length a weight equal to half the weight of the beam.

The explanations are the same as those of Tables II. and III.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from the first pier, to half the length of the beam). Second part of the beam, beyond the supported weight (continued from last Table).										
	1·0	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0
1·0	0·00 +112·50 0° 0′	0·00 +108·00 0° 0′	0·00 +102·00 0° 0′	0·00 +94·50 0° 0′	0·00 +85·50 0° 0′	0·00 +75·00 0° 0′	0·00 +63·00 0° 0′	0·00 +49·50 0° 0′	0·00 +34·50 0° 0′	0·00 +18·00 0° 0′	0·00 0·00 0° 0′
0·9	+ 0·07 + 90·00 − 0 30	+ 0·06 + 86·41 − 0 40	+ 0·05 + 81·62 − 0 50	+ 0·04 +75·63 − 1 10	+ 0·03 +68·45 − 1 30	0·00 +60·07 − 2 0	− 0·03 +50·51 − 2 40	− 0·09 +39·77 − 3 40	− 0·22 +29·79 − 5 50	− 0·57 +15·04 −11 40	−3·34 +3·41 −44 40
0·8	+ 0·08 + 67·52 − 1 0	+ 0·05 + 64·84 − 1 30	+ 0·02 + 61·28 − 2 0	− 0·03 +56·82 − 2 10	− 0·09 +51·49 − 3 30	− 0·19 +45·29 − 4 30	− 0·34 +38·24 − 6 10	− 0·59 +30·39 − 8 30	− 1·07 +21·87 −13 0	− 2·24 +13·14 −23 0	−5·95 +6·05 −44 50
0·7	+ 0·03 + 45·06 − 2 0	− 0·03 + 43·31 − 3 0	− 0·11 + 41·00 − 4 0	− 0·23 +38·12 − 5 10	− 0·40 +34·69 − 6 50	− 0·65 +30·73 − 8 50	− 1·01 +26·30 −11 30	− 1·59 +21·48 −15 40	− 2·59 +16·47 −22 0	− 4·43 +11·71 −32 0	−7·83 +7·91 −44 50
0·6	− 0·10 + 22·64 − 4 30	− 0·24 + 21·89 − 6 40	− 0·46 + 20·90 − 8 50	− 0·75 +19·70 −11 20	− 1·15 +18·30 −14 20	− 1·70 +16·75 −18 0	− 2·44 +15·09 −22 10	− 3·46 +13·40 −27 10	− 4·83 +11·77 −32 50	− 6·64 +10·29 −39 0	−8·98 +9·02 −45 0
0·5	− 1·88 + 1·88 −45 0	− 2·63 + 2·63 −45 0	− 3·38 + 3·38 −45 0	− 4·13 + 4·13 −45 0	− 4·88 + 4·88 −45 0	− 5·63 + 5·63 −45 0	− 6·32 + 6·32 −45 0	− 7·12 + 7·12 −45 0	− 7·88 + 7·88 −45 0	− 8·62 + 8·62 −45 0	−9·38 +9·38 −45 0
0·4	− 22·64 + 0·10 −85 30	− 21·89 + 0·24 −83 20	− 20·90 + 0·46 −81 10	−19·70 + 0·75 −78 40	−18·30 + 1·15 −75 40	−16·75 + 1·70 −72 0	−15·09 + 2·44 −67 50	−13·40 + 3·46 −62 50	−11·77 + 4·83 −57 10	−10·29 + 6·64 −51 0	−9·02 +8·98 −45 0
0·3	− 45·06 − 0·03 −88 0	− 43·31 + 0·03 −87 0	− 41·00 + 0·11 −86 0	−38·12 + 0·23 −84 50	−34·69 + 0·40 −83 10	−30·73 + 0·65 −81 10	−26·30 + 1·01 −78 30	−21·48 + 1·59 −74 20	−16·47 + 2·59 −68 0	−11·71 + 4·43 −58 0	−7·92 +7·83 −45 10
0·2	− 67·52 − 0·08 −89 0	− 64·84 − 0·05 −88 30	− 61·28 − 0·02 −88 0	−56·82 + 0·03 −87 20	−51·49 + 0·09 −86 30	−45·29 + 0·19 −85 30	−38·24 + 0·34 −83 50	−30·39 + 0·59 −81 30	−21·87 + 1·07 −77 0	−13·14 + 2·24 −67 0	−6·05 +5·95 −45 10
0·1	− 90·01 − 0·07 −89 30	− 86·41 − 0·06 −89 20	− 81·62 − 0·05 −89 10	−75·63 − 0·04 −88 50	−68·45 − 0·03 −88 30	−60·07 0·00 −88 0	−50·51 + 0·03 −87 20	−39·77 + 0·09 −86 20	−27·79 + 0·22 −84 10	−15·04 + 0·57 −78 20	−3·41 +3·34 −45 20
0·0	−112·50 0·00 −90 0	−108·00 0·00 −90 0	−102·00 0·00 −90 0	−94·50 0·00 −90 0	−85·50 0·00 −90 0	−75·00 0·00 −90 0	−63·00 0·00 −90 0	−49·50 0·00 −90 0	−34·50 0·00 −90 0	−18·00 0·00 −90 0	0·00 0·00

TABLE V.—Strains on the interior points of a beam whose ends rest upon piers, and in which a strain (of the nature of a moment or couple) is impressed on each end, as in the interior tubes of the Britannia Bridge.

The explanations are the same as those of Tables II. and III.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from the first pier, to half the length of the beam).										
	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
1·0	−37·50 0·00 90° 0′	−23·25 0·00 90° 0′	−10·50 0·00 90° 0′	0·00 + 0·75 0° 0′	0·00 +10·50 0° 0′	0·00 +18·75 0° 0′	0·00 +25·50 0° 0′	0·00 +30·75 0° 0′	0·00 +34·50 0° 0′	0·00 +36·75 0° 0′	0·00 +37·50 0° 0′
0·9	−30·24 + 0·31 84 50	−18·91 + 0·38 82 40	− 8·92 + 0·59 76 30	− 1·57 + 2·25 41 0	− 0·23 + 8·70 10 40	− 0·05 +15·12 5 10	+ 0·02 +20·46 3 0	+ 0·05 +24·63 1 50	+ 0·06 +27·61 1 10	+ 0·07 +29·40 0 30	+ 0·07 +30·00 0 0
0·8	−23·48 + 1·07 78 30	−15·17 + 1·32 74 10	− 8·10 + 1·90 64 50	− 3·09 + 3·64 43 30	− 1·04 + 7·13 21 30	− 0·40 +11·75 12 40	− 0·11 +15·54 7 10	− 0·02 +18·56 4 30	+ 0·05 +20·75 2 40	+ 0·09 +22·06 1 20	+ 0·10 +22·50 0 0
0·7	−17·29 + 2·37 70 0	−11·97 + 2·75 64 50	− 7·53 + 3·12 56 30	− 4·22 + 4·60 44 20	− 2·16 + 6·45 30 40	− 1·07 + 8·66 20 10	− 0·51 +10·79 13 10	− 0·22 +12·60 8 30	−0·03 +13·92 5 10	+ 0·06 +14·73 2 30	+ 0·08 +15·00 0 0
0·6	−11·86 + 4·40 58 50	− 9·19 + 4·59 55 0	− 6·89 + 4·83 50 20	− 4·94 + 5·14 44 40	− 3·37 + 5·51 38 20	− 2·15 + 5·95 31 20	− 1·26 + 6·41 24 20	− 0·64 + 6·84 17 40	− 0·24 + 7·19 11 20	− 0·02 + 7·42 5 30	+ 0·05 + 7·50 0 0
0·5	− 7·50 + 7·50 45 0	− 6·75 + 6·75 45 0	− 6·00 + 6·00 45 0	− 5·25 + 5·25 45 0	− 4·50 + 4·50 45 0	− 3·75 + 3·75 45 0	− 3·00 + 3·00 45 0	− 2·25 + 2·25 45 0	− 1·50 + 1·50 45 0	− 0·75 + 0·75 45 0	0·00 0·00
0·4	− 4·40 +11·86 31 10	− 4·59 + 9·19 35 0	− 4·83 + 6·89 39 40	− 5·14 + 4·94 45 20	− 5·51 + 3·37 51 40	− 5·95 + 2·15 58 40	− 6·41 + 1·26 65 40	− 6·81 + 0·64 72 20	− 7·19 + 0·24 78 40	− 7·42 + 0·02 84 30	− 7·50 − 0·05 90 0
0·3	− 2·34 +17·29 20 0	− 2·75 +11·97 25 10	− 3·42 + 7·53 33 30	− 4·60 + 4·22 45 40	− 6·45 + 2·16 59 20	− 8·66 + 1·07 69 50	−10·79 + 0·51 76 50	−12·60 + 0·22 81 30	−13·92 + 0·03 84 50	−14·73 − 0·06 87 30	−15·00 − 0·08 90 0
0·2	− 1·07 +23·48 11 30	− 1·32 +15·17 15 50	− 1·90 + 8·10 25 10	− 3·64 + 3·09 46 30	− 7·43 + 1·04 68 30	−11·75 + 0·40 77 20	−15·54 + 0·14 82 50	−18·56 + 0·02 85 30	−20·75 − 0·05 87 20	−22·06 − 0·09 88 40	−22·50 − 0·10 90 0
0·1	− 0·31 +30·24 5 10	− 0·38 +18·91 7 20	− 0·59 + 8·92 13 30	− 2·25 + 1·57 49 0	− 8·70 + 0·23 79 20	−15·12 + 0·05 81 50	−20·46 − 0·02 87 0	−24·63 − 0·05 88 10	−27·61 − 0·06 88 50	−29·40 − 0·07 89 30	−30·00 − 0·07 90 0
0·0	0·00 +37·50 0 0	0·00 +23·25 0 0	0·00 +10·50 0 0	− 0·75 0·00 90 0	−10·50 0·00 90 0	−18·75 0·00 90 0	−25·50 0·00 90 0	−30·75 0·00 90 0	−34·50 0·00 90 0	−36·75 0·00 90 0	−37·50 0·00 90 0

The numbers from $v = 0·0$ to $v = 1·0$ apply also from $v = 2·0$ to $v = 1·0$, changing the sign of the angle.

Table VI. Part I.—Strains on the interior points of a beam whose ends rest upon piers, and on one end of which a strain (of the nature of a moment or couple) is impressed, as in the exterior tubes of the Britannia Bridge. (Continued in Part II.)

The explanations are the same as those of Tables II. and III.

Values of w (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from the first pier, to half the length of the beam). The strain is impressed where $v=0$.										
	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
1·0	−37·50 0·00 90° 0′	−21·38 0·00 90° 0′	− 6·75 0·00 90° 0′	0·00 + 6·38 0° 0′	0·00 +18·00 0° 0′	0·00 +28·13 0° 0′	0·00 +36·75 0° 0′	0·00 +43·88 0° 0′	0·00 +49·50 0° 0′	0·00 +53·63 0° 0′	0·00 +56·25 0° 0′
0·9	−30·30 + 0·38 84 20	−17·54 + 0·51 81 0	− 6·37 + 1·04 68 50	− 0·77 + 5·95 20 50	− 0·19 +14·66 7 40	− 0·05 +22·63 4 20	0·00 +29·17 2 50	+ 0·03 +35·14 1 50	+ 0·05 +39·62 1 20	+ 0·06 +42·91 0 50	+ 0·07 +45·00 0 30
0·8	−23·72 + 1·32 77 10	−14·48 + 1·76 71 20	− 6·88 + 2·92 57 30	− 2·15 + 6·34 32 20	− 0·94 +11·83 16 20	− 0·43 +17·40 9 50	− 0·19 +22·34 6 30	− 0·06 +26·48 4 30	+ 0·01 +29·78 3 0	+ 0·06 +32·21 2 0	+ 0·09 +33·76 1 0
0·7	−17·81 + 2·89 68 20	−12·00 + 3·54 61 50	− 7·30 + 4·68 51 40	− 4·02 + 6·66 38 20	− 2·15 + 9·43 26 0	− 1·17 +12·50 17 40	− 0·63 +15·41 12 10	− 0·32 +17·95 8 30	− 0·13 +20·01 5 50	− 0·01 +21·54 3 50	+ 0·06 +22·53 2 0
0·6	−12·66 + 5·21 57 30	− 9·80 + 5·58 53 10	− 7·35 + 6·05 48 0	− 5·31 + 6·63 42 0	− 3·69 + 7·34 35 40	− 2·46 + 8·13 29 10	− 1·56 + 8·95 23 0	− 0·92 + 9·74 17 30	− 0·48 +10·43 12 40	− 0·19 +10·97 8 30	− 0·02 +11·32 4 30
0·5	− 8·44 + 8·44 45 0	− 7·69 + 7·69 45 0	− 6·94 + 6·94 45 0	− 6·19 + 6·19 45 0	− 5·44 + 5·44 45 0	− 4·69 + 4·69 45 0	− 3·94 + 3·94 45 0	− 3·19 + 3·19 45 0	− 2·44 + 2·44 45 0	− 1·69 + 1·69 45 0	− 0·94 + 0·94 45 0
0·4	− 5·21 +12·66 32 30	− 5·58 + 9·80 36 50	− 6·05 + 7·35 42 0	− 6·63 + 5·31 48 0	− 7·34 + 3·69 51 20	− 8·13 + 2·46 60 50	− 8·95 + 1·56 67 0	− 9·74 + 0·92 72 30	−10·43 + 0·48 77 20	−10·97 + 0·19 81 30	−11·32 + 0·02 85 30
0·3	− 2·89 +17·81 21 40	− 3·54 +12·00 28 10	− 4·68 + 7·30 38 20	− 6·66 + 4·03 51 40	− 9·43 + 2·15 64 0	−12·50 + 1·17 72 20	−15·41 + 0·61 77 50	−17·95 + 0·32 81 30	−20·01 + 0·13 84 10	−21·54 + 0·01 86 10	−22·53 − 0·06 88 0
0·2	− 1·32 +23·72 12 50	− 1·76 +14·48 18 40	− 2·92 + 6·88 32 30	− 6·34 + 2·15 57 40	−11·83 + 0·94 73 30	−17·40 + 0·43 80 10	−22·34 + 0·19 83 30	−26·48 + 0·06 85 30	−29·78 − 0·01 87 0	−32·21 − 0·06 88 0	−33·76 − 0·09 89 0
0·1	− 0·38 +30·30 5 40	− 0·51 +17·54 9 0	− 1·04 + 6·37 21 10	− 5·95 + 0·77 69 10	−14·66 + 0·19 82 20	−22·63 + 0·05 85 40	−29·47 0·00 87 10	−35·14 − 0·03 88 10	−39·62 − 0·05 88 40	−42·91 − 0·06 89 10	−45·00 − 0·07 89 30
0·0	0·00 +37·50 0 0	0·00 +21·38 0 0	0·00 + 6·75 0 0	− 6·38 0·00 90 0	−18·00 0·00 90 0	−28·13 0·00 90 0	−36·75 0·00 90 0	−43·88 0·00 90 0	−49·50 0·00 90 0	−53·63 0·00 90 0	−56·25 0·00 90 0

Table VI. Part II.—Strains on the interior points of a beam whose ends rest upon piers, and on one end of which a strain (of the nature of a moment or couple) is impressed, as in the exterior tubes of the Britannia Bridge. (Continued from Part I.)

The explanations are the same as those of Tables II. and III.

Values of u (the proportion of the vertical ordinate of a point, measured upwards from the lower edge, to the depth of the beam).	Values of v (the proportion of the horizontal ordinate of a point, measured from the first pier, to half the length of the beam). No strain is impressed where v=2·0.										
	1·1	$\frac{9}{8}$	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0
1·0	0·00 +57·38 0° 0′	0·00 +57·42 0° 0′	0·00 +57·00 0° 0′	0·00 +55·13 0° 0′	0·00 +51·75 0° 0′	0·00 +46·88 0° 0′	0·00 +40·50 0° 0′	0·00 +32·63 0° 0′	0·00 +23·25 0° 0′	0·00 +12·38 0° 0′	0·00 0·00 ° ′
0·9	+ 0·07 +45·90 0 10	+ 0·07 +45·94 0 0	+ 0·07 +45·60 − 0 20	+ 0·07 +44·11 − 0 40	+ 0·06 +41·41 − 1 0	+ 0·04 +37·53 − 1 30	+ 0·02 +32·45 − 2 20	− 0·02 +26·19 − 3 20	− 0·11 +18·78 − 5 30	− 0·36 +10·33 −11 30	− 2·33 + 2·40 −45 30
0·8	+ 0·10 +34·43 0 10	+ 0·10 +34·45 0 0	+ 0·09 +34·20 − 0 40	+ 0·08 +33·10 − 1 30	+ 0·04 +31·11 − 2 30	− 0·02 +28·24 − 3 40	− 0·12 +24·51 − 5 20	− 0·29 +19·96 − 7 50	− 0·62 +14·67 −12 30	− 1·46 + 8·98 −22 40	− 4·15 + 4·25 −45 20
0·7	+ 0·08 +22·95 0 20	+ 0·08 +22·97 0 0	+ 0·07 +22·81 − 1 10	+ 0·03 +22·11 − 3 0	− 0·06 +20·84 − 4 50	− 0·21 +19·04 − 7 10	− 0·45 +16·74 −10 10	− 0·86 +13·99 −14 40	− 1·58 +10·96 −21 20	− 2·94 + 7·97 −31 50	− 5·47 + 5·56 −45 10
0·6	+ 0·05 +11·48 0 50	+ 0·05 +11·48 0 0	+ 0·02 +11·43 − 2 40	− 0·10 +11·17 − 6 30	− 0·32 +10·72 −10 30	− 0·68 +10·10 −15 0	− 1·21 + 9·36 −20 10	− 1·97 + 8·54 −26 0	− 3·03 + 7·73 −32 20	− 4·45 + 6·97 −38 50	− 6·28 + 6·32 −45 10
0·5	− 0·19 + 0·19 45 0	0·00 0·00	− 0·56 + 0·56 −45 0	− 1·31 + 1·31 −45 0	− 2·06 + 2·06 −45 0	− 2·81 + 2·81 −45 0	− 3·56 + 3·56 −45 0	− 4·31 + 4·31 −45 0	− 5·06 + 5·06 −45 0	− 5·81 + 5·81 −45 0	− 6·56 + 6·56 −45 0
0·4	−11·48 − 0·05 89 10	−11·48 − 0·05 ±90 0	−11·43 − 0·02 −87 20	−11·17 + 0·10 −83 30	−10·72 + 0·32 −79 30	−10·10 + 0·68 −75 0	− 9·36 + 1·21 −69 50	− 8·54 + 1·97 −64 0	− 7·73 + 3·03 −57 40	− 6·97 + 4·45 −51 10	− 6·32 + 6·28 −44 50
0·3	−22·95 − 0·08 89 40	−22·97 − 0·08 90 0	−22·81 − 0·07 −88 50	−22·11 − 0·03 −87 0	−20·84 + 0·06 −85 10	−19·04 + 0·21 −82 50	−16·74 + 0·45 −79 50	−13·99 + 0·86 −75 20	−10·96 + 1·58 −68 40	− 7·97 + 2·94 −58 10	− 5·56 + 5·47 −44 50
0·2	−34·42 − 0·10 89 50	−34·45 − 0·10 90 0	−34·20 − 0·09 −89 20	−33·10 − 0·08 −88 30	−31·11 − 0·04 −87 30	−28·24 + 0·02 −86 20	−24·51 + 0·12 −81 40	−19·96 + 0·29 −82 10	−14·67 + 0·62 −77 30	− 8·98 + 1·46 −67 20	− 4·25 + 4·15 −44 40
0·1	−45·90 − 0·07 89 50	−45·94 − 0·07 90 0	−45·60 − 0·07 −89 40	−44·11 − 0·07 −89 20	−41·41 − 0·06 −89 0	−37·53 − 0·04 −88 30	−32·45 − 0·02 −87 40	−26·19 + 0·02 −86 40	−18·78 + 0·11 −84 30	−10·33 + 0·36 −78 30	− 2·40 + 2·33 −44 30
0·0	−57·38 0·00 90 0	−57·42 0·00 90 0	−57·00 0·00 −90 0	−55·13 0·00 −90 0	−51·75 0·00 −90 0	−46·88 0·00 −90 0	−40·50 0·00 −90 0	−32·63 0·00 −90 0	−23·25 0·00 −90 0	−12·38 0·00 −90 0	0·00 0·00

V. *On the Reflexion of Polarized Light from Polished Surfaces, Transparent and Metallic.*

By the Rev. SAMUEL HAUGHTON, *M.A., F.R.S., Fellow of Trinity College, Dublin.*

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Introduction.

AMONG the experimenters who have made the reflexion of polarized light the object of their researches, there is no one to whom science is more indebted than to M. JAMIN, whose accurate observations are a model for subsequent observers. His first paper on this subject was published (1847) in the 19th volume of the 'Annales de Chimie et de Physique,' 3rd series, p. 296, on Metallic Reflexion.

In this remarkable paper M. JAMIN verified many of the previous observations of BREWSTER, and added many of his own. He employed two distinct methods in these experiments,—

1st. *The method of Comparative Intensities*—by observing the relative intensities of the same beam of light reflected from a polished surface, composed partly of glass and partly of the substance to be examined.

2nd. *The method of Multiple Reflexions*, previously known from the researches of BREWSTER.

The optical constants used by JAMIN in this paper are—

(*a*) The angle (i_1) of maximum polarization.

(*b*) The angle (A) whose tangent is the ratio of I to J, the square roots of the intensities reflected in the plane of incidence, and in the perpendicular plane.

(*c*) The coefficient (ε) used by CAUCHY, which is connected with the other two constants by means of theoretical equations.

By the first method of observation, M. JAMIN determines the constants i_1 and ε for the following substances:—

1. Steel,
2. Speculum metal;

and by the second method of observation, he determines i_1 and A for

3. Silver,

and i_1 for

4. Zinc,

and gives the details of experiments on

5. Copper,

from which the optical constants may be found.

M. JAMIN'S next paper on Metallic Reflexion appeared in 1848, in the Annales de Chim. et de Phys. 3rd series, vol. xxii. p. 311. In this paper he makes use of the second method of observation, by multiple reflexion, and gives valuable tables of the results of his experiments with the various colours of the spectrum on the seven following metallic substances:—

1. Steel.
2. Speculum metal.
3. Silver.
4. Zinc.
5. Copper.
6. Brass.
7. Bell metal.

From these Tables the constants i_1 and A may be inferred.

In 1850 M. JAMIN published his well known paper "On the Reflexion of Light at the Surface of Transparent Bodies," in the Ann. de Chim. et de Phys. 3rd series, vol. xxix. p. 263. In this series of experiments he used a new method of observation, founded on the Quartz Compensator of BABINET. In this elaborate and important paper he publishes the details of his experiments on the following substances:—

1. Fire Opal,
2. Hyalite,
3. Realgar,
4. Blende,
5. Diamond,
6. Fluor-spar,
7, 8. Two kinds of glass,

and, in addition, gives in a Table at the end of the paper the constants of many other transparent bodies.

M. JAMIN has also published, in 1851, in the Ann. de Chim. et de Phys. vol. xxxi. p. 165, a memoir "On the Reflexion of Light at the Surface of Liquids," in which he determines the optical constants of many liquids.

It occurred to me that the method of observation employed by JAMIN for transparent bodies might be advantageously used in the case of metals; and I was thus led to commence the series of experiments the results of which are recorded in the following pages.

In these experiments I have added many metallic substances to JAMIN'S list, and have re-examined the metals observed by him by a different method.

In transparent bodies I have examined a few not experimented on by JAMIN, and investigated in detail the form of the reflected ellipse, under varying conditions of incidence and azimuth.

In the course of my paper I have employed for the second optical constant one more readily determined than those usually employed, but which is readily deduced from the constants A and k of JAMIN.

At the close of the paper I shall give a Table containing a comparison of the constants found by JAMIN and myself for all the bodies which we have both examined.

Some years ago, in making observations on polarized light, I found that by adjusting properly the azimuth of the incident polarized beam, and allowing it to fall at the angle of principal incidence, I could obtain a reflected beam of circularly polarized light.

On repeating the experiment with different polished surfaces, I found that the *coefficient of reflexion*, or whatever property it is that gives a surface a metallic reflexion, might be conveniently expressed by the cotangent of the azimuth at which an incident beam of plane-polarized light should be placed so as to give, on reflexion at the principal incidence, a reflected beam of circularly polarized light.

The following paper contains an account of my experiments on many substances, and a Table of their Coefficients of Reflexion and Refraction, determined with as much accuracy as I was able to attain with the instruments at my disposal.

The apparatus used by me consisted of a large graduated circle (horizontal), provided with two moveable arms, each furnished with graduated circles (vertical); and the large horizontal circle was capable of being hung vertically, so as to allow of experiments being made on liquids as well as solids. The substance to be examined was placed on a stage provided with adjusting screws, so as to bring the surface exactly into the centre, or intersection of the axes of the polarizing and analysing arms. These arms were mounted with Nicol prisms, made for me by DUBOSCQ of Paris, and without sensible deviation. The light employed was generally sunlight, but I sometimes used a moderator lamp with colza oil.

I employed the quartz compensator described by M. JAMIN*, for the purpose of converting the elliptically polarized reflected light into plane-polarized light, before allowing it to pass through the analyser.

The instrument used by me in making my observations on the reflexion of polarized light, was made by Mr. GRUBB of Dublin for the late Professor McCULLAGH, and was presented to me, shortly after McCULLAGH's death, by his brother. It is substantially the same as that described by M. JAMIN in vol. xxix. Ann. de Chim. et de Phys. sér. 3. I procured from M. DUBOSCQ SOLEIL, of Paris, a compensator of JAMIN'S pattern, and had it adapted to my own apparatus.

In making my observations I used the following precautions:—

1. The zero of both polarizer and analyser was determined by direct observation with red sunlight, reflected at the angle of polarization of several glasses found to give a reflected beam capable of being completely cut off by the Nicol prism.

2. The Nicol prisms themselves were carefully tested and found to have no deviation.

3. Each of my recorded observations is the mean of four or five; and when these differed from each other by more than 20′, I took the precaution of repeating them

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* Annales de Chimie et de Physique, sér. 3. vol. xxix. p. 263 *et seq.*

again, on another day, with my eye fresh and unfatigued, before I finally adopted my mean.

4. I frequently repeated the observations, with the incident light polarized at an equal angle, at the opposite side of the plane of incidence; and also reversing the polarizer and analyser, so as to read the opposite sides of their scales.

The following definitions will explain the sense in which I use certain terms.

The *Azimuth* of a beam of plane-polarized light is the angle which its plane of polarization makes with the plane of incidence.

The *Index of Refraction* is the ratio which the sine of the angle of incidence bears to the sine of the angle of refraction.

The *Coefficient of Refraction* is the tangent of the *Principal Incidence.*

The *Principal Incidence* is that angle of incidence at which rays polarized in any azimuth have the major axis of the reflected elliptic light in the plane of incidence; or at which the components of the reflected beam, in and perpendicular to the plane of incidence, differ by 90° in phase.

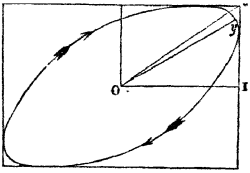
This angle is nearly the same as BREWSTER'S Angle of Polarization or Maximum Polarization.

The *Coefficient of Reflexion* is the Cotangent of the Azimuth of an incident beam of plane-polarized light, which after reflexion at the principal incidence becomes circularly polarized.

The *Principal Components* of the incident and reflected light are the components in and perpendicular to the plane of incidence.

The following preliminary investigation will serve to show the principles on which I have tabulated the results of my experiments:—

Let the elliptically polarized reflected beam be represented, as in the annexed figure, inscribed in a rectangle, whose sides are parallel to O I and O P, the plane of incidence and perpendicular plane.



Let O x be the diagonal of the circumscribed rectangle, and O y the axis of the ellipse; it is required, from the difference of phase of the light in the planes O I and O P, and knowing the direction of the line O x, to find the direction of O y and the ratio of the axes of the ellipse.

The angle x O I $=\alpha'$ is the azimuth of the reflected beam, measured by the analyser, after it has lost its elliptic polarization in the compensator; and the difference of phase of O I and O P is measured in the compensator itself, by the displacement necessary to reduce the elliptically-polarized to plane-polarized light.

We may imagine, to aid our conception, but without hypothesis, that a material point traverses the ellipse, and that its coordinates are

$$\xi = \mathrm{A}\sin(kt+e),$$
$$\eta = \mathrm{B}\sin(kt+e'),$$

where $e'-e$ is the difference of phase between the beams O I and O P, and A, B are the lines O I and O P.

Eliminating t, we find

$$\frac{\xi^2}{A^2}+\frac{\eta^2}{B^2}-2\cos(e'-e)\frac{\xi\eta}{AB}=\sin^2(e'-e). \qquad (1.)$$

In this ellipse, the angle ϕ made by the axis with the plane of incidence is found from the well-known expression

$$\tan 2\phi=\frac{2E}{D-F},$$

belonging to the ellipse

$$Dx^2+2Exy+Fy^2=\text{const.}$$

Substituting for D, E, F their values from (1.), we find

$$\tan 2\phi=\tan 2\alpha'\cos(e'-e); \qquad (2.)$$

ϕ denoting the angle y O I, and α' the angle x O I. But if a and b denote the axes of the ellipse, it can be proved that

$$\frac{b^2}{a^2}=\frac{(D+F)+(D-F)\sec 2\phi}{(D+F)-(D-F)\sec 2\phi};$$

or substituting from (1.) and (2.),

$$\left.\begin{aligned}\frac{a}{b}&=\sqrt{-\cot(\phi+\alpha')\cot(\phi-\alpha')}\\ \frac{b}{a}&=\sqrt{-\tan(\phi+\alpha')\tan(\phi-\alpha')}\end{aligned}\right\} \qquad (3.)$$

From equations (2.) and (3.), I calculate the position of the elliptic axes and their ratio.

The angle α' is obtained by direct measurement with the analysing prism; and $e'-e$ may be found, as follows, from the compensator.

In the compensator made for me by M. Duboscq, I find that 39·43 represents the zero, *i. e.* the position in which the compensator affects equally the light in and perpendicular to the plane of incidence; the number of graduations corresponding to a difference of half a wave (180°) I found to be

Red lamplight (colza oil)	15·43
Red sunlight ·	15·37
White lamplight (colza oil) . . .	13·29

If, therefore, C denote the reading of the compensator in any experiment, the difference of phase of the two principal beams will be expressed for red sunlight, in degrees, by the expression

$$(C-39{\cdot}43)\times\frac{180^\circ}{15{\cdot}37},$$

and by a corresponding formula for the other kinds of light.

The angle thus measured by the compensator is not the difference of phase between the principal components of the reflected light until it is increased by 180°, because experiment shows that in the act of reflexion there is this constant difference between the two components, in addition to the varying difference of phase, depending on incidence, azimuth, and nature of polished surface. We therefore use the formula

$$e'-e=180°+\left(C-39{\cdot}43\right)\times\frac{180°}{i}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4.)$$

where i denotes the interval corresponding to 180° for the light used.

In tabulating my experiments, I give the original measurements of the analyser and compensator, and use the equations (2.), (3.), and (4.) to calculate the other columns.

I. MUNICH GLASS (*a*).

The first experiments I shall record were made with glass procured from Munich by the late Professor McCULLAGH. I have four rhombs made of it, whose index of refraction I determined by the following experiments:—

TABLE I.—Munich Glass (*a*).

Rhomb.	Angle.	Minimum deviation* of red light.	Refractive index.
No. 1.	44 56 0	31 43 30	1·6229
No. 2.	54 28 30	41 28 0	1·623 0
No. 3.	39 50 0	27 17 0	1·6227
No. 4.	59 58 30	48 22 0	1·6221
		Mean	1·6227

Calculated Angle of Polarization =58° 21′.

I also found the refractive indices of No. 2 for the extreme red and violet rays to be 1·6190 and 1·6555, which indicates a dispersive power in the glass of 0·0573.

This glass was found to contain the following constituents:—

Silica	42·25
Oxide of Lead . . .	46·35
Lime	0·45
Alkalies (by diff.) . .	10·95
	100·00

The following Tables contain my observations on this glass:—

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* In all my experiments the red light used was passed through the same piece of red glass, which was very

TABLE II.—Munich Glass (*a*). (September 20, 1854.)
Azimuth of Polarizer=20°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$\epsilon'-\epsilon-180°$.	ϕ.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
84° 30′	39·69	11° 35′	2° 41′	+11° 34′	89·66	29° 23′
52 30	42·55	2 5	36 30	+ 1 40	45·80	5 42
53 30	44·09	2 0	54 31	+ 1 10	35·25	5 29
54 30	45·81	2 0	74 38	+ 0 32	29·71	5 29
55 30	48·68	2 6	108 13	− 0 39	28·68	5 45
56 30	49·86	2 37	122 2	− 1 24	25·89	7 9
73 30	53·88	11 45	169 4	−11 34	26·93	29 45

The principal incidence is therefore 54° 57′.

The last column of this Table is thus found:—

Let the principal components of the incident polarized beam, in and perpendicular to the plane of incidence, be $\cos\alpha$ and $\sin\alpha$ (unity denoting the incident beam); and let I and J denote what a unit of light becomes after reflexion, in and perpendicular to the plane of incidence respectively; then the principal components of the reflected beam are $I\cos\alpha$ and $J\sin\alpha$, and therefore

$$\frac{J}{I}=\tan\alpha'\cot\alpha; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

and the angle $\tan^{-1}\left(\frac{J}{I}\right)$ may be found from this equation without any difficulty.

According to the theory of FRESNEL,

$$\frac{J}{I}=\frac{\cos(i+r)}{\cos(i-r)},$$

an expression which vanishes at the polarizing angle ($i+r=90°$), and therefore $\tan^{-1}\left(\frac{J}{I}\right)$ ought at this angle of incidence to vanish also; but we find, not only in this experiment, but in those which follow, that it does not vanish, but only reaches a minimum, the tangent of which is sensibly equal to what I have called the Coefficient of Reflexion*.

In fact, let λ denote the angle whose cotangent is this coefficient. Then $I\cos\lambda$, $J\sin\lambda$ are the principal components of the reflected light, which by definition is circularly polarized, and therefore $I\cos\lambda=J\sin\lambda$, and

$$\cot\lambda=\frac{J}{I}.$$

The coefficients of Refraction and Reflexion, as determined by this experiment, are therefore

Coefficient of Refraction $=\tan 54° 57'=1{\cdot}4255$.
Coefficient of Reflexion $=\cot 84° 31'=0{\cdot}0960$.

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* Strictly speaking the angle of incidence at which the maximum is reached is found to be *somewhat* less than the Principal Incidence.

TABLE III.—Munich Glass (*a*). (June 26, 1854.)

Azimuth of Polarizer =45°. White lamplight (Colza oil).

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
43° 37′	39·54	18° 30′	3° 11′	+18° 28′	47·79	18° 30′
48 37	40·07	10 55	10 17	+10 45	29·42	10 55
50 45	40·61	8 10	17 36	+ 7 48	23·37	8 10
51 45	41·11	6 45	24 22	+ 6 10	20·70	6 45
52 45	42·67	6 10	45 29	+ 4 21	13·03	6 10
53 52	43·15	5 22	50 21	+ 3 27	13·86	5 22
54 20	44·46	5 1	69 44	+ 1 45	12·15	5 1
55 20	46·30	5 36	94 38	− 0 27	10·23	5 36
56 20	48·18	6 15	120 6	− 3 10	10·58	6 15
57 40	50·07	7 35	145 41	− 6 19	13·52	7 35
58 40	51·00	9 39	158 16	− 9 0	16·16	9 39
60 35	51·60	11 10	166 24	−10 53	22·34	11 10
65 40	51·98	18 11	171 33	−18 2	22·84	18 11
75 35	52·50	30 25	178 40	−30 25	∞	30 25

The principal incidence is therefore 55° 8′, and the minimum value of $\tan^{-1}\left(\frac{J}{I}\right)$ is 5° 1′, or Circular limit=84° 59′. Therefore the

Coefficient of Refraction=1·4352.
Coefficient of Reflexion =0·0877.

TABLE IV.—Munich Glass (*a*). (July 28, 1854.)

Azimuth of Polarizer =80°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
34° 30′	39·72	74° 0′	3° 23′	+74° 1′	88·20	31° 35′
43 30	40·13	61 45	8 11	+61 53	16·90	18 10
48 30	40·76	49 15	15 34	+49 24	7·62	11 34
50 30	42·15	38 0	31 49	+36 49	3·62	7 51
51 30	42·80	33 30	39 26	+30 36	3·09	6 40
52 30	43·85	28 0	51 42	+21 17	2·70	5 22
53 30	45·41	25 45	69 57	+11 39	2·28	4 52
54 30	47·20	26 34	90 54	− 0 36	1·99	5 2
55 30	48·80	28 45	109 37	−13 53	2·02	5 31
56 30	50·30	34 0	127 10	−28 7	2·26	6 47
57 30	51·20	40 0	137 42	−38 18	2·64	8 25
58 30	51·21	42 30	137 49	−41 38	2·82	9 10
60 30	52·14	53 45	148 42	−55 8	3·76	13 31
65 30	52·77	66 30	156 4	−67 47	6·59	22 5
70 45	53·18	75 15	160 52	−75 56	12·30	33 49

Principal Incidence =54° 27′. $\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$=4° 57′, or Circular limit =85° 3′.

Coeff. of Refraction =1·3993. Coeff. of Reflexion =0·0866.

TABLE V.—Munich Glass (*a*). (August 7, 1854.)

Azimuth of Polarizer =85°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	Tan$^{-1}\left(\frac{J}{I}\right)$.
34° 30′	39·68	80° 12′	2° 55′	+80° 13′	98·30	26° 52′
52 30	43·37	44 30	46 5	+44 17	2·37	4 55
53 30	44·33	39 54	57 19	+35 47	1·86	4 11
54 30	46·12	38 24	78 16	+20 28	1·36	3 58
55 30	48·13	41 50	101 47	−30 44	1·26	4 28
56 30	49·45	44 0	117 13	−42 49	1·64	4 50
57 30	50·42	51 0	128 34	−54 24	2·15	6 10
73 30	53·62	81 0	166 1	−81 15	26·76	28 55

Principal Incidence =54° 59′. Tan$^{-1}\left(\frac{J}{I}\right)$=3° 58′, or Circular limit =86° 2′.

Coeff. of Refraction =1·4272. Coeff. of Reflexion =0·0693.

TABLE VI.—Munich Glass (*a*). (September 27, 1854.)

Azimuth of Polarizer =85° 45′. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	Tan$^{-1}\left(\frac{J}{I}\right)$.
54° 30′	46·05	43° 26′	77° 23′	+37° 56′	1·25	4° 2′
54 45	46·75	43 20	85 38	+26 18	1·09	4 1
55 0	46·90	43 8	87 24	+17 24	1·08	3 59
55 15	47·53	43 15	94 46	−26 49	1·11	4 0
55 30	48·05	45 30	100 51	−47 39	1·21	4 20

Principal Incidence =55° 6′. Tan$^{-1}\left(\frac{J}{I}\right)$=3° 59′, or Circular limit =86° 1′.

Coeff. of Refraction =1·4334. Coeff. of Reflexion =0·0696.

TABLE VII.—Munich Glass (*a*). (September 27, 1854.)

Azimuth of Polarizer =85° 55′. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	Tan$^{-1}\left(\frac{J}{I}\right)$.
54° 30′	45·96	45° 30′	76° 24′	+47° 7′	1·27	4° 10′
54 45	46·65	45 12	84 28	+47 4	1·10	4 7
55 0	47·00	45 5	88 34	+48 19	1·02	4 6
55 15	47·72	45 40	96 59	−50 25	1·13	4 11
55 30	48·00	46 30	100 16	−53 12	1·20	4 18

Principal Incidence =55° 7′. Tan$^{-1}\left(\frac{J}{I}\right)$=4° 6′, or Circular limit =85° 54′.

Coeff. of Refraction =1·4343. Coeff. of Reflexion =0·0717.

TABLE VIII.—Munich Glass (*a*). (September 26, 1854.)

Azimuth of Polarizer =86°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
34° 30′	39·77	83° 20′	3° 58′	+83° 21′	120·50	30° 53′
52 30	43·03	53 30	42 7	+56 16	2·73	5 24
53 30	44·67	47 0	61 18	+49 8	1·70	4 17
54 0	45·25	46 20	68 5	+48 34	1·48	4 11
54 30	46·13	46 11	78 23	+50 48	1·23	4 10
54 45	46·48	46 0	82 28	+52 28	1·14	4 8
55 0	46·91	45 45	87 30	+60 29	1·06	4 6
55 15	47·33	47 0	92 25	−74 27	1·08	4 17
55 30	48·07	48 20	101 5	−60 39	1·25	4 30
56 0	48·86	50 15	111 30	−58 25	1·53	4 48
56 30	49·40	51 30	116 38	−58 37	1·71	5 2
57 30	50·20	57 0	126 0	−63 34	2·26	6 9
73 30	53·69	83 15	166 50	−83 25	37·96	30 35

Principal Incidence =55° 8′. $\text{Tan}^{-1}\left(\frac{J}{I}\right)=4° 6'$, or Circular limit =85° 54′.

Coeff. of Refraction =1·4352. Coeff. of Reflexion =0·0717.

TABLE IX.—Munich Glass (*a*). (September 21, 1854.)

Azimuth of Polarizer =87°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
34° 30′	39·48	84° 50′	0° 27′	+84° 50′	∞	30° 6′
52 30	42·87	60 0	40 14	+63 33	3·27	5 11
53 30	44·06	55 0	54 10	+60 56	2·16	4 17
54 0	44·93	54 20	64 21	+63 59	1·78	4 11
54 30	45·82	54 0	74 45	+70 30	1·52	4 8
54 45	46·34	53 54	80 50	+76 48	1·43	4 7
55 0	46·60	53 55	83 53	+80 50	1·40	4 7
55 15	47·23	53 34	91 15	−87 59	1·35	4 4
55 30	47·92	55 30	99 20	−78 33	1·48	4 22
56 0	48·40	56 30	104 57	−74 21	1·64	4 32
56 30	49·08	59 30	112 54	−72 28	1·97	5 5
57 30	50·00	64 30	123 40	−72 48	2·72	6 16
73 30	53·34	85 0	165 5	−85 10	44·50	30 55

Principal Incidence =55° 13′. $\text{Tan}^{-1}\left(\frac{J}{I}\right)=4° 4'$, or Circular limit =85° 56′.

Coeff. of Refraction =1·4397. Coeff. of Reflexion =0·0711.

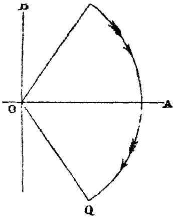
Collecting together the preceding results, and denoting by λ the azimuth of the plane of polarization of the incident light, which on reflexion at the principal incidence will produce, on reflexion, circularly polarized light, and calling it the Circular Limit, we obtain

Table X.—Constants of Munich Glass (a).

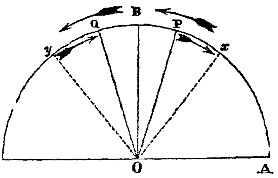
Azimuth of Polarizer.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
20° 0′	54° 57′	84° 31′	1·4255	0·0960
45 0	55 8	84 59	1·4352	0·0877
80 0	54 27	85 3	1·3993	0·0866
85 0	54 59	86 2	1·4272	0·0693
85 45	55 6	86 1	1·4334	0·0696
85 55	55 7	85 54	1·4343	0·0717
86 0	55 8	85 54	1·4352	0·0717
87 0	55 13	85 56	1·4397	0·0711
Means.........	55° 0′ 37″	85° 32′ 30″	1·4287	0·0780

The movement of the axis of the reflected ellipse differs according as the azimuth of the incident light is less or greater than the circular limit. This is shown in Plate VIII. fig. A, on which the values of ϕ are laid down for different angles of incidence in the two cases in which the azimuth of the incident light is 80° and 87°.

When the azimuth of the incident light is less than λ, the circular limit, the axis of the ellipse moves as in the annexed figure. Let P O A be the azimuth of the incident light, and Q O A equal to P O A; P O is the position of the axis major corresponding to 0° incidence; O A is the position of the axis major in the plane of incidence, corresponding to the principal incidence; and O Q is the position of the axis corresponding to 90° incidence.



When, however, the azimuth of the incident light is greater than the circular limit, the axis major moves from P to x, as in the annexed figure, then back from x to y, passing through B at the principal incidence, and finally from y to Q. Let P O A be the azimuth of the incident light, and Q O B equal to P O B. At the incidence 0°, O P is the position of the axis major; as the incidence increases from 0° to the principal incidence, the axis major moves from O P to O x and turns back, attaining the position O B at the principal incidence; and as the incident angle increases from the principal incidence to 90°, the axis major moves from O B to O y, and back again to O Q.



Having ascertained the truth of the preceding laws of the movement of the axis major of the elliptically polarized light, I made the following experiments. Having removed the compensator, I set the polarizer at 88° and 89°, and found that the analyser gave a minimum of light at 90°, showing that the axis major of the ellipse was perpendicular to the plane of incidence.

All the experiments already given were made with the Munich glass (*a*) No. 1. I made the following experiment with (*a*) No. 2, in order to establish fully the identity of the pieces of glass with regard to reflexion, as they are certainly identical in their refractive indices.

TABLE XI.—Munich Glass (*a*). (October 11, 1854.)
Azimuth of Polarizer =86°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	φ.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
34° 30′	39·70	83° 20′	3° 8′	+83° 20′	∞	30° 53′
52 30	43·33	53 30	45 37	+56 48	2·53	5 24
53 30	44·70	50 0	61 39	+55 11	1·73	4 46
54 0	45·60	49 0	72 11	+57 20	1·41	4 36
54 30	46·30	47 30	80 22	+58 48	1·21	4 22
54 45	46·79	46 45	85 58	+65 30	1·10	4 15
55 0	47·31	46 0	92 11	−66 15	1·05	4 9
55 15	47·70	47 30	96 45	−63 20	1·16	4 22
55 30	48·00	48 30	100 16	−62 17	1·24	4 31
56 0	48·83	50 30	109 58	−59 50	1·50	4 51
56 30	49·10	52 30	113 8	−62 9	1·64	5 12
57 30	49·95	57 0	123 5	−64 32	2·15	6 9
73 30	53·40	82 30	163 27	−82 48	26·98	27 59

Principal Incidence =54° 53′. Coeff. of Refraction =1·4220.
Circular Limit =85° 51′. Coeff. of Reflexion =0·0725.

The agreement of these values with those given for No. (1) in Table X. is sufficiently satisfactory.

II. MUNICH GLASS (*b*).

The glass now to be described is a rhomb which gave me the following values:—

Angle of rhomb . . . .=54° 30′
Minimum deviation of standard red . . . 34 2

Hence the refractive index of this red is 1·5244, and the angle of polarization 56° 44′.

TABLE XII.—Munich Glass (*b*). (October 10, 1854.)
Azimuth of Polarizer =45°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	φ.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
54° 15′	39·43	1° 0′	0° 0′	+1° 0′	∞	1° 0′
54 30	39·43	0 43	0 0	+0 43	∞	0 43
54 45	39·43	0 30	0 0	+0 30	∞	0 30
55 0	39·43	0 10	0 0	+0 10	∞	0 10
55 15	54·13	2 15	172 0	−2 14	209·4	2 15

Principal Incidence =55° 1′. Coeff. of Refraction =1·4290.
Circular Limit =89° 50′. Coeff. of Reflexion =0·0029.

TABLE XIII.—Munich Glass (*b*). (September 29, 1854.)

Azimuth of Polarizer $=80°$. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
34° 30′	39·43	72° 30′	0° 0′	+72° 30′	∞	29° 13′
52 30	39·57	20 15	1 38	+20 15	∞	3 38
53 30	39·65	11 30	2 34	+11 29	127·3	2 3
53 45	39·65	10 15	2 34	+10 14	135·6	1 49
54 0	39·65	7 30	2 34	+ 7 29	160·2	1 20
54 15	39·65	6 0	2 34	+ 5 59	179·9	1 4
54 30	39·73	1 20	3 30	+ 1 20	171·8	0 13
54 45	54·25	4 15	173 23	− 4 13	114·9	0 45
55 0	54·25	6 0	173 23	− 5 57	80·6	1 4
55 15	54·25	8 30	173 23	− 8 27	54·8	1 30
55 30	54·35	9 37	174 33	− 9 34	62·8	1 43
56 30	54·35	17 30	174 33	−17 26	37·5	3 11
57 30	54·36	25 25	174 41	−25 21	28·3	4 47
58 30	54·45	31 0	175 44	−30 58	30·3	6 3
73 30	54·68	70 0	178 25	−70 0	90·5	25 51
83 30	54·68	76 30	178 25	−76 30	164·3	36 18

Principal Incidence $=54°\ 35'$. Coeff. of Refraction $=1·4063$.
Circular Limit $=89°\ 47'$. Coeff. of Reflexion $=0·0037$.

TABLE XIV.—Munich Glass (*b*). (October 10, 1854.)

Azimuth of Polarizer $=87°$. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
54° 15′	39·43	15° 40′	0° 0′	+15° 40′	∞	0° 51′
54 30	39·43	2 15	0 0	+ 2 15	∞	0 7
54 45	54·13	3 45	172 0	− 3 43	114·5	0 12
55 0	54·13	7 40	172 0	− 7 36	56·1	0 24
55 15	54·13	14 20	172 0	−14 13	30·0	0 46

Principal Incidence $=54°\ 36'$. Coeff. of Refraction $=1·4071$.
Circular Limit $=89°\ 53'$. Coeff. of Reflexion $=0·0020$.

TABLE XV.—Munich Glass (*b*). (October 1, 1855.)

Azimuth of Polarizer $=88°$. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
33° 37′	39·43	86° 45′	0° 0′	+86° 45′	∞	31° 36′
43 37	39·43	83 45	0 0	+83 45	∞	17 41
53 37	39·43	37 30	0 0	+37 30	∞	1 32
54 37	39·43	359 45	180 0	− 0 15	∞	0 1
55 37	39·43	322 0	180 0	−38 0	∞	1 34
56 37	39·43	301 30	180 0	−58 30	∞	3 16
63 37	39·43	278 0	180 0	−82 0	∞	13 57
73 37	39·43	274 30	180 0	−85 30	∞	23 55

Principal Incidence $=54°\ 37'$. Coeff. of Refraction $=1·4080$.
Circular Limit $=89°\ 59'$. Coeff. of Reflexion $=0·0001$.

TABLE XVI.—Munich Glass (*b*). (October 5, 1855.)
Azimuth of Polarizer =89°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$	$\tan^{-1}\left(\frac{J}{I}\right)$.
33° 37′	39·43	88° 0′	0° 0′	+88° 0′	∞	26° 34′
43 37	39·43	85 25	0 0	+85 25	∞	12 17
53 37	39·43	55 0	0 0	+55 0	∞	1 26
54 37	39·43	358 0	180 0	− 2 0	∞	0 2
55 37	39·43	307 30	180 0	−52 30	∞	1 2
56 37	39·43	292 0	180 0	−68 0	∞	2 28
63 37	39·43	275 0	180 0	−85 0	∞	11 17
73 37	39·43	272 15	180 0	−87 45	∞	23 57

Principal Incidence =54° 35′. Coeff. of Refraction =1·4063.
Circular Limit = 0° 2′. Coeff. of Reflexion =0·0006.

From this and the four preceding Tables the following results may be collected.

TABLE XVII.—Constants of Munich Glass (*b*).

Azimuth of Polarizer.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
45°	55° 1′	89° 50′	1·4290	0·0029
80	54 35	89 47	1·4063	0·0037
87	54 36	89 53	1·4071	0·0020
88	54 37	89 59	1·4080	0·0001
89	54 35	89 58	1·4063	0·0006
Means.........	54° 40′ 48″	89° 53′ 24″	1·4113	0·0019

III. PARIS GLASS.

This glass was supplied to me by M. DUBOSCQ of Paris. Its refractive constants were found to be as follows:—

Angle of prism	59° 55′
Minimum deviation of extreme red	37 35
Minimum deviation of extreme violet	39 13
Index of refraction of extreme red	=1·5059
Index of refraction of extreme violet . . .	=1·5246

TABLE XVIII.—Paris Glass. (October 1, 1855.)
Azimuth of Polarizer $=88°$. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
33° 37′	39·43	86° 20′	0° 0′	+86° 20′	∞	28° 35′
43 37	39·43	84 30	0 0	+84 30	∞	19 56
53 37	39·43	63 45	0 0	+63 45	∞	4 3
54 37	40·19	50 20	8 53	+50 24	12·76	2 25
55 37	42·43	25 30	35 6	+22 39	4·24	0 57
55 52	44·44	19 30	58 36	+11 26	3·41	0 43
56 7	46·75	17 30	85 38	+ 1 32	3·18	0 38
56 22	48·39	18 45	104 41	− 5 30	3·07	0 41
56 37	50·22	27 30	126 14	−20 5	2·65	1 2
57 37	52·70	48 0	155 15	−48 18	4·59	2 13
58 37	53·27	60 0	161 55	−60 38	7·32	3 28
63 37	54·00	80 0	170 28	−80 7	36·85	11 12
73 37	54·40	87 0	175 8	−87 1	180·90	33 41

Principal Incidence $=56°\ 10'$. Coeff. of Refraction $=1·4919$.
Circular Limit $=89°\ 22'$. Coeff. of Reflexion $=0·0110$.

The compensator was then set at 47·12, which corresponds with a difference of phase of 90°, between the principal components of the reflected light; and the compensator being thus set, the angle of incidence was determined by trial, for which the dark band was centrally placed. The incidence so found is the principal incidence. Having thus found the principal incidence, I changed the azimuth of the polarizer, and read the analyser, obtaining the following results.

TABLE XIX.—Paris Glass. (October 1, 1855.)
Compensator $=47·12=90°$. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
89° 30′	48° 0′	1·110	0° 33′
89 0	32 0	1·600	0 37
88 0	18 0	3·077	0 39
87 0	13 20	4·219	0 43
86 0	10 0	5·671	0 43
85 0	9 0	6·314	0 48
80 0	4 30	12·706	0 48
70 0	2 45	20·819	1 0
60 0	1 40	34·367	0 58
50 0	1 10	49·103	0 59
40 0	0 54	63·656	1 4
30 0	0 47	73·139	1 21
20 0	0 37	92·908	1 42
10 0	0 24	143·237	2 16

Principal Incidence $=56°\ 7'$. Coeff. of Refraction $=1·4891$.
Circular Limit $=89°\ 24'$. Coeff. of Reflexion $=0·0104$.

The last column of this Table shows that the value of $\left(\frac{J}{I}\right)$ increases slightly as the azimuth of the polarizer diminishes.

Combining the preceding results, we find

TABLE XX.—Constants of Paris Glass.

No.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
XVIII.	56° 10′	89° 22′	1·4919	0·0110
XIX.	56 7	89 24	1·4891	0·0104
Means............	56° 8′ 30″	89° 23′	1·4905	0·9107

IV. FLUOR-SPAR.

The specimen of fluor-spar on which I made my experiments was transparent and blue. The following are the results I obtained.

TABLE XXI.—Fluor-Spar. (September 11, 1855.)

Azimuth of Polarizer = 80°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$	$\tan^{-1}\left(\frac{J}{I}\right)$.
33° 37′	39·43	73° 30′	0° 0′	+73° 30′	∞	30° 46′
43 37	39·43	60 45	0 0	+60 45	∞	17 29
53 37	39·43	10 30	0 0	+10 30	∞	1 53
54 37	39·43	0 30	0 0	+ 0 30	∞	0 5
55 37	39·43	351 45	180 0	− 8 15	∞	1 28
58 37	39·43	327 0	180 0	−33 0	∞	6 32
63 37	39·43	305 0	180 0	−55 0	∞	14 8
73 37	39·43	288 15	180 0	−71 45	∞	28 8

Principal Incidence = 54° 40′. Coeff. of Refraction = 1·4106.
Circular Limit = 89° 55′. Coeff. of Reflexion = 0·0014.

TABLE XXII.—Fluor-Spar. (September 20, 1855.)

Azimuth of Polarizer $=88°$. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	φ.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
53° 37′	39·43	43° 30′	0° 0′	+43° 30′	∞	1° 54′
54 7	39·43	31 0	0 0	+31 0	∞	1 12
54 37	39·43	15 0	0 0	+15 0	∞	0 32
55 7	39·43	344 30	180 0	−15 30	∞	0 33
55 37	39·43	331 0	180 0	−29 0	∞	1 7

Principal Incidence $=54°\ 52'$. Coeff. of Refraction $=1·4211$.
Circular Limit $=89°\ 28'$. Coeff. of Reflexion $=0·0093$.

From the preceding results combined, we obtain the following constants of fluor-spar.

TABLE XXIII.—Constants of Fluor-Spar.

No.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
XXI.	54° 40′	89° 55′	1·4106	0·0014
XXII.	54 52	89 28	1·4211	0·0093
Means............	54 46	89° 41′ 30″	1·4158	0·0053

V. GLASS OF ANTIMONY.

The specimen of this glass with which I experimented was given to me by Professor APJOHN.

TABLE XXIV.—Glass of Antimony. (October 5, 1855.)

Azimuth of Polarizer $=80°$. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	φ.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
33° 37′	39·43	75° 0′	0° 0′	+75° 0′	∞	33° 21′
43 37	39·43	66 20	0 0	+66 20	∞	21 55
53 37	39·82	38 50	4 33	+38 49	27·43	8 5
55 37	40·16	25 10	8 32	+25 1	17·84	4 44
57 37	41·90	10 30	28 54	+ 8 50	9·89	1 52
58 7	43·14	8 30	43 24	+ 6 16	9·86	1 31
58 37	46·19	6 20	79 5	+ 1 13	9·18	1 7
59 7	49·50	7 30	117 48	− 3 34	8·62	1 20
59 37	51·18	10 15	137 22	− 7 41	8·30	1 50
61 37	53·21	26 30	161 13	−25 46	7·77	5 2
63 37	53·70	39 30	166 57	−39 21	13·92	8 16
73 37	54·25	68 20	173 33	−68 25	27·04	23 56

Principal Incidence $=58°\ 44'$. Coeff. of Refraction $=1·6468$.
Circular Limit $=88°\ 53'$. Coeff. of Reflexion $=0·0195$.

Table XXV.—Glass of Antimony. (October 5, 1855.)
Azimuth of Polarizer =89°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$\epsilon'-\epsilon-180°$.	ϕ.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
33° 37′	39·43	89° 0′	0° 0′	+89° 0′	∞	45° 0′
43 37	39·43	87 45	0 0	+87 45	∞	23 57
53 37	39·65	83 20	2 34	+83 20	∞	8 30
55 37	39·91	80 0	5 36	+80 3	56·19	5 39
57 37	41·25	64 30	21 17	+65 30	6·93	2 6
58 7	42·43	55 40	35 6	+57 46	3·44	1 28
58 37	45·72	48 0	73 35	+55 12	1·36	1 7
58 47	47·00	47 0	88 34	+72 11	1·09	1 4
59 7	48·76	49 30	109 9	−67 26	1·24	1 10
59 37	50·35	60 0	127 45	−60 23	9·36	1 44
61 37	52·80	78 30	156 26	−79 22	12·75	6 9
63 37	53·68	82 15	166 43	−82 27	32·36	7 19
73 37	54·10	88 45	171 38	−88 46	281·50	38 40

Principal Incidence =58° 50′.	Coeff. of Refraction =1·6533.
Circular Limit =88° 56′.	Coeff. of Reflexion =0·0186.

Table XXVI.—Glass of Antimony. (October 5, 1855.)
Compensator =47·12=90°. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
89°	50° 0′	1·192	1° 11′
88	28 45	1·823	1 6
87	21 15	2·571	1 10
85	12 40	4·449	1 9
80	6 25	8·892	1 8
70	3 10	18·075	1 9
50	1 35	36·177	1 20
30	0 52	66·105	1 30
10	0 37	92·908	3 29
			Mean = 1° 28′ 0″

Principal Incidence =58° 52′.	Coeff. of Refraction =1·6555.
Circular Limit =88° 46′.	Coeff. of Reflexion =0·0215.

It is to be remarked, that in Table XXV., in which the azimuth of the polarizer is greater than the circular limit, the movement of the axis of the ellipse follows the same law as that of the Munich glass already described.

From the foregoing Tables, the optical constants of Glass of Antimony may be thus inferred:—

TABLE XXVII.—Constants of Glass of Antimony.

No.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
XXIV.	58° 44′	88° 53′	1·6468	0·0195
XXV.	58 50	88 56	1·6533	0·0186
XXVI.	58 52	88 46	1·6555	0·0215
Means	58° 48′ 40″	88° 51′ 40″	1·6519	0·0199

VI. QUARTZ (*a*). *Natural surface. Plane of incidence perpendicular to optical axis.*

TABLE XXVIII.—Quartz (*a*). (October 13, 1855.)

Azimuth of Polarizer =88°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	φ.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
33° 37′	39·43	87° 10′	0° 0′	+87° 10′	∞	35° 12′
43 37	39·43	84 20	0 0	+84 20	∞	19 23
53 37	39·90	67 45	3 9	+67 46	59·16	4 52
55 37	41·02	42 0	18 36	+41 50	6·09	1 48
56 7	43·40	32 0	46 27	+27 21	2·70	1 15
56 37	46·82	24 30	86 27	+ 4 4	2·20	0 55
57 7	50·53	35 0	129 52	−30 12	2·34	1 16
57 37	51·28	48 0	138 38	−49 0	2·65	2 13
58 37	52·10	63 15	148 14	−65 31	4·50	3 58
63 37	53·30	80 45	162 16	−81 10	20·52	12 6
73 37	53·50	86 30	164 36	−86 37	63·78	29 43

Principal Incidence =56° 40′. Coeff. of Refraction =1·5204.
Circular Limit =89° 5′. Coeff. of Reflexion =0·0160.

TABLE XXIX.—Quartz (*a*). (October 15, 1855.)

Compensator =47·12=90°. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
89° 30′	64° 0′	2·050	1° 1′
89 0	48 0	1·110	1 7
88 0	28 0	1·881	1 4
85 0	11 10	5·066	1 0
			Mean = 1° 3′ 0″

Principal Incidence =56° 40′. Coeff. of Refraction =1·5204.
Circular Limit =88° 51′. Coeff. of Reflexion =0·0200.

Hence we obtain

TABLE XXX.—Constants of Quartz (*a*).

No.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
XXVIII.	56° 40′	89° 5′	1·5204	0·0160
XXIX.	56 40	88 51	1·5204	0·0200
Means	56 40	88 58	1·5204	0·0180

VII. QUARTZ (*b*). *Natural surface. Plane of incidence contains optical axis.*

TABLE XXXI.—Quartz (*b*). (October 16, 1855.)

Azimuth of Polarizer =88°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
33° 37′	39·43	86° 30′	0° 0′	+86° 30′	∞	29° 43′
43 37	39·43	84 30	0 0	+84 30	∞	19 56
53 37	39·90	66 0	5 30	+66 4	27·85	4 29
55 37	40·72	45 15	15 5	+45 16	5·57	2 1
56 7	41·82	36 30	27 57	+35 27	4·22	1 29
56 37	43·87	23 30	51 57	+16 44	3·15	0 52
57 7	48·14	27 30	101 54	− 8 12	1·99	1 2
57 37	50·97	39 0	135 0	−36 39	2·47	1 37
63 37	53·85	79 30	168 42	−82 34	7·17	10 40
73 37	54·27	86 50	173 37	−86 51	176·20	32 15

Principal Incidence =56° 57′. Coeff. of Refraction =1·5369.
Circular Limit =89° 8′. Coeff. of Reflexion =0·0151.

TABLE XXXII.—Quartz (*b*). (October 16, 1855.)

Compensator =47·12=90°. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
89° 30′	44° 30′	1·017	0° 30′
89 0	41 10	1·143	0 52
88 0	26 20	2·020	0 59
85 0	13 20	4·219	1 11
			Mean =0° 53′ 0″

Principal Incidence =56° 52′. Coeff. of Refraction =1·5320.
Circular Limit =89° 34′. Coeff. of Reflexion =0·0076.

From which we obtain

TABLE XXXIII.—Constants of Quartz (*b*).

No.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
XXXI.	56° 57′	89° 8′	1·5369	0·0151
XXXII.	56 52	89 34	1·5320	0·0076
Means	56° 54′ 30″	89 21	1·5344	0·0108

The following experiments were made on the metallic bodies.

VIII. SPECULUM METAL.

TABLE XXXIV.—Speculum Metal. (July 29, 1854.)
Azimuth of Polarizer $=45°$. Red Lamplight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
35° 7′	40·25	42° 30′	9° 42′	+42° 27′	10·06	42° 30′
44 7	40·73	42 45	15 12	+42 40	7·42	42 45
49 7	41·19	41 30	20 35	+41 16	5·58	41 30
51 7	41·46	40 50	23 45	+40 27	4·78	40 50
53 7	41·62	40 20	25 36	+39 50	4·45	40 20
55 7	41·89	40 3	28 46	+39 22	3·96	40 3
57 7	42·10	39 20	31 14	+38 24	3·65	39 20
59 7	42·63	39 6	37 26	+37 38	3·03	39 6
61 7	43·06	39 20	42 28	+37 24	2·64	39 20
64 7	43·68	38 50	49 42	+35 40	2·24	38 50
69 7	44·96	37 45	64 42	+29 25	1·69	37 45
74 7	46·90	37 45	87 24	+ 4 58	1·29	37 45

Principal Incidence $=75°\ 27'$ (?). Circular Limit $=52°\ 15'$ (?).

In this experiment the angle of incidence was not made at any time equal to the principal incidence.

TABLE XXXV.—Speculum Metal. (August 30, 1855.)
Azimuth of Polarizer $=50°$. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
33° 37′	40·00	47° 15′	6° 40′	+47° 16′	16·48	42° 53′
43 37	40·95	45 40	17 47	+45 42	6·40	40 39
53 37	41·80	43 0	27 44	+42 45	4·13	38 3
63 37	43·34	40 15	45 44	+38 16	2·42	35 24
68 37	44·60	39 40	60 29	+34 32	1·77	34 50
73 37	46·26	39 0	79 59	+19 38	1·32	34 12
74 37	46·40	39 32	81 33	+18 38	1·27	34 42
75 37	46·56	40 15	83 18	+17 26	1·23	35 24
76 37	47·03	39 20	87 30	+ 6 8	1·23	34 40
77 37	47·50	39 43	94 25	−11 13	1·22	34 53
78 37	48·00	39 45	100 16	−21 56	1·29	34 55
83 37	50·85	43 0	133 36	−42 6	2·34	38 2
88 37	53·50	49 45	164 37	−49 55	7·65	44 45

Principal Incidence $=75°\ 51'$. Coeff. of Refraction $=3·9665$.
Circular Limit $=55°\ 48'$. Coeff. of Reflexion $=0·6796$.

TABLE XXXVI.—Speculum Metal. (August 28, 1855.)

Azimuth of Polarizer $=60°$. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	φ.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
33° 37′	40·17	57° 30′	8° 39′	+57° 38′	14·20	42° 11′
43 37	40·76	55 15	15 33	+55 36	7·89	39 46
53 37	41·76	53 30	27 15	+54 30	4·31	37 58
63 37	43·40	52 10	46 27	+55 10	2·44	36 38
68 37	44·60	51 0	60 29	+56 40	1·79	35 29
73 37	46·13	49 30	78 23	+64 6	1·29	34 3
74 37	46·30	49 30	80 22	+66 43	1·26	34 3
75 37	46·55	48 45	83 18	+69 14	1·19	33 22
76 37	46·91	49 15	87 30	+81 52	1·17	33 49
77 37	47·36	48 55	92 47	−80 17	1·16	33 31
78 37	48·25	51 0	103 11	−66 30	1·37	35 29
83 37	50·86	54 30	133 43	−58 14	2·53	38 59
88 37	54·00	60 0	170 28	−60 10	14·13	45 0

Principal Incidence $=75° \ 57'$. Coeff. of Refraction $=3·9959$.
Circular Limit $=56° \ 38'$. Coeff. of Reflexion $=0·6585$.

In this Table, the polarizer having been set at an angle exceeding the circular limit, the axis major of the ellipse passes through 90° at the principal incidence, and behaves exactly as in the transparent bodies.

TABLE XXXVII.—Speculum Metal. (June 29, 1855.)

Azimuth of Polarizer $=80°$. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	φ.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
33° 37′	40·14	80° 0′	8° 18′	+80° 6′	39·78	45° 0′
43 37	40·97	79 15	18 1	+79 44	17·57	42 53
53 37	41·81	77 30	27 50	+78 48	10·02	38 30
63 37	43·28	76 30	45 2	+80 6	6·06	36 18
68 37	44·44	76 0	58 36	+82 16	4·78	35 16
73 37	46·00	76 0	76 52	+86 33	4·13	35 16
74 37	46·32	75 47	80 36	+87 28	4·01	34 50
75 37	46·57	75 50	83 32	+88 16	3·99	34 56
76 37	46·94	76 0	87 52	+89 26	4·01	35 16
77 37	47·21	75 55	91 1	−89 44	3·99	35 6
78 37	48·15	76 30	102 1	−86 58	4·27	36 18
81 37	49·38	77 30	116 24	−84 9	5·45	38 30
83 37	50·20	77 45	126 0	−82 30	5·89	39 5
85 37	51·30	79 20	128 53	−81 48	8·24	43 7
88 37	54·19	80 0	172 41	−80 4	47·68	45 0

Principal Incidence $=76° \ 7'$. Coeff. of Refraction $=4·0458$.
Circular Limit $=55° \ 10'$. Coeff. of Reflexion $=0·6959$.

TABLE XXXVIII.—Speculum Metal (fresh polished with rouge). (Sept. 11, 1855.)

Compensator =47·12=90°. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	75° 40′	3·732	34° 37′
70	62 45	1·942	35 15
60	49 45	1·181	34 18
50	38 0	1·279	33 15
40	29 30	1·767	33 57
30	21 45	2·506	34 39
20	14 10	3·961	34 43
10	7 30	7·596	36 45
			Mean = 34° 41′ 7″

Principal Incidence =78° 7′. Coeff. of Refraction =4·7522.

Circular Limit =55° 19′. Coeff. of Reflexion =0·6920.

This experiment shows that the fresh polishing of the surface affected the coefficient of refraction more than the coefficient of reflexion, on which the elliptic polarization altogether depends.

The angle $\tan^{-1}\left(\frac{J}{I}\right)$ is not constant, but attains a minimum at the circular limit.

Additional direct experiments with speculum metal, such as setting the compensator at 90°, making the incidence 76°, setting the analyser at 45°, and then determining the azimuth of the polarizer, gave for the circular limit 54° 45′.

Combining all together, we find

TABLE XXXIX.—Constants of Speculum Metal.

No.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
XXXV.	75° 51′	55° 48′	3·9665	0·6796
XXXVI.	75 57	56 38	3·9959	0·6585
XXXVII.	76 7	55 10	4·0458	0·6959
XXXVIII.	78 7	55 19	4·7522	0·6920
Direct Ex.	……	54 45	……	0·7067
Means ………	76 33	55° 32′ 0″	4·1901	0·6865

IX. Silver.

I examined three descriptions of silver,—

(*a*) Fine silver, rolled.
(*b*) Fine silver, cast.
(*c*) Standard silver, rolled.

Table XL.—Silver (*a*). (September 3, 1855.)
Compensator $=47·12=90°$. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	Tan$^{-1}\left(\frac{J}{I}\right)$.
80°	79° 30′	5·395	43° 34′
70	69 0	2·605	43 29
60	56 40	1·520	41 17
50	46 15	1·044	41 14
40	36 10	1·368	41 3
30	27 15	1·941	41 44
20	18 0	3·077	41 45
10	9 40	5·870	44 1
			Mean=42° 15′ 52″

Principal Incidence =72° 37′. Coeff. of Refraction =3·1942.
Circular Limit =48° 46′. Coeff. of Reflexion =0·8765.

Table XLI.—Fine Silver (*a*) (newly polished). (September 7, 1855.)
Compensator $=47·12=90°$. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	Tan$^{-1}\left(\frac{J}{I}\right)$.
80°	79° 40′	5·484	44° 2′
70	68 15	2·506	42 23
60	54 45	1·415	39 15
50	46 15	1·045	41 14
45	42 45	1·082	42 45
40	38 0	1·280	42 57
30	27 40	1·907	42 15
20	19 0	2·904	43 25
10	9 20	6·084	42 59
			Mean=43° 28′ 20″

Principal Incidence =71° 37′. Coeff. of Refraction =3·0090.
Circular Limit =48° 13′. Coeff. of Reflexion =0·8936.

Having set the angle of incidence at 72° 37′, the compensator at 47·12=90°, and the analyser at 45°, I found, by trial, the polarizer or circular limit to be 48° 0′.

TABLE XLII.—Silver (*b*).* (September 6, 1855.)

Compensator $=47{\cdot}12=90°$. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	79° 35′	5·439	43° 48′
70	69 5	2·616	43 36
60	58 40	1·642	43 29
50	47 50	1·104	42 49
45	42 45	1·082	42 45
40	37 30	1·303	42 27
30	28 10	1·867	42 51
20	18 40	2·960	42 52
10	9 50	5·769	44 31
			Mean=43° 14′ 13″

Principal Incidence =78° 7′.	Coeff. of Refraction =4·7522.
Circular Limit =47° 13′.	Coeff. of Reflexion =0·9255.

TABLE XLIII.—Silver (*c*). (September 7, 1855.)

Compensator $=47{\cdot}12=90°$. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	79° 30′	5·395	43° 34′
70	68 30	2·538	42 44
60	57 30	1·570	42 11
50	47 15	1·082	42 14
45	42 30	1·091	42 30
40	37 0	1·327	41 55
30	28 0	1·881	42 39
20	19 25	2·837	44 5
10	9 50	5·769	44 30
			Mean=42° 55′ 47″

Principal Incidence =78° 22′.	Coeff. of Refraction =4·8573.
Circular Limit =47° 38′.	Coeff. of Reflexion =0·9120.

By direct experiment, as before described, I found the circular limit to be 46° 45′.

On the day preceding that on which the experiments were made on *Silver* (*c*), I examined it before polishing, when evidently tarnished with sulphuret, and found

Principal Incidence =67° 37′.	Coeff. of Refraction =2·4282.
Circular Limit =52° 30′.	Coeff. of Reflexion =0·7673.

Combining the preceding results into one Table, we find, TABLE XLIV.—Constants of Silver.

SILVER (*a*).	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
XL.	72° 37′	48° 46′	3·1942	0·8765
XLI.	71 37	48 13	3·0090	0·8936
Direct exp.	......	48 0	......	0·9004
Means	72 7	48° 19′ 40″	3·1016	0·8901
SILVER (*b*). XLII.	78 7	47° 13′	4·7522	0·9255
Silver (*c*). XLIII.	78 22	47 38	4·8573	0·9120
Direct exp.	......	46 45	......	0·9407
Means	78 22	47° 11′ 30″	4·8573	0·9263

Note.—In all the experiments on silver, the minimum value of $\tan^{-1}\left(\frac{J}{I}\right)$, corresponding to the circular limit, is apparent, although, if the surface were mathematically smooth, it ought to be constant, being a function of the incidence only.

X. GOLD (Standard).

TABLE XLV.—(September 20, 1855.)

Compensator $=47{\cdot}12=90°$. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	79° 45′	5·530	44° 17′
70	68 45	2·571	43 6
60	58 15	1·616	43 0
50	47 0	1·072	42 38
45	42 30	1·091	42 30
40	37 45	1·291	42 42
30	28 0	1·881	42 39
20	19 10	2·876	43 41
10	9 40	5·870	44 1
			Mean=43° 10′ 26″

Principal Incidence $=75°\ 37'$. Coeff. of Refraction $=3{\cdot}8994$.
Circular Limit $=47°\ 47'$. Coeff. of Reflexion $=0{\cdot}9073$.

The minimum value of $\tan^{-1}\left(\frac{J}{I}\right)$ is here also evident.

XI. MERCURY (Distilled).

TABLE XLVI.—(November 1, 1860.)

Compensator =47·12=90°. Red Lamplight.

Polarizer.	Analyser.	$\frac{a}{b}$.	Tan$^{-1}\left(\frac{J}{I}\right)$.
80°	76° 0′	4·011	35° 16′
70	62 42	1·937	35 11
60	51 35	1·260	36 3
50	41 0	1·150	36 6
40	32 2	1·598	36 43
30	23 25	2·309	36 53
20	14 49	3·780	36 1
10	7 19	7·788	36 4
0	0 0	∞	
			Mean=36° 2′ 7″

Principal Incidence =81° 4′. Coeff. of Refraction =6·3616.
Circular Limit =53° 46′. Coeff. of Reflexion =0·7328.

By a direct experiment, I obtained, as before described,

Circular Limit =53° 52′. Coeff. of Reflexion =0·7301.

The value of $\tan^{-1}\left(\frac{J}{I}\right)$ appears to be constant in mercury: can this be due to its being a liquid?

XII. PLATINUM.

TABLE XLVII.—(September 21, 1855.)

Compensator =47·12=90°. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	Tan$^{-1}\left(\frac{J}{I}\right)$.
80°	76° 10′	4·061	35° 36′
70	63 0	1·962	35 32
60	52 15	1·291	36 43
50	40 10	1·185	35 19
40	32 0	1·600	36 41
30	22 15	2·444	35 19
20	14 45	2·798	35 53
10	8 0	7·115	38 33
			Mean=36° 12′ 0″

Principal Incidence =76° 37′. Coeff. of Refraction =4·2030.
Circular Limit =54° 0′. Coeff. of Reflexion =0·7265.

XIII. PALLADIUM.

TABLE XLVIII.—(September 21, 1855.)

Compensator $=47{\cdot}12=90°$. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	Tan$^{-1}\left(\frac{J}{I}\right)$.
80°	75° 15′	3·798	33° 49′
70	65 40	2·211	38 50
60	50 10	1·199	34 41
50	40 15	1·181	35 23
40	29 30	1·631	33 59
30	22 50	2·375	36 6
20	15 0	3·732	36 21
10	8 0	7·115	38 33
			Mean $=35°\ 57'\ 45''$

Principal Incidence $=77°\ 37'$.
Circular Limit $=54°\ 47'$.

Coeff. of Refraction $=4{\cdot}5546$.
Coeff. of Reflexion $=0{\cdot}7058$.

XIV. COPPER.

TABLE XLIX.—Copper. (October 6, 1857.)

Azimuth of Polarizer $=46°\ 15'$. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	Tan$^{-1}\left(\frac{J}{I}\right)$.
63° 30′	44·96	42° 40′	64° 42′	+39° 35′	1·589	41° 25′
68 30	46·19	42 30	79 0	+32 41	1·236	41 15
69 30	46·49	42 45	82 36	+29 17	1·164	41 30
70 30	46·77	42 25	85 52	+19 17	1·123	41 10
71 30	47·17	42 16	90 34	− 2 57	1·101	41 5
72 30	47·25	42 20	91 30	− 7 50	1·102	41 5
73 30	47·63	42 15	95 57	−23 33	1·152	41 4
74 30	47·81	42 32	98 4	−29 12	1·180	41 17
75 30	48·54	43 50	106 35	−37 34	1·174	42 35
76 30	48·76	43 1	109 9	−39 2	1·416	41 46
78 30	49·84	43 20	121 48	−41 51	1·804	42 5
83 30	51·74	45 24	144 2	−45 30	3·000	44 9

Principal Incidence $=71°\ 21'$.
Circular Limit $=48°\ 55'$.

Coeff. of Refraction $=2{\cdot}9629$.
Coeff. of Reflexion $=0{\cdot}8718$.

Table L.—Copper. (October 6, 1857.)

Azimuth of Polarizer = 47° 45′. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
63° 30′	44·93	45° 10′	64° 24′	+45° 23′	1·593	42° 25′
68 30	46·17	44 20	78 55	+41 34	1·218	41 35
69 30	46·38	44 25	81 18	+41 10	1·166	41 40
70 30	46·67	42 45	84 42	+24 47	1·129	40 1
71 30	47·15	43 50	90 20	− 7 58	1·043	41 6
72 30	47·68	43 20	96 32	−31 27	1·137	40 36
73 30	47·85	43 20	98 31	−34 16	1·174	40 36
74 30	48·23	43 45	102 58	−39 29	1·261	41 1
75 30	48·45	43 45	105 32	−40 22	1·320	41 1
76 30	48·84	43 45	110 6	−41 23	1·435	41 1
78 30	49·71	44 45	120 17	−44 30	1·732	42 0
83 30	51·85	45 50	145 19	−46 1	3·177	43 5

Principal Incidence = 71° 27′. Coeff. of Refraction = 2·9800.
Circular Limit = 49° 59′. Coeff. of Reflexion = 0·8396.

Table LI.—Copper. (October 6, 1857.)

Azimuth of Polarizer = 55°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
63° 30′	44·72	50° 15′	61° 39′	+55° 40′	1·734	40° 7′
68 30	45·93	50 15	76 3	+63 46	1·362	40 7
69 30	46·17	50 0	78 51	+66 11	1·301	39 51
70 30	46·68	50 20	84 49	+77 12	1·231	40 11
71 30	46·90	50 7	87 24	+82 57	1·204	39 58
72 30	47·24	50 26	91 22	−86 27	1·226	40 16
73 30	47·59	49 30	95 28	−74 29	1·203	39 21
74 30	47·77	49 28	97 34	−70 1	1·228	39 19
75 30	48·23	49 30	102 57	−62 38	1·320	39 21
76 30	48·67	49 47	108 6	−59 15	1·282	39 38
78 30	49·33	51 20	115 49	−58 39	1·683	41 11
83 30	51·37	53 45	139 42	−56 14	2·896	43 41

Principal Incidence = 71° 6′. Coeff. of Refraction = 2·9207.
Circular Limit = 50° 40′. Coeff. of Reflexion = 0·8194.

TABLE LII.—Copper. (September 21, 1855.)

Compensator $=47{\cdot}12=90°$. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
80°	79° 50′	5·576	44° 31′
70	69 15	2·639	43 51
60	59 30	1·697	44 25
50	48 0	1·110	42 59
45	43 0	1·072	43 0
40	37 30	1·303	42 27
30	28 40	1·829	42 47
20	19 30	2·824	44 13
10	9 50	5·769	44 31
			Mean=43° 38′ 13″

Principal Incidence =73° 37′.
Circular Limit =47° 0′.

Coeff. of Refraction =3·4013.
Coeff. of Reflexion =0·9325.

Combining the preceding results, we obtain

TABLE LIII.—Constants of Copper.

No.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
XLIX.	71° 21′	48° 55′	2·9629	0·8718
L.	71 27	49 59	2·9800	0·8396
LI.	71 6	50 40	2·9207	0·8194
LII.	73 37	47 0	3·4013	0·9325
Means	71° 52′ 45″	49° 8′ 30″	3·0662	0·8656

XV. ZINC.

TABLE LIV.—Zinc. (April 22, 1858.)

Azimuth of Polarizer =53°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
63° 30′	43·71	47° 0′	50° 4′	48° 7′	2·143	38° 57′
68 30	44·58	46 0	60 15	47 1	1·723	37 58
75 30	46·93	45 45	87 45	61 51	1·048	37 43
76 30	46·98	45 0	88 20	+45 0	1·000	37 0
77 30	47·19	45 30	90 47	−70 56	1·022	37 29
78 30	47·84	45 45	98 23	−50 5	1·161	37 43
79 30	48·49	45 30	106 0	−46 49	1·327	37 29
80 30	48·65	46 0	107 52	−48 15	1·375	37 58
81 30	49·20	46 0	114 12	−47 26	1·548	37 58
83 30	50·47	46 30	129 10	−47 22	2·114	38 27
88 30	54·15	52 0	172 13	−52 4	13·078	43 58

Principal Incidence =77° 11′. Coeff. of Refraction =4·3956.
Circular Limit =53° 0′. Coeff. of Reflexion =0·7535.

TABLE LV.—Zinc. (May 7, 1858.)

Azimuth of Polarizer =57°. Red Sunlight.

Incidence.	Compensator.	Analyser.	$e'-e-180°$.	ϕ.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
63° 30′	43·37	53° 0′	46° 6′	+56° 14′	2·485	40° 45′
68 30	44·43	50 30	58 30	+55 12	1·849	38 14
73 30	45·94	48 30	76 9	+58 35	1·315	36 17
75 30	46·49	49 30	82 36	+70 26	1·227	37 15
76 30	46·95	49 0	87 58	+82 55	1·169	36 46
77 30	47·49	50 0	94 18	−78 29	1·224	37 44
78 30	47·74	50 0	97 13	−72 16	1·244	37 44
79 30	48·34	49 30	104 14	−61 5	1·354	37 15
80 30	48·79	50 30	109 30	−60 6	1·491	38 14
81 30	49·32	50 30	115 42	−57 4	1·658	38 14
83 30	50·17	52 0	125 39	−56 35	2·048	39 44
88 30	54·06	54 0	171 10	−54 5	14·983	41 48

Principal Incidence =76° 49′. Coeff. of Refraction =4·2691.
Circular Limit =53° 29′. Coeff. of Reflexion =0·7404.

TABLE LVI.—Zinc. (September 20, 1855.)

Compensator $=47\cdot12=90°$. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$	Tan$^{-1}\left(\frac{J}{I}\right)$.
80°	75° 0′	3·732	33° 21′
70	62 0	1·881	34 23
60	49 30	1·171	34 3
50	39 45	1·202	34 54
40	29 15	1·785	33 43
30	22 30	2·414	35 39
20	15 0	3·732	36 21
10	7 30	7·596	36 45
			Mean=34° 53′ 37″

Principal Incidence $=78°$ 7′. Coeff. of Refraction $=4\cdot7522$.
Circular Limit $=55°$ 23′. Coeff. of Reflexion $=0\cdot6903$.

Combining the preceding results, we obtain the following Table for zinc.

TABLE LVII.—Constants of Zinc.

No.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
LIV.	77° 11′	53° 0′	4·3956	0·7535
LV.	76 49	53 29	4·2691	0·7404
LVI.	78 7	55 23	4·7522	0·6903
Means	77° 22′ 20″	53° 57′ 20″	4·4723	0·7281

XVI. LEAD (polished).

TABLE LVIII. (September 20, 1855.)

Compensator $=47\cdot12=90°$. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	Tan$^{-1}\left(\frac{J}{I}\right)$.
80°	64° 0′	2·050	19° 52′
70	40 30	1·171	17 16
60	29 45	1·750	18 16
50	22 30	2·414	19 10
40	15 0	3·732	17 43
30	10 0	5·671	16 59
20	7 15	7·861	19 16
10	2 30	22·903	13 54
			Mean=17° 48′ 15″

Principal Incidence $=69°$ 37′. Coeff. of Refraction $=2\cdot6913$.
Circular Limit $=71°$ 55′. Coeff. of Reflexion $=0\cdot3265$.

XVII. BISMUTH.

TABLE LIX. (September 25, 1855.)

Compensator $=47\cdot12=90°$. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
80°	76° 15'	4·086	35° 43'
70	60 45	1·785	33 1
60	50 25	1·209	34 56
50	39 30	1·213	34 40
40	30 30	1·697	35 4
30	21 0	2·605	33 37
20	14 25	3·890	35 14
10	6 30	8·777	32 52
			Mean=34° 23' 22"

Principal Incidence =73° 37'. Coeff. of Refraction =3·4013.
Circular Limit =55° 2'. Coeff. of Reflexion =0·6993.

XVIII. TIN.

TABLE LX. (September 25, 1855.)

Compensator $=47\cdot12=90°$. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
80°	76° 30'	4·165	36° 18'
70	64 10	2·065	36 56
60	52 35	1·307	37 2
50	40 30	1·171	35 38
40	32 0	1·600	36 40
30	22 30	2·414	35 40
20	15 20	3·647	36 59
10	8 10	6·968	39 8
			Mean=36° 47' 37"

Principal Incidence =75° 7'. Coeff. of Refraction =3·7627.
Circular Limit =53° 43'. Coeff. of Reflexion =0·7341.

XIX. IRON.

TABLE LXI.—Hard Steel. (September 29, 1855.)

Compensator =47·12=90°. Red Sunlight.

Polariser.	Analyser.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	72° 15′	3·124	28° 51′
70	55 30	1·455	27 54
60	42 35	1·088	27 57
50	33 0	1·540	28 36
40	24 35	2·186	28 36
30	17 30	3·171	28 39
20	11 50	4·773	29 57
10	6 15	9·131	31 51
			Mean = 29° 2′ 37″

Principal Incidence =78° 7′. Coeff. of Refraction =4·7522.
Circular Limit =61° 52′. Coeff. of Reflexion =0·5347.

TABLE LXII.—Soft Steel. (September 29, 1855.)

Compensator =47·12=90°. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	70° 45′	2·863	26° 48′
70	55 50	1·473	28 12
60	41 45	1·120	27 16
50	31 45	1·616	27 27
40	23 30	2·300	27 24
30	17 50	3·108	29 7
20	11 20	4·989	28 50
10	6 0	9·514	30 48
			Mean = 28° 14′ 0″

Principal Incidence =77° 7′. Coeff. of Refraction =4·3721.
Circular Limit =63° 13′. Coeff. of Reflexion =0·5048.

Swedish Iron (cut perpendicular to the grain).

TABLE LXIII. (September 29, 1855.)

Compensator $=47 \cdot 12=90°$. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
80°	71° 35′	3·003	27° 54′
70	55 10	1·437	27 37
60	40 45	1·160	26 27
50	32 0	1·600	27 40
40	22 45	2·385	26 33
30	17 0	3·271	27 54
20	11 0	5·144	28 6
10	5 30	10·385	28 38
			Mean = 27° 36′ 7″

Principal Incidence = 76° 7′. Coeff. of Refraction = 4·0458.
Circular Limit = 62° 57′. Coeff. of Reflexion = 0·5106.

Swedish Iron (cut parallel to the grain).

TABLE LXIV. (September 29, 1855.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
80°	71° 45′	3·032	28° 8′
70	55 20	1·446	27 46
60	41 40	1·124	27 12
50	32 0	1·600	27 40
40	24 0	2·246	27 57
30	17 30	3·171	28 39
20	11 0	5·144	28 6
10	5 35	10·229	29 0
			Mean = 28° 3′ 30″

Principal Incidence = 76° 7′. Coeff. of Refraction = 4·0458.
Circular Limit = 62° 26′. Coeff. of Reflexion = 0·5220.

Combining the preceding results, we find

TABLE LXV.—Constants of Steel and Iron.

No.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
LXI. Hard steel.	78° 7′	61° 52′	4·7522	0·5347
LXII. Soft steel.	77 7	63 13	4·3721	0·5048
LXIII. Iron (*a*).	76 7	62 57	4·0458	0·5106
LXIV. Iron (*b*).	76 7	62 26	4·0458	0·5220

XX. ALUMINIUM.

TABLE LXVI. (May 10, 1856.)

Compensator =47·12=90°. Red Sunlight.

Polarizer.	Analyser.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
80°	75° 45′	3·937	34° 46′
70	60 30	1·767	32 45
60	48 0	1·110	32 40
50	37 30	1·303	32 47
40	28 15	1·861	32 38
30	20 30	2·674	32 56
20	13 45	4·086	33 55
10	7 10	7·953	35 30
			Mean=33° 29′ 37″

Principal Incidence	=77° 7′.	Coeff. of Refraction	=4·3721.
Circular Limit	=57° 9′.	Coeff. of Reflexion	=0·6457.

By a direct experiment I found the circular limit to be 57° 15′.

XXI. ALLOYS OF COPPER AND ZINC.

The following experiments were made on fourteen alloys of copper and zinc prepared by Mr. ROBERT MALLET, in atomic proportions, as follow:—

No. 1 . . .	$10Cu+Zn$
No. 2 . . .	$9Cu+Zn$
No. 3 . . .	$8Cu+Zn$
No. 4 . . .	$7Cu+Zn$
No. 5 . . .	$6Cu+Zn$
No. 6 . . .	$5Cu+Zn$
No. 7 . . .	$4Cu+Zn$
No. 8 . . .	$3Cu+Zn$
No. 9 . . .	$2Cu+Zn$
No. 10 . . .	$Cu+Zn$
No. 11 . . .	$Cu+2Zn$
No. 12 . . .	$Cu+3Zn$
No. 13 . . .	$Cu+4Zn$
No. 14 . . .	$Cu+5Zn$

The chemical and physical properties of these alloys are fully described by Mr. MALLET in his "Report on the Action of Air and Water upon Iron" to the British Association for the Advancement of Science for the year 1840, p. 306.

In all the experiments red sunlight was used, and the compensator was placed at 47·12=90°.

Table LXVII.—Alloys of Copper and Zinc, No. 1. (September 16, 1856.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
80°	79° 0′	5·144	42° 13′
70	66 45	3·237	40 16
60	56 10	1·492	40 45
50	45 30	1·017	40 30
40	35 30	1·402	40 22
30	27 0	1·962	41 26
20	18 15	3·032	42 11
10	9 25	6·029	43 15
			Mean=41° 22′ 15″

Principal Incidence =72° 5′.
Circular Limit =49° 32′.

Coeff. of Refraction =3·0930.
Coeff. of Reflexion =0·8531.

Table LXVIII.—Alloys of Copper and Zinc, No. 2. (September 16, 1856.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
80°	79° 40′	5·484	44° 3′
70	67 35	2·424	41 25
60	58 0	1·600	42 44
50	45 35	1·020	40 35
40	35 25	1·406	40 17
30	27 30	1·921	42 2
20	17 40	3·140	41 11
10	9 30	5·976	43 30
			Mean=41° 58′ 22″

Principal Incidence =72° 15′.
Circular Limit =49° 32′.

Coeff. of Refraction =3·1240.
Coeff. of Reflexion =0·8531.

TABLE LXIX.—Alloys of Copper and Zinc, No. 3. (September 18, 1856.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	79° 10′	5·225	44° 20′
70	67 20	2·394	41 4
60	54 0	1·376	38 28
50	46 0	1·035	40 59
40	34 50	1·437	39 40
30	27 15	1·941	41 44
20	17 35	3·155	41 2
10	9 15	6·140	42 43
			Mean = 41° 2′ 30″

Principal Incidence = 73° 10′. Coeff. of Refraction = 3·3052.
Circular Limit = 49° 6′. Coeff. of Reflexion = 0·8662.

TABLE LXX.—Alloys of Copper and Zinc, No. 4. (September 18, 1856.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	78° 30′	4·915	40° 55′
70	67 45	2·444	41 40
60	57 0	1·540	41 38
50	46 0	1·035	40 59
40	35 30	1·402	40 22
30	27 0	1·962	41 26
20	18 0	3·077	41 45
10	8 50	6·435	41 23
			Mean = 41° 16′ 0″

Principal Incidence = 73° 8′. Coeff. of Refraction = 3·2983.
Circular Limit = 49° 3′. Coeff. of Reflexion = 0·8677.

TABLE LXXI.—Alloys of Copper and Zinc, No. 5. (September 18, 1856.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\text{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	79° 10′	5·225	42° 40′
70	67 0	2·356	40 37
60	56 15	1·496	40 50
50	45 55	1·032	40 55
40	35 52	1·383	40 45
30	26 50	1·977	41 13
20	18 15	3·032	42 11
10	9 45	5·819	44 16
			Mean = 41° 40′ 52″

Principal Incidence = 74° 5′. Coeff. of Refraction = 3·5066.
Circular Limit = 49° 5′. Coeff. of Reflexion = 0·8667.

TABLE LXXII.—Alloys of Copper and Zinc, No. 6. (September 19, 1856.)

Polariser.	Analyser.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	79° 25′	5·352	43° 20′
70	68 30	2·538	42 44
60	57 55	1·595	42 39
50	47 40	1·098	42 39
40	36 30	1·351	41 24
30	28 15	1·861	42 57
20	18 12	3·041	42 5
10	9 45	5·819	44 16
			Mean = 42° 45′ 30″

Principal Incidence = 74° 8′. Coeff. of Refraction = 3·5183.
Circular Limit = 47° 37′. Coeff. of Reflexion = 0·9126.

TABLE LXXIII.—Alloys of Copper and Zinc, No. 7. (September 19, 1856.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	79° 10′	5·225	42° 39′
70	67 30	2·414	41 18
60	57 0	1·540	41 39
50	45 40	1·023	40 39
40	35 0	1·428	39 51
30	26 10	2·035	40 24
20	17 45	3·124	41 20
10	9 25	6·029	43 15
			Mean = 41° 48′ 7″

Principal Incidence = 73° 16′. Coeff. of Refraction = 3·3261.
Circular Limit = 49° 23′. Coeff. of Reflexion = 0·8576.

TABLE LXXIV.—Alloys of Copper and Zinc, No. 8. (July 13, 1857.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	77° 30′	4·511	38° 30′
70	65 10	2·161	38 11
60	55 40	1·464	40 13
50	43 50	1·041	38 51
40	33 5	1·535	37 49
30	24 40	2·177	38 30
20	15 40	3·565	37 37
10	8 20	6·827	39 43
			Mean = 38° 40′ 30″

Principal Incidence = 73° 12′. Coeff. of Refraction = 3·3121.
Circular Limit = 50° 53′. Coeff. of Reflexion = 0·8132.

Table LXXV.—Alloys of Copper and Zinc, No. 9. (October 5, 1857.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	79° 0′	5·144	42° 13′
70	67 30	2·414	41 18
60	54 55	1·424	39 25
50	46 20	1·047	41 19
40	35 35	1·397	40 27
30	26 0	2·050	40 11
20	17 0	3·271	40 2
10	8 40	6·560	40 50
			Mean = 40° 43′ 7″

Principal Incidence = 72° 18′. Coeff. of Refraction = 3·1334.
Circular Limit = 48° 46′. Coeff. of Reflexion = 0·8764.

Table LXXVI.—Alloys of Copper and Zinc, No. 10. (October 5, 1857.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	79° 0′	5·144	42° 13′
70	65 0	2·144	37 58
60	54 20	1·393	38 49
50	43 45	1·044	38 46
40	34 35	1·450	39 35
30	25 15	2·120	39 15
20	16 30	3·375	39 9
10	8 45	6·497	41 7
			Mean = 39° 36′ 30″

Principal Incidence = 72° 15′. Coeff. of Refraction = 3·1240.
Circular Limit = 51° 11′. Coeff. of Reflexion = 0·8045.

Table LXXVII.—Alloys of Copper and Zinc, No. 11. (October 5, 1857.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\mathrm{Tan}^{-1}\left(\frac{J}{I}\right)$.
80°	78° 30′	4·915	40° 55′
70	65 30	2·194	38 37
60	54 0	1·376	38 28
50	43 0	1·072	38 3
40	33 30	1·511	38 16
30	24 30	2·194	38 17
20	16 15	3·431	38 41
10	8 35	6·625	40 34
			Mean = 38° 58′ 52″

Principal Incidence = 72° 15′. Coeff. of Refraction = 3·1240.
Circular Limit = 51° 49′. Coeff. of Reflexion = 0·7864.

TABLE LXXVIII.—Alloys of Copper and Zinc, No. 12. (October 5, 1857.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
80°	74° 40′	3·647	32° 45′
70	58 45	1·648	30 57
60	48 0	1·110	32 40
50	37 45	1·291	33 1
40	28 10	1·867	31 57
30	20 0	2·747	32 14
20	13 20	4·219	33 4
10	6 30	8·776	32 56
			Mean = 32° 26′ 45″

Principal Incidence = 76° 7′. Coeff. of Refraction = 4·0458.
Circular Limit = 57° 5′. Coeff. of Reflexion = 0·6473.

TABLE LXXIX.—Alloys of Copper and Zinc, No. 13. (October 6, 1857.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
80°	76° 15′	4·086	35° 46′
70	60 50	1·792	33 7
60	49 45	1·181	34 18
50	39 10	1·227	34 21
40	29 15	1·785	33 43
30	20 40	2·651	33 10
20	13 30	4·165	33 25
10	6 40	8·555	33 32
			Mean = [illegible]

Principal Incidence = 73° 52′. Coeff. of Refraction = 3·4570.
Circular Limit = 55° 31′. Coeff. of Reflexion = 0·6868.

TABLE LXXX.—Alloys of Copper and Zinc, No. 14. (October 6, 1857.)

Polarizer.	Analyser.	$\frac{a}{b}$.	$\tan^{-1}\left(\frac{J}{I}\right)$.
80°	75° 45′	3·937	34° 46′
70	61 45	1·861	34 7
60	48 40	1·136	33 17
50	39 0	1·235	34 12
40	28 40	1·829	33 5
30	20 50	2·628	33 23
20	13 55	4·036	34 15
10	7 10	7·953	35 30
			Mean = 34° 4′ 22″

Principal Incidence = 76° 0′. Coeff. of Refraction = 4·0108.
Circular Limit = 56° 12′. Coeff. of Reflexion = 0·6694.

The alloys from 1 to 11 are all yellowish, and from 12 to 14 are whitish.

The following Table shows that the Coefficients of Refraction from 1 to 11 increase gradually, reaching a maximum at No. 6 ($5Cu+Zn$), and then diminish to No. 11, in passing from which to No. 12 the coefficient suddenly increases. The Coefficient of Reflexion follows an order somewhat similar, but suddenly decreases in passing from 11 to 12, which is the limit at which the zinc begins to preponderate over the copper, in producing the optical properties of the alloy.

In Plate VIII. fig. B, I have tabulated the coefficients of refraction and reflexion of the alloys of copper and zinc, showing the progression of these constants, as just described.

TABLE LXXXI.—Optical Constants of all the Substances examined.

Substance.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.	Refractive Index.
(A.) *Transparent.*					
I. Munich Glass (*a*)	55° 0′ 37″	85° 32′ 30″	1·4287	0·0780	1·6227
II. Munich Glass (*b*)	54 40 48	89 53 24	1·4113	0·0019	1·5244
III. Paris Glass	56 8 30	89 23 0	1·4905	0.0107	1·5100
IV. Fluor-Spar	54 46 0	89 41 30	1·4158	0·0053	
V. Glass of Antimony	58 48 40	88 51 40	1·6519	0·0199	
VI. Quartz (*a*)	56 40 0	88 58 0	1·5204	0·0180	
VII. Quartz (*b*)	56 54 30	89 21 0	1·5344	0·0108	
(B.) *Metals.*					
VIII. Speculum	76 33 0	55 32 0	4·1901	0·6865	
IX. Silver (*a*)	72 7 0	48 19 40	3·1016	0·8901	
— Silver (*b*)	78 7 0	47 13 0	4·7522	0·9255	
— Silver (*c*)	78 22 0	47 11 30	4·8573	0·9263	
X. Gold	75 37 0	47 47 0	3·8994	0·9073	
XI. Mercury	81 4 0	53 49 0	6·3616	0·7315	
XII. Platinum	76 37 0	54 0 0	4·2030	0·7265	
XIII. Palladium	77 37 0	54 47 0	4·5546	0·7058	
XIV. Copper	71 52 45	49 8 30	3·0662	0·8656	
XV. Zinc	77 22 20	53 57 20	4·4723	0·7281	
XVI. Lead	69 37 0	71 55 0	2·6913	0·3265	
XVII. Bismuth	73 37 0	55 2 0	3·4013	0·6993	
XVIII. Tin	75 7 0	53 43 0	3·7627	0·7341	
XIX. Iron	76 7 0	62 41 30	4·0458	0·5163	
— Steel	77 37 0	62 32 30	4·5621	0·5197	
XX. Aluminium	77 7 0	57 9 0	4·3721	0·6457	
XXI. Alloys of Copper and Zinc:—					
No. 1	72 5 0	49 32 0	3·0930	0·8531	
No. 2	72 15 0	49 32 0	3·1240	0·8531	
No. 3	73 10 0	49 6 0	3·3052	0·8662	
No. 4	73 8 0	49 3 0	3·2983	0·8677	
No. 5	74 5 0	49 5 0	3·5066	0·8667	
No. 6	74 8 0	47 37 0	3·5183	0·9126	
No. 7	73 16 0	49 23 0	3·3261	0·8576	
No. 8	73 12 0	50 53 0	3·3121	0·8132	
No. 9	72 18 0	48 46 0	3·1334	0·8764	
No. 10	72 15 0	51 11 0	3·1240	0·8045	
No. 11	72 15 0	51 49 0	3·1240	0·7864	
No. 12	76 7 0	57 5 0	4·0458	0·6473	
No. 13	73 52 0	55 31 0	3·4570	0·6868	
No. 14	76 0 0	56 12 0	4·0108	0·6694	

In the preceding Table there are twelve pure metals; if we arrange these in two Tables, according to the magnitude of the Coefficients of Refraction and Reflexion, we obtain the following.

TABLE LXXXII.—Coefficient of Refraction of pure Metals.

Metal.	Coefficient of Refraction.
I. Mercury	6·3616
II. Silver	4·8047
III. Palladium	4·5546
IV. Zinc	4·4723
V. Aluminium	4·3721
VI. Iron	4·3039
VII. Platinum	4·2030
VIII. Gold	3·8994
IX. Tin	3·7627
X. Bismuth	3·4013
XI. Copper	3·0662
XII. Lead	2·6913

TABLE LXXXIII.—Coefficient of Reflexion of pure Metals.

Metal.	Coefficient of Reflexion.
I. Silver	0·9259
II. Gold	0·9073
III. Copper	0·8656
IV. Tin	0·7341
V. Mercury	0·7315
VI. Zinc	0·7281
VII. Platinum	0·7265
VIII. Palladium	0·7058
IX. Bismuth	0·6993
X. Aluminium	0·6457
XI. Iron	0·5180
XII. Lead	0·3265

The *brilliancy* of a metallic surface depends on the coefficient of refraction, and there is, doubtless, some sensible quality, not yet clearly defined, which corresponds to the coefficient of reflexion, which indicates the power of the surface to form elliptically polarized light from incident plane-polarized light. This quality might be provisionally named *lustre*.

It is very remarkable that gold, silver, and copper, which from time immemorial have pleased the eye of man, and been used as coins, should head the list of bodies possessing a high coefficient of reflexion. Mercury, which has so brilliant a surface, and therefore heads the list in Table LXXXII., occupies a comparatively low place in Table LXXXIII., probably owing to its being a liquid, and its surface, therefore, in a less favourable condition than that of a solid for imparting elliptic polarization to an incident beam.

M. JAMIN has examined optically several of the substances mentioned in the preceding

Tables—the metallic bodies by the methods of equal intensities and multiple reflexions, and the transparent bodies by the method employed in this paper, and originally used by him.

I have deduced from his original observations, the optical constants of the substances common to him and myself, and have recorded them for the purpose of comparison, in the two following Tables, LXXXIV. and LXXXV.*

TABLE LXXXIV.—Optical Constants of Metals, deduced from JAMIN'S experiments.

Substance.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
Steel (1)	76° 0′	59° 6′	4·0108	0·5985
Steel (2)	77 4	61 27	4·3546	0·5441
I. Means	76° 32′ 0″	60° 16′ 30″	4·1827	0·5713
Silver (3)	71 40	54 0	3·0178	0·7265
Silver (4)	75 0	47 1	3·7320	0·9320
II. Means	73° 20′ 0″	50° 30′ 30″	3·3747	0·8292
Zinc (5)	77 0	......	4·3314	......
Zinc (5)	79 13	....	5·2505	....
Zinc (6)	75 11	60 57	3·7804	0·5554
III. Means	77° 8′ 0″	60° 57′ 0″	4·4541	0·5554
Copper (7)	70 9	50 36	2·7700	0·8214
Copper (8)	71 21	53 41	2·9629	0·7350
IV. Means.................	70° 45′ 0″	52° 8′ 30″	2·8664	0·7782
Speculum metal (9)	75 50	56 45	3·9616	0·6556
Speculum metal (10)	......	56 40	......	0·6577
Speculum metal (11)	76 14	53 33	4·0815	0·7386
V. Means	76° 2′ 0″	55° 39′ 20″	4·0215	0·6839
VI. Brass (12)	71 31	52 57	2·9916	0·7549

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* These Tables were added during the printing of the paper.

TABLE LXXXV.—Optical Constants of Transparent Bodies, from JAMIN'S experiments.

Substance.	Principal Incidence.	Circular Limit.	Coefficient of Refraction.	Coefficient of Reflexion.
I. Glass of Antimony (13)	63° 34′	88° 26′	2·0115	0·0290
II. Quartz (13) …………	56 50	89 25	1·5301	0·0102
III. Fluor-Spar (13) ………	55 15	89 31	1·4415	0·0084

(1) Ann. de Chim. et de Phys. (sér. 3) vol. xix. p. 304.

From the two Tables in this page, I find at 75° incidence, I=0·946, and J=0·566, from which it follows that

$$\tan^{-1}\left(\frac{J}{I}\right)=30^\circ\ 54'.$$

(2) Ann. de Chim. et de Phys. (sér. 3) vol. xxii. p. 316 (mean red).

The azimuths given in this and the following page are arcs such that

$$\tan\ (\text{azimuth})\ =k^2$$
$$k=\tan^{-1}\left(\frac{J}{I}\right).$$

From this consideration the coefficient of reflexion is deduced.

(3) Ann. de Chim. et de Phys. (sér. 3) vol. xix. p. 315.

(4) Ann. de Chim. et de Phys. (sér. 3) vol. xxii. p. 316 (mean red).

(5) Ann. de Chim. et de Phys. (sér. 3) vol. xix. p. 320.

(6) Ann. de Chim. et de Phys. (sér. 3) vol. xxii. p. 316 (mean red).

(7) Ann. de Chim. et de Phys. (sér. 3) vol. xix. p. 337. I have calculated the value of the circular limit and coefficient of refraction from the experiment recorded as made with *two* reflexions.

(8) Ann. de Chim. et de Phys. (sér. 3) vol. xxii. p. 317 (red light).

(9) Ann. de Chim. et de Phys. (sér. 3) vol. xix. pp. 305, 306. The ratio of J to I at the principal incidence is found to be, from the Tables of these two pages, as 623 to 950, from which the circular limit is deduced.

(10) Ann. de Chim. et de Phys. (sér. 3) vol. xix. p. 330.

(11) Ann. de Chim. et de Phys. (sér. 3) vol. xxii. p. 316 (red light).

(12) Ann. de Chim. et de Phys. (sér. 3) vol. xxii. p. 317 (red light).

(13) Ann. de Chim. et de Phys. (sér. 3) vol. xxix. p. 303.

VI. *On the Exact Form of Waves near the Surface of Deep Water.*

By WILLIAM JOHN MACQUORN RANKINE, *C.E., LL.D., F.R.SS. L. & E.*

Received September 27,—Read November 27, 1862.

(1.) THE investigations of the Astronomer Royal and of some other mathematicians on straight-crested parallel waves in a liquid, are based on the supposition that the displacements of the particles of the liquid are small compared with the length of a wave. Hence it has been very generally inferred that the results of those investigations are approximate only, when applied to waves in which the displacements, as compared with the length of a wave, are considerable.

(2.) In the present paper I propose to prove that one of those results (viz., that in very deep water the particles move with a uniform angular velocity in vertical circles whose radii diminish in geometrical progression with increased depth, and consequently that surfaces of equal pressure, including the upper surface, are trochoidal) is exact for all displacements, how great soever.

(3.) I believe the trochoidal form of waves to have been first explicitly stated by Mr. SCOTT RUSSELL; but no demonstration of its exactly fulfilling the conditions of the question has yet been published, so far as I know.

(4.) In 'A Manual of Applied Mechanics' (first published in 1858), page 579, I stated that the theory of rolling waves might be deduced from that of the positions assumed by the surface of a mass of water revolving in a vertical plane about a horizontal axis; as the theory of such waves, however, was foreign to the subject of the book, I did not then publish the investigation on which that statement was founded.

(5.) Having communicated some of the leading principles of that investigation to Mr. WILLIAM FROUDE in April 1862, I learned from him that he had already arrived independently at similar results by a similar process, although he had not published them.

(6.) PROPOSITION I.—*In a mass of gravitating liquid whose particles revolve uniformly in vertical circles, a wavy surface of trochoidal profile fulfils the conditions of uniformity of pressure,—such trochoidal profile being generated by rolling, on the underside of a straight line, a circle whose radius is equal to the height of a conical pendulum that revolves in the same period with the particles of liquid.*

In fig. 1 (p. 128) let B be a particle of liquid revolving uniformly in a vertical circle of the radius C B, in the direction indicated by the arrow N; and let it make n revolutions in a second. Then the centrifugal force of B (taking its mass as unity) will be

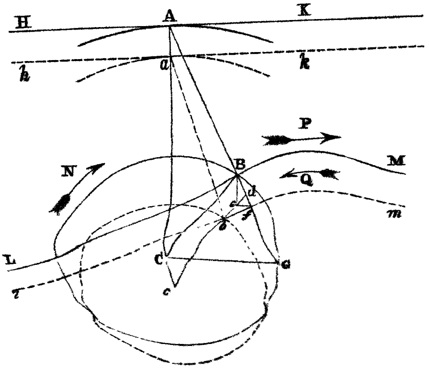
$$4\pi^2 n^2 \,.\, \mathrm{CB}.$$

Draw $\overline{CA}$ vertically upwards, and of such a length that centrifugal force : gravity :: $\overline{CB}:\overline{AC}$; that is to say, make

$$\overline{AC}=\frac{g}{4\pi^2n^2};$$

which is the well-known expression for the height of a revolving pendulum making n revolutions in a second.

Fig. 1.



Then $\overline{AC}$ being in the direction of and proportional to gravity, and $\overline{CB}$ in the direction of and proportional to centrifugal force, $\overline{AB}$ will be in the direction of and proportional to the resultant of gravity and centrifugal force; and the surface of equal pressure traversing B will be normal to $\overline{AB}$.

The profile of such a surface is obviously a trochoid LBM, traced by the point B, which is carried by a circle of the radius $\overline{CA}$ rolling along the underside of the horizontal straight line HAK. Q.E.D.

(7.) *Corollaries.*—The length of the wave whose period is one-nth of a second is equal to the circumference of the rolling circle; that is to say (denoting that length by λ),

$$\lambda=2\pi.\overline{CA}=\frac{g}{2\pi n^2}:$$

the period of a wave of a given length λ is given in seconds, or fractions of a second, by the equation

$$\frac{1}{n}=\sqrt{\frac{2\pi\lambda}{g}};$$

and the velocity of propagation of such a wave is

$$n\lambda=\frac{g}{2\pi n}=\sqrt{\frac{g\lambda}{2\pi}}=\sqrt{g.\overline{CA}};$$

results agreeing with those of the known theory.

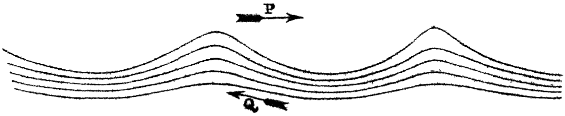
(8.) **Proposition II.**—*Let another surface of uniform pressure be conceived to exist indefinitely near to the first surface; then, if the first surface is a surface of continuity, so also is the second.*

By a surface of continuity is here meant one which always passes through the same set of particles of liquid, so that a pair of such surfaces contain between them a layer of particles which are always the same.

The perpendicular distance between a pair of surfaces of uniform pressure is in this case inversely proportional to the resultant of gravity and centrifugal force; that is to say, to the normal $\overline{\mathrm{AB}}$. Hence if a curve *lbfm* be drawn indefinitely near to the curve LBM, so that the perpendicular distance between them, $\overline{\mathrm{B}f}$, shall everywhere be inversely proportional to the normal $\overline{\mathrm{AB}}$, the second curve will also be the profile of a surface of uniform pressure.

Conceive now that the whole mass of liquid has, combined with its wave-motion, a uniform motion of translation, with a velocity equal and opposite to that of the propagation of the waves. The dynamical conditions of the mass are not in the least altered by this; but the forms of the waves are rendered stationary (as we sometimes see in a rapid stream), and, instead of a series of waves propagated in the direction shown by the arrow P, we have an *undulating current* running the reverse way, in the direction shown by the arrow Q. (This is further illustrated by fig. 2.) According to a well-known property of curves described by rolling, the velocity of the particle B in that current is proportional to the normal $\overline{\mathrm{AB}}$, and is given by the expression

Fig. 2.



$$2\pi n . \overline{\mathrm{AB}}.$$

Consider the layer of the current contained between the surfaces LBM and *lbm*. In order that the latter of those surfaces, as well as the former, may be a surface of continuity, it is necessary and sufficient that the thickness of the layer $\overline{\mathrm{B}f}$ at each point should be inversely as the velocity; and that condition is already fulfilled; for $\overline{\mathrm{B}f}$ varies inversely as $\overline{\mathrm{AB}}$, and $\overline{\mathrm{AB}}$ varies as the velocity of the current at B; therefore LBM and *lbm* are not only a pair of surfaces of uniform pressure, but a pair of surfaces of continuity also. Q.E.D.

(9.) *Corollary.*—The surfaces of uniform pressure are identical with surfaces of continuity throughout the whole mass of liquid.

(10.) *Corollary.*—Inasmuch as the resultant of gravity and centrifugal force at B is represented by

$$g \cdot \frac{\overline{\mathrm{AB}}}{\overline{\mathrm{AC}}},$$

the excess of the uniform pressure at the surface *lbm* above that at the surface LBM is given by the expression

$$dp = w \cdot \frac{\overline{\mathrm{AB}}}{\overline{\mathrm{AC}}} \cdot \overline{\mathrm{B}f},$$

in which w is the heaviness of the liquid, in units of weight per unit of volume. By omitting the factor w, the pressure is expressed in units of height of a column of the liquid.

(11.) PROPOSITION III.—*The profile of the lower surface of the layer referred to in the preceding proposition is a trochoid generated by a rolling circle of the same radius with that which generates the first trochoid; and the tracing-arm of the second trochoid is shorter than that of the first trochoid by a quantity bearing the same proportion to the depth of the centre of the second rolling circle below the centre of the first rolling circle, which the tracing-arm of the first rolling circle bears to the radius of that circle.*

At an indefinitely small depth $\overline{\mathrm{A}a}$ below the horizontal line HAK, draw a second horizontal line *hak*, on the under side of which let a circle roll with a radius $\overline{ca} = \overline{\mathrm{CA}}$, the radius of the first rolling circle; so that the indefinitely small depths $\overline{\mathrm{C}c} = \overline{\mathrm{A}a}$. To find the tracing-arm of the second rolling circle, draw cd parallel to $\overline{\mathrm{CB}}$, the tracing-arm of the first circle; in cd take $\overline{ce} = \overline{\mathrm{CB}}$, and cut off $\overline{eb} = \overline{ed}$; b will be the tracing-point, and $\overline{cb}$ the tracing-arm required; for, according to the principle laid down in the enunciation, we are to have

$$\overline{\mathrm{CB}} - \overline{cb} = \overline{eb} = \mathrm{C}c \cdot \frac{\overline{\mathrm{CB}}}{\overline{\mathrm{CA}}}.$$

Let the second circle roll; then b will trace a trochoid *lbm*. From b let fall bf perpendicular to $\overline{\mathrm{AB}}$ produced; Bf will be the indefinitely small thickness at B of the layer between the two trochoidal surfaces.

The proposition enunciated amounts to stating that Bf is everywhere inversely proportional to the normal $\overline{\mathrm{AB}}$; so that *lbm* is the profile of a surface of uniform pressure and of continuity.

To prove this, join $\overline{\mathrm{B}e}$ and $\overline{ef}$. Then Be is parallel to ACc, and equal to Cc, and def is evidently an isosceles triangle, $\overline{ef}$ being $= ed$. Let AB (produced if necessary) cut the circle of the radius CB in G; then CG is parallel to ef, and the indefinitely small triangle Bef is similar to the triangle ACG; consequently $\overline{\mathrm{AC}} : \overline{\mathrm{AG}} :: \overline{\mathrm{B}e} = \overline{\mathrm{C}c} : \mathrm{B}f$; or

$$\overline{\mathrm{B}f} = \overline{\mathrm{C}c} \cdot \frac{\overline{\mathrm{AG}}}{\overline{\mathrm{AC}}};$$

but, by a well-known property of the circle,

$$\overline{\mathrm{AG}} = \frac{\overline{\mathrm{AC}}^2 - \overline{\mathrm{CB}}^2}{\overline{\mathrm{AB}}};$$

and therefore

$$\overline{\mathrm{B}f} = \overline{\mathrm{C}c} \cdot \frac{\overline{\mathrm{AC}}^2 - \overline{\mathrm{CB}}^2}{\overline{\mathrm{AC}} \cdot \overline{\mathrm{AB}}};$$

that is to say, the thickness of the layer varies inversely as the normal $\overline{\mathrm{AB}}$; *and the second trochoid, lbm, is therefore the profile of a surface of uniform pressure and of continuity.* Q. E. D.

(12.) *Corollaries.*—The profiles of the surfaces of uniform pressure and of continuity form an indefinite series of trochoids, described by equal rolling circles, rolling with the same speed below an indefinite series of horizontal straight lines.

The tracing-arms of those circles (each of which arms is the radius of the circular orbit of the particles contained in the trochoidal surface which it traces) diminish in geometrical progression with increase of depth, according to the following laws:—

For convenience, let $\overline{\mathrm{C}c}$ be denoted by dk, $\overline{\mathrm{CB}}$ by r, and $\overline{cb}$ by $r-dr$; then

$$dr=dk.\frac{r}{\mathrm{AC}}=dk.\frac{r}{2\pi\lambda},$$

and the integration of this equation gives the following result:—

Let k denote the vertical depth of the centre of the generating circle of a given surface below the centre of the generating circle of the free upper surface of the liquid;

r_0 the tracing-arm of the free upper surface (= half the amplitude of disturbance);

r_1 the tracing-arm of the surface whose middle depth is k; then

$$r_1=r_0e^{-\frac{k}{\mathrm{AC}}}=r_0e^{-\frac{2\pi k}{\lambda}},$$

a formula exactly agreeing with that found for indefinitely small disturbances by previous investigators.

(13.) PROPOSITION IV.—*The centres of the orbits of the particles in a given surface of equal pressure stand at a higher level than the same particles do when the liquid is still, by a height which is a third proportional to the diameter of the rolling circle and the tracing-arm or radius of the orbits of the particles, and which is equal to the height due to the velocity of revolution of the particles.*

If the liquid were still, the given surface of equal pressure would become horizontal. To find the level at which it would stand, we must first find what relation the mean vertical depth of a given layer of particles bears to the depth $\overline{\mathrm{C}c}=dk$, between the centres of the rolling circles that generate its boundaries.

The length of the arc of the curve LBM described in an indefinitely short interval of time dt is

$$2\pi n.\overline{\mathrm{AB}}.dt,$$

and the thickness of the layer being

$$\overline{\mathrm{B}f}=dk.\frac{\overline{\mathrm{AC}}^2-\overline{\mathrm{CB}}^2}{\overline{\mathrm{AC}}.\overline{\mathrm{AB}}},$$

let the product of those quantities be divided by the distance through which the centre of the rolling circle moves in the same time, viz.

$$2\pi n.\overline{\mathrm{AC}}.dt,$$

and the result will be the mean vertical depth of the layer, which being denoted by dk_0 we have

$$dk_0=dk.\left(1-\frac{\overline{CB}^2}{\overline{AC}^2}\right)=dk.\left(1-\frac{r^2}{\overline{AC}^2}\right)=dk.\left(1-\frac{r_0^2}{\overline{AC}^2}.e^{-\frac{2k}{AC}}\right).$$

The difference by which the mean vertical thickness of the layer falls short of the difference of level of the rolling circles of its upper and lower surfaces is given by the following expression,

$$dk-dk_0=\frac{r_0^2}{\overline{AC}^2}e^{-\frac{2k}{AC}}dk;$$

and this being integrated from ∞ to k, gives the depth of the position of a given particle, when the liquid is still, below the level of the centre of the orbit of the same particle when disturbed, viz.

$$k_0-k=\frac{r_0^2}{2AC}.e^{-\frac{2k}{AC}}=\frac{r^2}{2AC}=\frac{\pi r^2}{\lambda},$$

or *a third proportional to the diameter of the rolling circle and the radius of the orbit of the particle; also* $\frac{r^2}{2AC}=\frac{4\pi^2n^2r^2}{2g}$ *is the height due to the velocity of revolution of the particles.* Q.E.D.

(13 A.) *Corollary.*—The mechanical energy of a wave is half actual and half potential,—half being due to motion, and half to elevation. In other words, the mechanical energy of a wave is double of that due to the motion of its particles only, there being an equal amount due to the mean elevation of the particles above their position when the water is still.

(14.) *Corollary.*—The crests of the waves rise higher above the level of still water than their hollows fall below it; and the difference between the elevation of the crest and the depression of the hollow is double of the quantity mentioned in Proposition IV., that is to say, it is

$$\frac{r^2}{AC}=\frac{2\pi r^2}{\lambda}.$$

(15.) *Corollary as to Pressures.*—An expression has already been given in art. 10 for the difference of pressure at the upper and under surfaces of a given layer. Substituting in that expression the value of the thickness of the layer, we find

$$dp=w.\frac{\overline{AB}}{\overline{AC}}.dk.\frac{\overline{AC}^2-\overline{CB}^2}{\overline{AC}.\overline{AB}}=w.dk\left(1-\frac{\overline{CB}^2}{\overline{AC}^2}\right)=w.dk_0$$

(as the preceding corollary shows), being precisely the same as if the liquid were still; and hence it follows that *the hydrostatic pressure at each individual particle during wave-motion is the same as if the liquid were still.*

(16.) In Proposition III. it has been shown, by geometrical reasoning from the mechanical construction of the trochoid, that a wave consisting of trochoidal layers satisfies the condition of continuity. It may be satisfactory also to show the same thing by the use of algebraic symbols. For that purpose the following notation will be used.

Let the origin of coordinates be assumed to be in the horizontal line containing the centre of the circle which is rolled to trace the profile of *cycloidal* waves, having cusps, and being (as Mr. SCOTT RUSSELL long ago pointed out) the highest waves that can exist without breaking. In such waves, the tracing-arm, or radius vector, of the uppermost particles is equal to the radius of the rolling circle; and that arm diminishes for each successive layer proceeding downwards.

Let x and y be the coordinates of any particle, x being measured horizontally, *against* the direction of propagation, and y vertically downwards.

Let k (as before) be the vertical coordinate of the centre of the given particle's orbit; h the horizontal coordinate of the same centre.

Let R be the radius of the rolling circle, a the angular velocity of the tracing-arm $(=2\pi n)$, so that

$$2\pi \mathrm{R}=\lambda$$

is the length of a wave, and

$$a\mathrm{R}=n\lambda\sqrt{\frac{g\lambda}{2\pi}}=\sqrt{g\mathrm{R}}$$

is the velocity of propagation.

Let θ denote the *phase* of the wave at a given particle, being the angle which its radius vector, or tracing-arm, makes with the direction of $+y$, that is, with a line pointing vertically downwards.

Let t denote time, reckoned from the instant at which all the particles for which $h=0$ are in the axis of y; then

$$\theta=at+\frac{h}{\mathrm{R}}. \quad \ldots \quad (1.)$$

Then the following equations give the coordinates of a given particle at a given instant:

$$x=h+\mathrm{R}e^{-\frac{k}{\mathrm{R}}}\sin\theta; \quad \ldots \quad (2.)$$

$$y=k+\mathrm{R}e^{-\frac{k}{\mathrm{R}}}\cos\theta. \quad \ldots \quad (3.)$$

Let u and v denote the vertical and horizontal components of the velocity of the particle at the given instant; then

$$u=\frac{dx}{dt}=\quad a\mathrm{R}\,.\,e^{-\frac{k}{\mathrm{R}}}\cos\theta=\quad a(y-k); \quad \ldots \quad (4.)$$

$$v=\frac{dy}{dt}=-a\mathrm{R}\,.\,e^{-\frac{k}{\mathrm{R}}}\sin\theta=-a(x-h). \quad \ldots \quad (5.)$$

The well-known equation of continuity in a liquid in two dimensions is

$$\frac{du}{dx}+\frac{dv}{dy}=0; \quad \ldots \quad (6.)$$

and from equations (4.) and (5.) it appears that we have in the present case

$$\frac{du}{dx}+\frac{dv}{dy}=a\left(-\frac{dk}{dx}+\frac{dh}{dy}\right)=a\left(-\frac{dk}{dx}+\frac{\mathrm{R}d\theta}{dy}\right). \quad \ldots \quad (7.)$$

In the original formulæ, k and θ are the independent variables. When x and y are made the independent variables instead, we have, by well-known formulæ,

$$\left.\begin{aligned} \frac{dk}{dx}&=1\div\left\{\frac{dx}{dk}-\frac{dx}{d\theta}\cdot\frac{\frac{dy}{dk}}{\frac{dy}{d\theta}}\right\}=\frac{e^{-\frac{k}{\mathrm{R}}}\sin\theta}{1-e^{-\frac{2k}{\mathrm{R}}}} \\ \text{and}\qquad & \\ \frac{d\theta}{dy}&=1\div\left\{\frac{dy}{d\theta}-\frac{dy}{dk}\cdot\frac{\frac{dx}{d\theta}}{\frac{dx}{dk}}\right\}=\frac{e^{-\frac{k}{\mathrm{R}}}\sin\theta}{\mathrm{R}\left(1-e^{-\frac{2k}{\mathrm{R}}}\right)} \end{aligned}\right\}, \quad \ldots\ldots \quad (8.)$$

so that the equation of continuity (6.) is exactly verified.

(17.) Another mode of testing algebraically the fulfilment of the condition of continuity is the following. It is analogous to that employed by Mr. AIRY; but inasmuch as the disturbances in the present paper are regarded as considerable compared with the length of a wave, it takes into account quantities which, in Mr. AIRY's investigation, are treated as inappreciable.

Consider an indefinitely small rhomboidal particle, bounded by surfaces for which the values of h and k are respectively h, $h+dh$, k, $k+dk$. Then the area of that rhomboid is

$$\left(\frac{dx}{dh}\cdot\frac{dy}{dk}-\frac{dx}{dk}\cdot\frac{dy}{dh}\right)dh\,.\,dk;$$

and the condition of continuity is that this area shall be at all times the same; that is to say, that

$$\frac{d}{dt}\left(\frac{dx}{dh}\cdot\frac{dy}{dk}-\frac{dx}{dk}\cdot\frac{dy}{dh}\right)=0. \quad \ldots\ldots \quad (9.)$$

Upon performing the operations here indicated upon the values of the coordinates in equations (2.) and (3.), the value of the quantity in brackets is found to be

$$1-e^{-\frac{2k}{\mathrm{R}}}; \quad \ldots\ldots \quad (10.)$$

which is obviously independent of the time, and therefore fulfils the condition of continuity.

APPENDIX.

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On the Friction between a Wave and a Wave-shaped Solid.

Conceive that the trough between two consecutive crests of the trochoidal surface of a series of waves is occupied, for a breadth which may be denoted by z, by a solid body with a trochoidal surface, exactly fitting the wave-surface; that the solid body moves forward with a uniform velocity equal to that of the propagation of the waves, so as to continue always to fit the wave-surface, and that there is friction between the solid surface and the contiguous liquid particles, according to the law which experiment has shown to be at least approximately true, viz. varying as the surface of contact, and as the square of the velocity of sliding.

Conceive, further, that each particle of the liquid has that pressure applied to it which is required in order to keep its motion sensibly the same as if there were no friction; the solid body must of course be urged forwards by a pressure equal and opposite to the resultant of all the before-mentioned pressures.

The action, amongst the liquid particles, of pressures sufficient to overcome the friction, will disturb to a certain extent the motions of the liquid particles, and the figures of the surfaces of uniform pressure; but it will be assumed that those disturbances are small enough to be neglected, for the purposes of the present inquiry. The smallness of the pressures producing such disturbances, and consequently the smallness of those disturbances themselves, may be inferred from the fact, that the friction of a current of water over a surface of painted iron of a given area is equal to the weight of a layer of water covering the same area, and of a thickness which is only about ·0036 of the height due to the velocity of the current.

Those conditions having been assumed, let it now be proposed, *to find approximately the amount of resultant pressure required to overcome the friction between the wave and the wave-shaped solid.*

This problem is to be solved by finding the mechanical work expended in overcoming friction in an indefinitely small time dt, and dividing that work by the distance through which the solid moves in that time.

Taking, as before, as an independent variable the *phase* θ, being the angle which the tracing-arm $\overline{CB}=r$ (fig. 1) makes with a line pointing vertically downwards, the length of the elementary arc corresponding to an indefinitely small increment of phase $d\theta$ is

$$qd\theta,$$

where q is taken, for brevity's sake, to denote the normal AB.

The area of the corresponding element of the solid surface is

$$zqd\theta.$$

The velocity of sliding of the liquid particles over that elementary surface is

$$aq,$$

in which a, as before, denotes $\frac{d\theta}{dt}$, the angular velocity of the tracing-arm. Hence let ϱ denote the heaviness (or weight of unity of volume) of the liquid, and f its coefficient of friction when sliding over the given solid surface; the intensity of the friction per unit of area is

$$\frac{f\varrho a^2q^2}{2g}.$$

That friction has to be overcome, during the time dt, through the distance

$$aqdt=qd\theta.$$

Multiplying now together the elementary area, the intensity of the friction, and the distance through which it is overcome in the time dt, we find the following value for the work performed in that time in overcoming the friction at the given elementary surface,

$$zqd\theta\times\frac{f\varrho a^2q^2}{2g}\times qd\theta=\frac{f\varrho a^2}{2g}\cdot q^4zd\theta^2.$$

Now during the time dt, the solid advances through the distance

$$a\mathrm{R}dt=\mathrm{R}d\theta$$

(R, as before, being the radius of the rolling circle); and dividing the elementary portion of work expressed above by that distance, we find the following value for an elementary portion of the pressure required to overcome the friction,

$$d\mathrm{P}=\frac{f\varrho a^2}{2g}\cdot\frac{q^4z}{\mathrm{R}}\cdot d\theta.\quad\ldots\ldots\ldots\ldots\quad(1.)$$

The total pressure required to overcome the friction is found by integrating the preceding expression throughout an entire revolution, that is to say,

$$\mathrm{P}=\frac{f\varrho a^2z}{2g\mathrm{R}}\int_0^{2\pi}q^4d\theta.\quad\ldots\ldots\ldots\ldots\quad(2.)$$

To obtain this integral the following value of the square of the normal q or AB is to be substituted,

$$q^2=\mathrm{R}^2+r^2+2\mathrm{R}r.\cos\theta,$$

whence

$$\int_0^{2\pi}q^4d\theta=\mathrm{R}^4\int_0^{2\pi}\left(1+\frac{2r^2}{\mathrm{R}^2}+\frac{r^4}{\mathrm{R}^4}+4\left(1+\frac{r^2}{\mathrm{R}^2}\right)\frac{r}{\mathrm{R}}\cdot\cos\theta+4\,\frac{r^2}{\mathrm{R}^2}\cos^2\theta\right)d\theta=2\pi\mathrm{R}^4\left(1+4\frac{r^2}{\mathrm{R}^2}+\frac{r^4}{\mathrm{R}^4}\right),$$

and

$$\mathrm{P}=\frac{2\pi f\varrho a^2\mathrm{R}^3z}{2g}\cdot\left(1+4\frac{r^2}{\mathrm{R}^2}+\frac{r^4}{\mathrm{R}^4}\right).\quad\ldots\ldots\ldots\quad(3.)$$

The following modification of this expression is sometimes convenient:—

Let $\mathrm{V}=a\mathrm{R}$ denote the velocity of advance of the solid;

$\lambda=2\pi\mathrm{R}$, as before, its length, being the length of a wave;

$\sin\beta=\frac{r}{\mathrm{R}}$ the sine of the greatest angle made by a tangent to the trochoidal surface with the direction of advance; then

$$\mathrm{P}=\frac{f\varrho\mathrm{V}^2}{2g}\cdot\lambda z(1+4\sin^2\beta+\sin^4\beta\,{}^{*}).\quad\ldots\ldots\ldots\quad(4.)$$

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[*] This formula (neglecting $\sin^4\beta$ as unimportant in practice) has been used to calculate approximately the resistance of steam-vessels, and its results have been found to agree very closely with those of experiment, and have also been used since 1858 by Mr. JAMES R. NAPIER and the author with complete success in practice, to calculate beforehand the engine-power required to propel proposed vessels at given speeds. The formula has been found to answer approximately, even when the lines of the vessel are not trochoidal, by putting for β

It is to be observed that the resistance P, as determined by the preceding investigation, being deduced from the amount of work performed against friction, includes not only the longitudinal components of the direct action of friction on each element of the surface of the solid, but the longitudinal components of the excess of the hydrostatic pressure against the front of the solid above that against its rear, which is the indirect effect of friction. The only quantities neglected are those arising from the disturbances of the figures of the surfaces of equal pressure, which quantities are assumed to be unimportant, for reasons already stated. The consideration of such quantities would introduce terms into the resistance varying as the fourth and higher powers of the velocity.

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NOTE, *added in October* 1862.

The investigation of Mr. STOKES (Camb. Trans. vol. viii.) proceeds to the second degree of approximation in shallow water, and to the third degree in water indefinitely deep. In the latter case he arrives at the result, that the crests of the waves rise higher above the level of still water than the troughs sink below that level, by a height agreeing with that stated in art. 14 of this paper, and that the profile of the waves is *approximately* trochoidal.

Mr. STOKES also arrives at the conclusion, that, when the disturbance is considerable compared with the length of a wave, there is combined with the orbital motion of each particle a *translation* which diminishes rapidly as the depth increases. No such translation has been found amongst the results of the investigation in the present paper; and hence it would appear that Mr. STOKES's results and mine represent two different possible modes of wave-motion *.

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the mean of the values of the greatest angle of obliquity for a series of water-lines. The method of using the formula in practice, and a Table showing comparisons of its results with those of experiment, were communicated to the British Association in 1861, and printed in the Civil Engineer and Architect's Journal for October of that year, and in part also in the 'Mechanics' Magazine,' 'The Artisan,' and 'The Engineer.' The ordinary value of the coefficient of friction f appears to be about ·0036 for water gliding over painted iron. The quantity $\lambda z(1+4\sin^2\beta+\sin^4\beta)$ corresponds to what is called, in the paper referred to, the *augmented surface.*

* NOTE *added in June* 1863.

The difference between the cases considered by Mr. STOKES and by me is the following:—In Mr. STOKES's investigation, the *molecular rotation* is null; that is to say,

$$\tfrac{1}{2}\left(\frac{dv}{dx}-\frac{du}{dy}\right)=0\,;$$

while in my investigation it is constant in each layer, being the following function of k,

$$\tfrac{1}{2}\left(\frac{dv}{dx}-\frac{du}{dy}\right)=\frac{ar_0^2e^{-\frac{2k}{R}}}{R^2-r_0^2e^{-\frac{2k}{R}}}. \quad \ldots\ldots\ldots\ldots\ldots \quad (11.)$$

From this last equation it follows that

$$\frac{d}{dt}\left(\frac{dv}{dx}-\frac{du}{dy}\right)=0\,;$$

and therefore that the condition of continuity of pressure is verified.

The simplicity with which an exact result is obtained in the present paper, is entirely due to the following peculiarity:—Instead of taking for independent variables (besides the time) the *undisturbed* coordinates of a particle of liquid, there are taken two quantities, h and k, which are *functions* of those coordinates, of forms which are left indeterminate until the end of the investigation. h then proves to be identical with the undisturbed horizontal coordinate; but k proves to be a function of the undisturbed vertical coordinate for which there is no symbol in our present notation, being the root of the transcendental equation

$$k_0 - k - \frac{k_0^2}{2R} \cdot e^{-\frac{2k}{R}} = 0,$$

in which k_0 is the undisturbed vertical coordinate (see art. 13). Hence it is evident that, had k_0 instead of k been taken as the independent variable, the question of wave-motion considered in this paper could not have been solved except by a complex and tedious process of approximation.

VII. *Photo-chemical Researches.*—Part V. *On the Direct Measurement of the Chemical Action of Sunlight.*

By Robert Bunsen, *For. Mem. R.S., Professor of Chemistry in the University of Heidelberg, and* Henry E. Roscoe, *B.A., Ph.D., Professor of Chemistry in Owens College, Manchester.*

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The photo-chemical action exerted by direct sunlight and by diffuse daylight upon a horizontal portion of the earth's surface, varies with the time of year and with the latitude of the place, and constitutes an important link in the chain of physical relations which connects the organic with the inorganic world.

In former communications* made to the Royal Society we have endeavoured experimentally to determine the distribution of these chemical actions on the earth's surface, as varying with the time of day and year, and with the geographical position of the place, when the sky is perfectly unclouded. The methods of measurement there adopted are, unfortunately, not applicable to the determination of the variations in photo-chemical intensity when, as is most frequently the case, the transparency of the atmosphere is more or less obscured by clouds, mist, or rain. To enable us to estimate the alterations which occur in the amount of the chemically active rays falling on the earth's surface, we must, therefore, have recourse to a mode of measurement totally different from that employed in our former investigations.

In spite of the numerous futile attempts which have from time to time been made for the purpose of establishing a standard of measurement of the chemical action of light by means of photographic tints, it appeared to us not impossible in this way to gain the end we desired.

The proposals which have been made to register and measure the chemical action of light by help of photographic tints have been very numerous; amongst others we may mention those of Jordan, Hunt, Herschel†, and Claudet‡. All instruments based upon such principles must, however, afford totally unreliable results unless we know the conditions under which photographic surfaces of a constant degree of sensitiveness can be prepared, and unless the relations are determined which exist between the degree of tint produced and the time and the intensity of the light acting to produce such a tint.

Hence one of the first points of inquiry is, to determine whether the tints produced by the photographic action vary in shade in the direct ratios of the intensities of the incident light.

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* Philosophical Transactions, 1857, pp. 355, 381, 601; 1859, p. 879.

† Ibid. 1840, p. 46.

‡ London, Dublin, and Edinburgh Philosophical Magazine, Ser. 3. vol. xxxiii. p. 329.

For the purpose of measuring the degree of tint which the paper assumed, we employed a circular disc with black and white sectors whose relation to one another could be altered at pleasure. By allowing the black sector to cover $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, &c. of the surface of the disc, we obtained, on rotation, a disc having the tints $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, &c. The central portions of the disk were filled with the papers which had been tinted by the action of the light. It was soon found that very slight differences in the degree of shade could be detected so long as the tints are light-coloured, but that when deeper tints are employed the eye loses the power of estimating such differences. Experiments thus made proved that the shade produced upon photographic paper is not proportional to the intensity of the incident light; thus the intensities 5 and 1 were found to correspond to the shades 0·5 and 0·22. Hence we have altogether relinquished the idea of employing any mode of measurement founded upon a comparison of photographic papers of *different* shades.

We next had to examine whether *equal* shades of blackness produced by light of different intensities acting for different times can be used as the basis of the mode of measurement, under the supposition that equal shades of blackness always correspond to equal products of the intensity of the incident light into the times of insolation. The truth of this proposition, which was assumed some years ago by MALAGUTI[*], has recently been experimentally verified by HANKEL[†] within the slight variation of intensity from 1 to 2½. In order to prove the truth of this proposition for the wider limits needed in our measurements, it was necessary to determine the times of insolation very exactly to within small fractions of a second, and to be able to estimate with accuracy the points of equal shade. These ends were gained by employing the following arrangement.

The iron stand (Plate IX. fig. 1) carries the metal plate (A), which can be placed horizontally by three set-screws, and in which a straight slit, 15 millimetres broad and 190 millimetres long, is cut. Over this slit, which is shaded black in the drawing, is placed a very thin and elastic sheet of mica *b c d*, blackened at one end from *b* to *c*, and fastened at *d* to the curved drum (E) attached to the pendulum (F). When the pendulum is allowed to vibrate, the sheet of mica as it rolls on and off the curved drum (E) at each vibration, uncovers and again covers the slit, so that each point throughout the whole length of the slit is exposed for a different period. If we wish to use this instrument for the purpose of exposing a photographic surface to the action of the light for different times, the paper is gummed upon the white surface of the metallic slide (G, fig. 1); this is then covered by a metallic lid, which does not touch the paper, and the whole arrangement pushed into the dark groove *h*, placed directly under the slit, and protected from the entrance of light by a lappet of cloth, which hangs in front. The metallic lid is then withdrawn, the screw *k* turned, and thus the paper slightly pressed against the slit, so that no light can enter sideways between the paper and the thin metallic edges of the slit. By raising the lever $n\ m\ l$ at l, the pendulum is released from the catch at m, and, after completing a vibration, it is held fast by a lower catch at n. If it be required to double or to multiply the time of insolation, it is only necessary to repeat the vibration, once or several times, care being taken before each vibration to raise the rod of the pendulum so as to allow the end to fall into the upper catch. In order thus to set the pendulum in motion, and to stop it with certainty and ease, the lever is once for all balanced by a small weight at l, so that the arm $n\ m$ is but slightly heavier than the arm l.

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[*] Ann. de Chim. et de Phys. tom. lxii. p. 5.

[†] "Messungen über die Absorption der chemischen Strahlen des Sonnenlichts," Abhandl. d. Kön. Sächs. Gesellschaft der Wissenschaften. Leipsig, 1862. Bd. ix. p. 55.

The time which the sensitized paper is insolated at any point on the slit can be obtained when we know the duration and the amplitude of vibration of the pendulum. Let $\alpha\ \alpha$ (fig. 2) represent the end of the sheet of mica in the position in which it is found when the end of the pendulum is held in the upper catch m; let $\beta\ \beta$ (fig. 2) represent the position of the end of the sheet of mica when the pendulum is freely hanging and in equilibrium; and let $\gamma\ \gamma$ (fig. 2) be the position which the same end occupies during the vibration when the time t has elapsed since the start of the pendulum. If we call u the distance $\gamma\ \beta$, and τ the time during which the sensitized surface at $\gamma\gamma$ is insolated, the relation between u and τ is found as follows:—Let a represent the distance $\alpha\ \beta$, and let T represent the duration of a simple vibration, we have

$$u=a\,.\cos\left(\frac{t}{\mathrm{T}}\pi\right).$$

If t_1 signify the time at which the end of the sheet of mica returns to the position $\gamma\ \gamma$, we have

$$t_1=2\mathrm{T}-t,$$

and also

$$\mathrm{T}=t_1-\frac{\tau}{2},$$

and therefore

$$\tau=2(\mathrm{T}-t);$$

or, if we express τ in terms of u,

$$\tau=\frac{2\mathrm{T}}{\pi}\left(\pi-\cos^{-1}\frac{u}{a}\right),$$

or

$$u=-a\cos\left(\frac{\tau}{2\mathrm{T}}\pi\right). \qquad \text{(1.)}$$

In our instrument $a=105$ millimetres, and $\mathrm{T}=\frac{3}{4}$ second. From formula (1) a Table can easily be calculated, in which the time of insolation is given for every point on the slit; that is, the time for which, during one vibration of the pendulum, this point is uncovered by the blackened sheet of mica. For this purpose we have applied a millimetre scale, commencing at δ, to the slit, which in our instrument, reckoning from $\beta\ \beta$, extended 85 millims. in the direction of δ, and 105 millims. in the direction of ε (fig. 2). Column I. of the following Table contains the numbers of the millimetre scale, and column II. the corresponding times of insolation for one vibration expressed in seconds.

TABLE I.

I. Millims.	II. Seconds.	I. Millims.	II. Seconds.	I. Millims.	II. Seconds.	I. Millims.	II. Seconds.	I. Millims.	II. Seconds.	I. Millims.	II. Seconds.
0	1·200	32	1·003	64	0·846	96	0·700	128	0·549	160	0·369
1	1·193	33	0·998	65	0·841	97	0·695	129	0·544	161	0·363
2	1·186	34	0·993	66	0·837	98	0·691	130	0·539	162	0·357
3	1·179	35	0·988	67	0·832	99	0·686	131	0·534	163	0·350
4	1·172	36	0·983	68	0·828	100	0·682	132	0·528	164	0·343
5	1·165	37	0·977	69	0·823	101	0·677	133	0·523	165	0·336
6	1·158	38	0·972	70	0·819	102	0·672	134	0·518	166	0·329
7	1·151	39	0·967	71	0·814	103	0·668	135	0·513	167	0·321
8	1·144	40	0·962	72	0·809	104	0·663	136	0·508	168	0·314
9	1·137	41	0·957	73	0·805	105	0·659	137	0·502	169	0·309
10	1·131	42	0·952	74	0·800	106	0·654	138	0·497	170	0·300
11	1·125	43	0·947	75	0·796	107	0·650	139	0·492	171	0·291
12	1·119	44	0·942	76	0·791	108	0·645	140	0·487	172	0·283
13	1·113	45	0·937	77	0·786	109	0·640	141	0·482	173	0·274
14	1·106	46	0·932	78	0·782	110	0·635	142	0·476	174	0·266
15	1·100	47	0·927	79	0·777	111	0·631	143	0·470	175	0·257
16	1·094	48	0·922	80	0·773	112	0·626	144	0·465	176	0·249
17	1·087	49	0·917	81	0·768	113	0·621	145	0·459	177	0·240
18	1·081	50	0·912	82	0·764	114	0·617	146	0·453	178	0·229
19	1·076	51	0·907	83	0·759	115	0·612	147	0·448	179	0·219
20	1·070	52	0·903	84	0·755	116	0·607	148	0·442	180	0·208
21	1·064	53	0·898	85	0·750	117	0·603	149	0·436	181	0·198
22	1·058	54	0·893	86	0·745	118	0·598	150	0·431	182	0·187
23	1·053	55	0·888	87	0·741	119	0·593	151	0·425	183	0·176
24	1·047	56	0·884	88	0·736	120	0·588	152	0·419	184	0·161
25	1·041	57	0·879	89	0·732	121	0·583	153	0·413	185	0·146
26	1·036	58	0·874	90	0·727	122	0·578	154	0·407	186	0·131
27	1·030	59	0·870	91	0·723	123	0·573	155	0·401	187	0·116
28	1·025	60	0·865	92	0·718	124	0·568	156	0·394		
29	1·019	61	0·860	93	0·714	125	0·563	157	0·388		
30	1·014	62	0·856	94	0·709	126	0·558	158	0·382		
31	1·009	63	0·851	95	0·704	127	0·553	159	0·376		

The paper insolated in the slit whilst the pendulum is vibrating, exhibits throughout its whole length a regularly diminishing shade. In Table I. the time of insolation for each one of these different shades is to be found.

If we wish to determine which of these shades on the strip of paper corresponds to another tint produced by a separate insolation, we cannot make the comparison of the two shades either by daylight or by lamp- or candle-light, as the weakest light of this kind, which must be used in order to make such a comparison accurately, produces a sensible alteration in the tint of the paper during the process of reading. Still less does it appear advisable to fix the paper with hyposulphite of soda, or other material, before comparison, as such a treatment frequently causes irregular alteration of tone. We have overcome this difficulty by employing an intense soda-flame for illuminating the surfaces to be compared. This light is chemically so inactive that the rays proceeding from the flame can be concentrated by a convex lens and allowed to fall for several hours upon the sensitized paper without producing the least change of colour. This mode of illumination possesses another important advantage, inasmuch as small differences in colour, which render the comparison of shades by the eye with ordinary white light so difficult, entirely disappear when the monochromatic soda-flame is employed.

In order to avoid the necessity of carrying the instrument into the dark after each insolation, a millimetre scale, similar to the one upon the slit, is fastened upon a wooden board (fig. 3, *a*) covered with paper, and moveable in a groove across a fixed wooden stand. The strip of photographically tinted paper is then cut off from the slide G and gummed upon the board (*a*, fig. 3), so that it has the same position relative to the scale on the board as it had to the scale on the slit. A, fig. 4 represents a small square wooden block having a circular hole in the middle 5 to 6 millims. in diameter, the lower half being covered by the paper of which the degree of shade has to be determined. This block is pressed by means of a spring, as is seen in fig. 3, in a fixed position against the strip of paper. On throwing the image of the soda-flame C, by help of the convex lens D, upon the circular opening in the block, it is easy, by drawing the slide backwards and forwards, to determine the exact point at which the upper and lower halves of the circular hole appear equally dark. It is then only necessary to read off on the scale the number representing the time which the paper at that point has been insolated in order to determine the degree of shade which the paper in that time has attained. To ensure accuracy in the observations, it is necessary that the eye should always be placed in one and the same position; most advantageously in a direction nearly perpendicular to the surface of the strip of paper.

Having proved by experiment with the above instrument that we were able to measure the length of time required to effect equal degrees of shade within hundredths of a second, it became necessary to obtain a series of lights of accurately known degrees of intensity, and varying as much as possible from each other, in order to produce various shades on photographic paper. We employed for this purpose direct sunlight, and in the following manner we avoided any errors arising from the varying intensity of the light caused by the alteration of the sun's zenith-distance, and the changes of transparency in the atmosphere. In the roof of the darkened loft of the laboratory at Heidelberg, a brass plate was inserted, in which were bored round holes varying in size, and countersunk from the outside. The diameters of these holes were accurately measured with a micrometer, and through the holes sunlight was allowed to fall upon the sensitized paper, the surface of which was placed perpendicularly to the incident rays, and at such a distance from the brass plate that, seen from this position, the holes presented a smaller apparent diameter than the sun. The several intensities of the small pictures of the sun thrown upon the paper, which were so well defined that the larger solar spots were distinctly seen, are thus obtained quite independently of any alteration which change in the height of the sun, or variation in the transparency of the air may produce, and are directly proportional to the areas of the openings through which the light passes.

By allowing these pictures of the sun having the intensities I_0, I_1, I_2..., to act upon the paper for the times t_0, t_1, t_2, the products I_0t_0, I_1t_1, I_2t_2 were obtained for the various tints thus produced. A second piece of the same photographic paper was then exposed to diffuse daylight, of the intensity i, in the pendulum-instrument, and thus a tinted strip was produced containing a successive gradation of shade. For each particular shade the time of insolation τ_0, τ_1, τ_2 could be read off on the scale, and hence the products $i\tau_0$, $i\tau_1$, $i\tau_2$, &c. could be formed. The points upon the strip were next determined (by reading off with the soda-flame) which corresponded in shade to the several tints produced by the sun-pictures. On the supposition that equal products of intensity into time of insolation correspond to equal shades of blackness, we have the following equations:—

$$I_0 t_0 = i\tau_0,$$

$$I_1 t_1 = i\tau_1,$$

and

$$\frac{I_0}{I_1} = \frac{t_1 \tau_0}{t_0 \tau_1};$$

or, if we take a special case,

$$\tau_1 = t_1 I_1 \,.\, \text{const.} \quad \ldots \ldots \ldots \ldots \quad (2.)$$

The following experiments prove that this equation is satisfied.

Experiment I.

8th August 1860, at 12ʰ 30ᵐ P.M. Cloudless sky.

I. I.	II. t.	III. τ. Observed.	IV. τ. Calculated.	V. Difference.
1·00	20″	2″·55	2″·47	−0″·08
1·69	20	4·06	4·17	+0·11
2·78	20	7·01	6·86	−0·15
4·00	20	9·92	9·87	−0·05
5·44	20	13·46	13·43	−0·03
7·47	20	18·26	18·44	+0·18

Experiment II.

2nd August 1862, at 12ʰ 11ᵐ P.M. Cloudless sky.

I. I.	II. t.	III. τ. Observed.	IV. τ. Calculated.	V. Difference.
1·00	150″	4″·53	4″·75	+0″·22
1·71	150	8·71	8·15	−0·56
36·81	10	11·62	11·66	+0·04
45·04	10	14·08	14·28	+0·20

The first column contains the intensity of the sun-pictures employed to darken the paper; the second column, the length of time which the paper was thus exposed; the third, the times, measured with the pendulum-instrument, which the diffuse light of day took to produce the same degree of shade; and lastly, the fourth column gives the same times calculated from equation (2.), in which the constant is equal to 0·12339.

As the intensity of the light in these experiments varied from 1 to nearly 50 without a greater deviation from the calculated results occurring than that which may fairly be ascribed to the unavoidable experimental errors, we conclude

> That equal products of the intensity of the light into the time of insolation correspond, within very wide limits, to equal shades of darkness produced on chloride-of-silver paper of uniform sensitiveness.

Upon this important proposition a method may be founded for measuring the chemical action of light by means of simple observations. For, if we assume as the unit of photo-chemical action that intensity of light which produces in the unit of time a given degree of shade, we have only to determine on a strip of paper, blackened in the pendulum-apparatus, the point where the shade of the strip coincides with the given unalterable tint. The reciprocals of the times which correspond to these points of equal shade give the intensity of the light expressed in terms of the above unit.

It is clear that this method is available only under the suppositions,

(I.) That the phenomena of induction, accompanying the light of the intensities employed in the measurement of the total daylight, are of so short a duration that the variations thus produced fall within the limits of the necessary experimental errors;

(II.) That it is possible to prepare a photographic surface possessing a perfectly constant degree of sensitiveness;

(III.) That an unchangeable shade of blackness is obtainable which can be easily prepared, and can be exactly compared to a photographically tinted paper.

(I.)

In order to investigate the influence of photo-chemical induction upon the blackening of the chloride-of-silver paper, we have employed the following method. By means of the pendulum-apparatus we exposed strips of the same sensitive paper quickly one after the other to the light of a cloudless sky, insolating the first strip during n_0 vibrations of the pendulum, the second strip during n_1 vibrations, and determining on each of these strips the points of equal shade. The times of insolation, $t_0, t_1, t_2 \ldots$, corresponding to each of these points are obtained from Table I. If no appreciable induction occurs, the products n_0t_0, n_1t_1, n_2t_2, &c. must, in accordance with the former proposition, be equal. If, on the contrary, the chemical action continued for a certain length of time after each vibration, as is the case with photo-chemical induction, the products $n_0\ t_0$ must regularly alter with increasing n. The following experiments prove that this is not the case.

Experiment III.—Intensity No. 1.

n.	t.	nt.	Deviation from Mean.
4	1·024	4·096	−0·067
4	1·041	4·164	+0·001
4	1·063	4·252	+0·089
8	0·532	4·256	+0·093
8	0·525	4·200	+0·037
12	0·341	4·092	−0·071
12	0·340	4·080	−0·083

The same readings by a second observer.

n.	*t*.	*nt*.	Deviation from Mean.
4	1·048	4·192	−0·002
4	1·054	4·216	+0·026
4	1·054	4·216	+0·026
8	0·515	4·120	−0·070
8	0·520	4·160	−0·030
12	0·342	4·104	−0·086
12	0·360	4·320	+0·130

The same observations, giving the mean of seven readings.

n.	*t*.	*nt*.	Deviation from Mean.
4	1·028	4·112	−0·028
4	1·036	4·144	+0·004
4	1·036	4·144	+0·004
8	0·513	4·104	−0·036
8	0·501	4·008	−0·132
12	0·354	4·248	+0·108
12	0·352	4·224	+0·084

Intensity No. 2.

n.	*t*.	*nt*.	Deviation from Mean.
12	1·022	12·264	+0·505
12	0·982	11·784	+0·025
18	0·654	11·772	+0·013
18	0·655	11·790	+0·031
24	0·479	11·496	−0·263
24	0·477	11·448	−0·311

Intensity No. 3.

n.	*t*.	*nt*.	Deviation from Mean.
3	0·975	2·925	−0·011
3	0·975	2·925	−0·011
4	0·739	2·956	+0·020
4	0·735	2·940	+0·004
6	0·487	2·922	−0·014
6	0·492	2·952	+0·016

Intensity No. 4.

n.	*t*.	*nt*.	Deviation from Mean.
2	1·053	2·106	−0·004
2	1·057	2·114	+0·004
4	0·523	2·092	−0·018
4	0·532	2·128	+0·018

Intensity No. 5.

n.	*t.*	*nt.*	Deviation from Mean.
9	0·810	7·290	+0·074
9	0·793	7·137	−0·079
12	0·603	7·236	+0·020
12	0·600	7·200	−0·016

Intensity No. 6.

n.	*t.*	*nt.*	Deviation from Mean.
1	1·061	1·061	+0·032
1	1·050	1·050	+0·021
2	0·502	1·004	−0·025
2	0·502	1·004	−0·025

Intensity No. 7.

n.	*t.*	*nt.*	Deviation from Mean.
2	1·129	2·258	−0·008
6	0·379	2·274	+0·008

Considering the importance of this question, we deem it advisable to record another series of experiments, made for the purpose of investigating the influence of photo-chemical induction on the results of our measurements. They were made by allowing a circular disc of metal, having a sector cut out, to revolve for the same length of time, but at different rates, over two papers of the same degree of sensibility. The disc revolved at the rate of 30 revolutions per minute over one paper, and at the rate of 366 revolutions per minute over the other paper; the object being to determine whether the shade produced by the same intensity of the light and the same length of insolation remained constant, and was independent of the rate of rotation of the disk. Inasmuch as the results of these experiments, which were made on August 1, 1859, at 12^h noon, with the light of a cloudless sky, coincide with those of the former series, we do not think it necessary to enter into a full description of the experiments.

We may therefore conclude

> That photo-chemical induction does not exert any prejudical effect with intensities of light such as are employed in the measurements under consideration.

(II.)

The next question upon which the successful solution of our problem materially depends, concerns the possibility of preparing a photographic paper which shall always possess the same degree of sensitiveness. In describing this portion of our investigation, we have thought it necessary to enter more minutely into the experimental details than perhaps may be consistent with the reader's patience, because we felt that, unless a complete description of the experiments were given, it would be impossible to remove the most weighty objection which can be urged against photometric measurements based upon a comparison of photographic shades.

It appeared most rational to avoid all complicated photographic receipts for the preparation of our sensitive surface, and we therefore limited our investigation to the case of a simple paper covered with a film of pure chloride of silver.

For the purpose of comparing the sensitiveness of papers prepared in various ways and under varying conditions, we employed a strip of paper which had been photographically shaded in the pendulum-photometer, and afterwards fixed in a bath of hyposulphite of soda. The strip exhibited a gradually increasing shade from its white to its dark end, and was furnished with an arbitrary scale, so that the particular shade corresponding to a given number could easily at any time be read off by the soda-flame.

In order to test whether any given papers possessed an equal degree of sensitiveness, they were exposed for equal lengths of time to the same light, and then, by means of the arrangement represented in fig. 3, they were examined to see whether they exhibited the same degree of shade; that is, whether they corresponded to the same number on the scale adapted to the fixed strip.

We always employed a solution of chemically pure crystallized nitrate of silver as the silvering liquid. The pure chloride of sodium required for obtaining a film of chloride of silver was prepared by passing gaseous hydrochloric acid into a concentrated solution of common salt, washing the precipitated chloride of sodium with water, and heating it strongly in a platinum basin.

We have made the following series of experiments for the purpose of determining the influence exerted on the sensitiveness of the paper by the concentration of the solution of salt, the quantity of silver contained in the silvering solution, the quality of paper used, and the changes of atmospheric temperature and moisture.

1. *Silvering the Paper.*

Pieces of the same perfectly homogeneous salted paper, of a quality such as is usually employed by photographers, prepared according to a method hereafter described, were allowed to lie for two minutes upon the surface of silver solutions of different strengths, as follows:—

Paper *a* on a solution containing 12 $AgNO_6$ to 100 of water.
" *b* " " 10 " "
" *c* " " 8 " "
" *d* " " 6 " "

The papers were then air-dried in the dark, exposed for one and the same time to the daylight, and their shade determined. The following numbers were obtained: the readings A were made by one independent observer, the readings B by another; and each number is the mean of several readings. Equality in the numbers denotes equality in the shade, that is, equality in the sensitiveness of the paper.

Experiment IV.

Intensity No. 1.

Parts of nitrate of silver to 100 of water.	Observations.	
	A.	B.
12	128·6	129·7
10	128·7	127·0
8	128·7	128·0
6	129·7	130·0

Intensity No. 2.

Parts of nitrate of silver to 100 of water.	Observations.	
	A.	B.
12	125·5	125·0
10	125·5	125·5
8	125·4	124·2
6	161·5	160·2

Intensity No. 3.

Parts of nitrate of silver to 100 of water.	Observations.	
	A.	B.
12	110·0	110·0
10	109·5	109·3
8	109·6	109·3
6	119·0	120·0

Intensity No. 4.

Parts of nitrate of silver to 100 of water.	Observations.	
	A.	B.
12	90·6	90·0
10	88·0	88·3
8	90·7	89·4
6	89·6	89·0

From these numbers it is seen that the sensitiveness of the paper remains unaltered when the concentration of the silver solution varies from 8 to 10 or 12 parts of nitrate of silver to 100 of water, but that when a solution containing 6 parts of this salt to 100 of water is employed, the point at which alteration occurs is approached.

The influence of the concentration of the silver solution having thus been determined, it was next necessary to examine the dependence of the sensitiveness of the paper upon the length of time during which it remained upon the silver solution. For this purpose pieces of the same homogeneous salted paper were laid for various times upon the surface of a silver solution containing 12 parts of nitrate of silver to 100 of water:—

Paper *a* silvered for $\frac{1}{4}$ of a minute.
„ *b* „ 1 minute.
„ *c* „ 8 minutes.

On determining the shades of paper thus prepared and insolated for an equal time, the following results were obtained:—

Experiment V.

Intensity No. 1.

Duration of the silvering.	Observations.	
	A.	B.
0′ 15″	140·6	140·5
1 0	139·0	140·0
8 0	139·6	139·0

Intensity No. 2.

Duration of the silvering.	Observations.	
	A.	B.
0′ 15″	91·0	91·0
1 0	91·5	90·5
8 0	91·5	92·0

Experiment V. (*continued*).

Intensity No. 3.

Duration of the silvering	Observations.
	A.
′ ″	
0 15	45·9
1 0	47·1
8 0	45·0

Intensity No. 4.

Duration of the silvering.	Observations.
	A.
′ ″	
0 15	89·9
1 0	90·0
8 0	89·2

Hence we may conclude that the time during which the paper lies on the surface of the silver-bath can vary from 15 seconds to 8 minutes without any difference in the sensitiveness of the paper being noticeable.

If the duration of the silvering be shortened below the 15 seconds, a film of chloride of silver is obtained which is much less sensitive than that obtained by a longer silvering.

It appeared to be of special importance to determine by experiment how long a silver-bath can be used without the quantity of nitrate of silver being reduced below 8 parts to 100 of water, at which point the sensitiveness of the paper may begin to alter. We found that when a paper was silvered, rather more nitrate of silver than water was removed from the silver-bath; that, however, two-thirds of a solution containing 12 of nitrate of silver to 100 of water may be used up before the quantity of silver salt sinks from 12 to 8. One square decimetre of paper does not absorb more than 0·01 grm. of nitrate of silver from a solution of the above strength.

Not only a diminution in the silver occurs on using the silver-bath, but likewise a formation of nitrate of soda takes place, which might, by its presence, affect the sensitiveness of the paper. We have therefore compared a freshly prepared silver-bath with one which had been long in use; and the results of this examination are seen in the following Tables, and show that the occurrence of the nitrate of soda produces no effect upon the sensitiveness of the paper.

Experiment VI.

Silver solution.	Intensity No. 1.		Intensity No. 2.
	A.	B.	A.
Long used	130.2	130·8	73·0
Freshly prepared ...	130·0	131·5	73·4
Freshly prepared ...	130·8	130·9	73·2
Long used	130·0	130·3	74·0

The next series of observations show the length of time which the silvered paper may be preserved in the dark before insolation without alteration of its sensitiveness. The paper employed was silvered in a solution containing 12 parts of nitrate of silver to 100 of water.

Experiment VII.

Kept in the dark for	Intensity No. 1.		Kept in the dark for	Intensity No. 2.	Kept in the dark for	Intensity No. 3.
	A.	B.		A.		A.
1 hour......	100·0	101·0	5 hours ...	99·3	5 hours ...	111·8
5 hours ...	98·9	99·0	6 hours ...	98·6	6 hours ...	109·8
9 hours ...	100·0	101·0	7 hours ...	98·8	7 hours ...	109·4
			8 hours ...	98·4	8 hours ...	109·8

Kept in the dark for	Intensity No. 4.		Kept in the dark for	Intensity No. 5.
	A.	B.		A.
4 hours ...	99·8	99·7	4 hours ...	99·2
15 hours ...	100·8	101·0	15 hours ...	100·0

2. *Salting the Paper.*

If the paper be allowed to float upon the surface of the solution of chloride of sodium as it is allowed to do upon the nitrate-of-silver solution, a paper is obtained which, after drying and silvering as already described, exhibits a sensitive surface of great irregularity, as is seen from the following experiments. In these, different parts of the same sheet of paper lying 1 decimetre from each other were examined, by two observers, A and B. The readings differ widely among themselves, a circumstance which could not occur if the sensitiveness of the film had been equal throughout the sheet.

Experiment VIII.

Part of Paper.	Intensity No. 1.		Intensity No. 2.		Intensity No. 3.		Intensity No. 4.	
	Solution containing 2 per cent. NaCl.		Solution containing 4 per cent. NaCl.		Solution containing 7 per cent. NaCl.		Solution containing 8 per cent. NaCl.	
	A.	B.	A.	B.	A.	B.	A.	B.
Upper part of sheet ...	100·0	100·0	96·3	......	114·4	115·0	94·0	93·0
Middle part of sheet ...	116·5	117·5	100·0	......	122·6	122·5	99·0	99·6
Lower part of sheet ...	......	......	122·2	......	141·0	140·8	109·6	109·6

From the above experiments it appears that the most sensitive portions of the sheet of paper were those which were lowest when the sheet was hung to dry vertically—that is, those parts by which the salt solution had been most thoroughly imbibed. We therefore endeavoured to obtain a homogeneously sensitive paper by immersing the paper in the solution of salt, and allowing it to soak for five minutes. The salt solution employed contained 4 per cent. of chloride of sodium, the silver solution contained 12 parts of nitrate of silver to 100 parts of water. The following experiments give the results obtained by this mode of treatment:—

Experiment IX.

Single Sheet of Paper.	Intensity No. 1.		Intensity No. 2.		Intensity No. 3.		Intensity No. 4.	
	A.	B.	A.	B.	A.	B.	A.	B.
Upper part.................	96·9	98·0	121·6	120·2	72·0	72·0	87·5	87·8
Middle part	97·0	95·2	121·6	120·0	72·6	72·0	87·0	87·8
Lower part.................	97·5	98·0	122·5	......	......	......	88·0	87·5

Three Sheets of Paper.	Intensity No. 1.		Intensity No. 2.	Intensity No. 3.	
	A.	B.	A.	A.	B.
No. 1. Upper part.........	83·6	83·8	69·7	87·0	86·5
No. 2. Middle part	84·2	84·0	69·7	87·3	87·8
No. 3. Lower part	85·5	85·6	69·2	88·0	88·0

These observations show that, in order to obtain a homogeneous sensitive film of chloride of silver, the paper must not be *laid upon* but *immersed in* the chloride-of-sodium solution.

From the following experiments we learn the influence which the concentration of the salt solution exerts upon the sensitiveness of the paper. The papers salted in different solutions were all silvered in a bath containing 12 parts of nitrate of silver to 100 parts of water.

Experiment X.

Intensity No. 1.			Intensity No. 2.			Intensity No. 3.		
Na Cl to 100 of water.	A.	B.	Na Cl to 100 of water.	A.	B.	Na Cl to 100 of water.	A.	B.
1	62·6	60·4	4	93·2	93·0	6	67·6	68·6
2	95·7	94·7	5	92·9	93·3	8	83·4	83·7
3	132·6	129·6	6	111·5	113·2	10	94·7	93·7
4	167·0	168·0				12	97·0	95·0

Intensity No. 4.		Intensity No. 5.		
Na Cl to 100 of water.	A.	Na Cl to 100 of water.	A.	B.
13	154·5	12	69·0	70·0
14·5	159·6	15	75·0	78·5
16	161·6	18	95·0	95·0
		21	94·5	95·0

An examination of the above Table shows

That the sensitiveness of the paper increases rapidly with increasing strength of the chloride-of-sodium solution, and that, as far as the observations extend, no

limit exists beyond which an increase or a diminution of the percentage of salt in solution ceases to affect the sensitiveness of the film.

In order to obtain constant results, it is therefore necessary to employ a solution of chloride of sodium of unvarying strength. We have decided upon using a solution which contains 3 per cent. of chloride of sodium. Such a solution is especially convenient, because the paper dipped into it removes salt and water almost exactly in the proportions in which they are contained in solution; thus 225 cubic centimetres of a 3 per cent. salt-bath was altered from 2·948 per cent. NaCl to 2·935 per cent. by impregnating 0·72 square metre of paper. In another experiment the strength of 10 litres of salt solution containing 2·97 per cent. NaCl, was only increased to 3·08 per cent. by impregnating 4½ square metres of paper. It is therefore possible to impregnate 5 square metres of paper with a solution containing 60 grammes of chloride of sodium, without any danger of reducing the strength of the salt solution below the point at which differences begin to appear.

3. *Influence of the description of Paper employed.*

In the examination of the effect of change of quality in the paper used, we have confined our experiments to three kinds of paper, differing extremely in thickness, from the thickest to the thinnest commonly in use among photographers.

One square decimetre of the first of these, called paper *a*, weighed 0·354 grm.; the same area of the second, called *b*, weighed 0·732 grm.; and the same quantity of the third sort, called *c*, weighed 0·876 grm. From the first series of experiments made with these papers, we thought that the varying thickness of the paper was of the greatest moment in determining the sensibility of the film; thus, for instance, the three sorts of papers, sensitized in exactly the same way, gave the following unequal readings:—

Papers.	Intensity No. 1.
a	90·0
b	75·3
c	72·5

We soon convinced ourselves, however, that this want of agreement was not caused by any difference in the sensitiveness of the film, but solely by the difference in the partial opacity of the papers. If the transparency was got rid of by placing a piece of thick white paper behind the tinted papers whilst reading off, the following numbers were obtained instead of the foregoing:—

Papers.	Intensity No. 1.
a	73·6
b	73·6
c	72·0

The following series of readings show still more clearly that no difference in the shade of these papers can be observed when a white background is placed behind the tinted papers:—

Experiment XI.

Paper.	Na Cl to 100 of water.	Ag N O^6 to 100 of water.	Intensity No. 1.		Intensity No. 2.
			A.	B.	
a	2	12	73·0	71·0	109·5
b	2	12	73·0	73·0	112·0
c	2	12	69·5	73·6	109·5

Paper.	Na Cl to 100 of water.	Ag N O^6 to 100 of water.	Intensity No. 1.	Intensity No. 2.	Intensity No. 3.
a	16	12	89·3	120·0	142·4
b	16	8	91·0	120·0	142·9
c	16	10	90·0	120·0	141·9

Hence we may conclude

That variation in the thickness of white paper, such as is usually employed for photographic purposes, is without influence upon the sensitiveness of the film of chloride of silver.

4. *Influence of the Changes of Atmospheric Temperature and Moisture.*

In order to become acquainted with the influence exerted by change of temperature and moisture upon the sensitiveness of the paper, we gummed portions of the same sheet of sensitized air-dried paper upon two tin boxes, filled with water of different temperatures, and exposed these two papers for the same length of time to the same intensity of light. No greater differences in shade were observed in the papers thus tinted than such as arose from the unavoidable experimental errors; this is seen from the following numbers:—

Experiment XII.

Intensity No. 1.								Intensity No. 2.			
Temperature.								Temperature.			
+3° C.		+50° C.		+3° C.		+50° C.		+4° C.		+30° C.	
A.	B.	A.	B.	A.	B.	A.	B.	A.	B.	A.	B.
89·2	88·3	88·0	89·0	88·2	88·5	88·0	88·5	81·6	80·6	81·6	80·6

Hence there can be no doubt

That the ordinary changes of atmospheric temperature and moisture do not affect the sensitiveness of the paper.

From the results of the foregoing series of experiments it is easy to select the conditions under which a paper of constant sensibility can be obtained.

Preparation of the Standard Paper.

We give the following receipt for preparing a constant standard paper, by help of which comparative measurements can be carried out at any time or at any place:—

300 grammes of pure chloride of sodium are dissolved in 10 litres of water, and the solution poured into a shallow zinc vessel large enough to float the paper. The sheets of paper, about 0·3 metre in area, are one by one laid flat in the solution, and the vessel slightly agitated in order to remove adhering bubbles of air. When the sheet has been immersed for 5 minutes, it is taken out of the solution and dried by hanging in the air in a vertical position. The solution thus prepared will serve to salt seventy sheets of paper, each having an area of 0·3 metre. This salted paper can be preserved for months without being in any way injured.

The silvering is managed, with the precautions ordinarily in use amongst photographers, by placing the paper, after each sheet has been cut into four pieces, upon the surface of a solution containing 120 grammes of nitrate of silver to 1 litre of water, placed in a flat glass dish. Each piece of paper is allowed to lie upon the surface of the silver-bath for 2 minutes. This litre of silver solution serves to sensitize 500 such pieces of paper, after which its volume will be reduced to one-half. The standard paper thus prepared can be preserved in the dark after drying in the air for from 15 to 24 hours without undergoing any appreciable change in its sensitiveness.

It is scarcely necessary to add that, by adhering to the proportions here given, smaller quantities of the standard paper may likewise be prepared.

We next give a series of experiments to show that such a standard paper, prepared at different times and under various conditions, can always be obtained homogeneous throughout its surface, and of a perfectly constant sensitiveness.

Experiment XIII.

Size of paper 0·3 square metre.	Intensity No. 1.		Intensity No. 2.	
	A.	B.	A.	B.
Upper part	115·0	115·0	129·4	126·6
Middle part	115·4	116·4	129·0	127·2
Lower part	116·0	115·4	129·5	128·0

The standard paper used for the above experiment consisted of a sheet 0·3 of a square metre in size; and the portions which were selected for trial lay at a distance of about 25 centimetres from each other. In order to show that no irregularities occurred in a large number of sheets prepared in the same salt solution, we add a comparison of the sensitiveness of eighteen sheets of standard paper, each of an area of 0·075 square metre, salted in a solution containing 2·95 per cent. of chloride of sodium.

Paper.	Intensity No. 1.		Paper.	Intensity No. 2.	
	A.	B.		A.	B.
Middle part of Sheet No. 1 ...	98·0	98·3	Upper part of Sheet No. 1 ...	88·0	88·0
Middle part of Sheet No. 9 ...	98·3	98·9	Middle part of Sheet No. 8 ...	89·5	89·6
Middle part of Sheet No. 18...	98·0	99·0	Lower part of Sheet No. 17 ...	89·5	89·5

Paper.	Intensity No. 3.		Paper.	Intensity No. 4.	
	A.	B.		A.	B.
Upper part of Sheet No. 2 ...	69·0	69·5	Lower part of Sheet No. 4 ...	70·5	70·0
Middle part of Sheet No. 4 ...	69·5	70·8	Lower part of Sheet No. 12 ...	70·2	70·0
Middle part of Sheet No. 11...	69·3	70·0	Upper part of Sheet No. 16 ...	70·0	71·0
			Upper part of Sheet No. 18 ...	70·5	70·5

Paper.	Intensity No. 5.		Paper.	Intensity No. 6.	
	A.	B.		A.	B.
Upper part of Sheet No. 4 ...	99·5	99·9	Lower part of Sheet No. 8 ...	63·9	62·5
Middle part of Sheet No. 6 ...	100·5	100·0	Middle part of Sheet No. 16...	62·5	62·5
Lower part of Sheet No. 9 ...	101·0	101·0	Upper part of Sheet No. 17 ...	61·8	61·5

The following Tables contain the results obtained from three salt solutions of the same approximate strength but prepared at different times, the exact composition of each being determined by silver analysis. In these solutions three sheets of paper, each 0·075 sq. metre in area, were prepared, the sheets being afterwards sensitized as described. Here, likewise, the same uniformity is strikingly seen.

Paper.	Na Cl to 100 parts of water.	Intensity No. 1.	Intensity No. 2.
Upper part of Sheet No. 2 ...	3·026	87·0	75·4
Middle part of Sheet No. 3 ...	2·950	86·3	74·4
Middle part of Sheet No. 2 ...	3·028	86·0	74·9
Lower part of Sheet No. 2 ...	3·000	85·9	74·4

Paper.	Na Cl to 100 parts of water.	Intensity No. 1.		Intensity No. 2.	
		A.	B.	A.	B.
Upper part of Sheet No. 2 ...	2·950	77·5	76·5	89·0	87·0
Middle part of Sheet No. 1 ...	3·026	77·0	77·5	89·0	88·0
Middle part of Sheet No. 3 ...	3·000	78·2	78·0	90·1	93·0
Lower part of Sheet No. 2 ...	3·026	78·9	78·5	89·9	90·9

Paper.	Na Cl to 100 parts of water.	Intensity No. 1.
Upper part of Sheet No. 3 ...	2·950	86·2
Middle part of Sheet No. 3 ...	3·028	87·0
Middle part of Sheet No. 2 ...	3·000	86·8
Lower part of Sheet No. 2 ...	3·028	87·5

Paper.	Na Cl to 100 parts of water.	Intensity No. 1.		Intensity No. 2.	
		A.	B.	A.	B.
Upper part of Sheet No. 2 ...	2·950	70·2	70·0	101·3	101·5
Lower part of Sheet No. 2 ...	3·026	70·6	69·3	101·5	101·7
Middle part of Sheet No. 1 ...	3·026	70·0	69·5	100·9	100·9
Middle part of Sheet No. 3 ...	3·000	70·0	70·4	101·0	100·0

We may, therefore, conclude that the standard photographic paper prepared according to the above receipt possesses a degree of sensitiveness sufficiently constant for all the purposes of our measurements.

(III.)

For the purpose of obtaining a unit of measurement, we need a perfectly fixed and unalterable shade of colour, which can be at any time easily prepared; this is made by mixing oxide of zinc and lamp-black in certain proportions, and grinding them together until no change of tint is produced by further rubbing. The oxide of zinc is prepared in the wet way chemically pure, and then ignited at a low red heat for 5 minutes in a closed platinum crucible. The lamp-black is prepared chemically pure by allowing a turpentine lamp to burn under a large porcelain dish filled with cold water, and igniting the soot, which is deposited on the outside of the dish, for 5 minutes in a covered platinum crucible. In this way a fine impalpable powder is obtained which burns without leaving the slightest trace of ash.

Experiment showed that the tint obtained by mixing 1 part of the lamp-black with 1000 parts of oxide of zinc is that about which the eye can distinguish between very slight alterations of shade, but that these slight alterations cannot be observed when the tint is either darker or lighter. Hence we have adopted the mixture containing 1000 parts of oxide of zinc to 1 part of lamp-black as the standard tint. In order that the colour may adhere to the paper, it is mixed with water containing 1 per cent. of isinglass. During the preparation of the colour it was noticed that the shade of the mixture became gradually darker when it was well ground on a glass plate and afterwards dried, but that after a time a point was reached at which no alteration in shade was produced by repeating the operations of grinding and drying. The following observations of points of equal shade made upon a fixed strip illustrate this gradual change.

	First Preparation.		
	A.	B.	Mean.
After the first grinding	66·0	66·2	66·1
After the second grinding ...	72·9	72·5	72·7
After the third grinding......	72·4	72·6	72·5
After the fourth grinding ...	72·8	73·0	72·9

In order to obtain a perfectly constant colour, the mixture must be well ground on a glass plate with water for an hour, then dried in the water-bath, and this operation repeated until no difference in shade can be detected between the mixture in various stages on examination in the usual manner upon a fixed strip.

Four separate mixtures, made at different times and with differently prepared constituents, gave the observations contained in the following Table, in which the constant nature of the tint produced is seen.

	1st Reading.		2nd Reading.		Mean.
	A.	B.	A.	B.	
1. First preparation	72·0	71·8	71·5	72·5	71·95
2. Second preparation ...	72·5	72·0	72·0	72·0	72·12
3. Third preparation ...	72·9	73·0	......	......	72·95
4. Fourth preparation ...	72·0	72·0	72·2	73·2	72·35

We may therefore consider it proved

> That the colour used as the measure of the standard tint can at any time be prepared of a constant and unalterable shade.

Having, in the foregoing, described the mode in which a standard photographic paper of constant sensitiveness, and a standard tint of unvarying shade can be prepared, we need only apply the proposition that equal products of the intensities into the times of insolation produce equal shades of blackness, in order to found a method of measurement and comparison of the chemical action of the total daylight. As the *unit of measurement*, we propose to adopt

> That intensity of the light which in one second of time produces the standard tint of blackness upon the standard paper.

When the standard paper is insolated in the pendulum-apparatus, a strip is obtained which is tinted with every gradation of shade from dark to white. If the point on this strip which coincides in shade with a piece of paper covered with the standard tint be determined by means of the arrangement (fig. 3), we have only to look for the corresponding reading of the millimetre scale in Table I. to obtain the time of insolation t in seconds which was necessary in order to produce this shade. If this time of insolation were found to be one second, the intensity of the light then acting would, according to definition, be $I=1$. For any other time of insolation, t for example, the intensity of the chemical rays would be $\frac{1}{t}$. The following Table (II.) gives in column II. the intensities, for one vibration of the pendulum, corresponding to the points of equal shade of the standard paper and standard tint, as read off on the millimetre scale in column I. The intensities corresponding to *n* vibrations of the pendulum are obtained by dividing the numbers in column II. by *n*.

Table II.

I. Millims.	II. Intensity.	I. Millims.	II. Intensity.	I. Millims.	II. Intensity.	I. Millims.	II. Intensity.	I. Millims.	II. Intensity.	I. Millims.	II. Intensity.
0	0·834	32	0·997	64	1·183	96	1·429	128	1·824	160	2·710
1	0·839	33	1·002	65	1·190	97	1·439	129	1·840	161	2·763
2	0·844	34	1·007	66	1·197	98	1·448	130	1·856	162	2·816
3	0·849	35	1·012	67	1·203	99	1·458	131	1·874	163	2·869
4	0·853	36	1·018	68	1·209	100	1·467	132	1·892	164	2·923
5	0·858	37	1·023	69	1·215	101	1·477	133	1·911	165	2·977
6	0·864	38	1·029	70	1·221	102	1·487	134	1·930	166	3·048
7	0·869	39	1·034	71	1·228	103	1·497	135	1·949	167	3·119
8	0·874	40	1·040	72	1·235	104	1·507	136	1·969	168	3·190
9	0·879	41	1·046	73	1·242	105	1·517	137	1·990	169	3·262
10	0·884	42	1·051	74	1·249	106	1·528	138	2·011	170	3·334
11	0·889	43	1·057	75	1·256	107	1·539	139	2·032	171	3·437
12	0·894	44	1·062	76	1·263	108	1·551	140	2·053	172	3·534
13	0·899	45	1·068	77	1·270	109	1·563	141	2·078	173	3·650
14	0·904	46	1·074	78	1·277	110	1·575	142	2·103	174	3·759
15	0·909	47	1·079	79	1·285	111	1·586	143	2·128	175	3·891
16	0·914	48	1·085	80	1·293	112	1·598	144	2·153	176	4·016
17	0·919	49	1·090	81	1·301	113	1·610	145	2·179	177	4·167
18	0·924	50	1·096	82	1·309	114	1·622	146	2·207	178	4·367
19	0·929	51	1·102	83	1·317	115	1·634	147	2·235	179	4·566
20	0·935	52	1·108	84	1·325	116	1·647	148	2·263	180	4·807
21	0·940	53	1·114	85	1·333	117	1·660	149	2·291	181	5·051
22	0·945	54	1·120	86	1·342	118	1·673	150	2·320	182	5·348
23	0·950	55	1·127	87	1·350	119	1·686	151	2·354	183	5·682
24	0·955	56	1·133	88	1·359	120	1·700	152	2·389	184	6·212
25	0·961	57	1·139	89	1·367	121	1·715	153	2·424	185	6·848
26	0·966	58	1·145	90	1·376	122	1·730	154	2·459	186	7·633
27	0·971	59	1·151	91	1·385	123	1·745	155	2·494	187	8·620
28	0·976	60	1·156	92	1·394	124	1·760	156	2·537		
29	0·981	61	1·163	93	1·402	125	1·776	157	2·580		
30	0·986	62	1·170	94	1·411	126	1·792	158	2·623		
31	0·992	63	1·176	95	1·420	127	1·808	159	2·666		

The observations are carried out in the manner fully described in the commencement of the present communication. In order to make several observations quickly after each other, the screw (*k*, fig. 1) is loosened, and, after each observation, the slide (G) drawn out rather more than the width of the slit. The readings are also made in the way described, with the arrangement fig. 3, half the hole in the block (fig. 4) being occupied with paper covered with a thick layer of the standard tint. Care must be taken that the white background upon which the strip is placed is free from spots or dirt, which by appearing through the paper may alter the readings. The paper on which the standard tint is painted must not be too thin, and must be thoroughly air-dried before use. Each comparison of shade is made five or six separate times, the scale being covered up during every reading, and the mean of these observations recorded.

As an example of such measurements, we append several observations representing the chemical action exerted upon a horizontal surface by the whole sunlight and diffuse daylight during the various hours of the day. These observations, which are contained in Table III., were carried on in Manchester, on the roof of the Laboratory of Owens College, and were made on days in which the sun sometimes shone, and sometimes was obscured by clouds. The observations are represented by the curves (fig. 5), and the maxima and minima correspond exactly with the appearance and disappearance of the sun. From these few observations an idea may be formed of the vast differences exhibited by the chemical activity of sun- and day-light about the periods of the longest and the shortest days.

Table III.

Wednesday, December 18, 1861.				Thursday, December 19, 1861.				Wednesday, July 30, 1862.			
t.	*i.*	*n.*	$\frac{i}{n}$=I.	*t.*	*i.*	*n.*	$\frac{i}{n}$=I.	*t.*	*i.*	*n.*	$\frac{i}{n}$=I.
h m				h m				h m			
10 6 A.M.	1·05	124	0·00847	9 39 A.M.	1·79	120	0·0149	7 0 A.M.	0·88	60	0·0147
10 16	2·49	170	0·0147	9 49	2·10	150	0·0140	7 20	0·85	32	0·0266
10 26	1·60	100	0·0160	10 1	1·89	120	0·0157	7 35	1·07	25	0·0428
10 36	1·49	90	0·0166	10 21	1·93	100	0·0193	7 50	0·89	16	0·0556
10 47	1·47	100	0·0147	10 31	1·72	80	0·0215	8 0	0·83	10	0·0830
10 56	1·34	80	0·0168	10 41	2·05	80	0·0256	8 35	0·92	12	0·0767
11 6	1·47	80	0·0184	10 51	1·66	80	0·0208	9 0	1·33	15	0·0887
11 16	1·59	100	0·0159	11 1	1·93	90	0·0215	9 5	1·20	10	0·120
11 26	1·41	80	0·0176	11 11	1·91	80	0·0239	9 30	1·22	7	0·174
11 36	1·39	75	0·0185	11 21	1·91	80	0·0239	10 10	1·12	5	0·224
11 46	1·25	80	0·0156	11 31	1·91	80	0·0239	10 20	0·91	5	0·182
11 56 A.M.	1·46	66	0·0221	11 41	1·73	80	0·0216	10 30	0·83	10	0·0830
12 6 P.M.	1·52	60	0·0253	11 51 A.M.	1·69	61	0·0277	11 0	0·86	11	0·0782
12 16	1·42	50	0 0284	12 1 P.M.	1·66	60	0·0277	11 30	0·86	4	0·215
12 26	1·42	45	0·0316	12 11	1·54	50	0·0308	12 0	0·86	3	0·287
12 36	1 20	40	0·0300	12 21	1·49	50	0·0298	12 30 P.M.	0·86	3	0·287
12 46	0·92	80	0·0115	12 41	1·10	50	0·0519	1 30	0·88	6	0·147
12 57	1·02	120	0·0085	12 51	1·37	65	0·0211	2 0	1·11	8	0·139
1 6	1·19	90	0·0132	1 1	1·02	50	0·0204	2 30	1·33	13	0·110
1 16	1·38	75	0·0184	1 11	1·12	65	0·0172	3 0	1·22	9	0·136
1 26	1·22	65	0·0188	1 21	1·56	90	0·0173	4 0	1·27	15	0·0846
1 36	1·05	50	0·0210	1 36	1·69	86	0·0197	4 35	1·22	18	0·0678
1 47	0·84	60	0·0140	1 46	1·75	100	0·0175	5 0	1·49	20	0·0745
1 56	1·26	100	0·0126	1 56	1·54	100	0·0154	5 30	1·34	25	0·0536
2 10	1·36	150	0·00906	2 6	1·22	100	0·0122	6 0	1·24	40	0·0310
2 22	1·34	150	0·00893	2 16	1·40	120	0·0117				
2 32	1·41	160	0·00881	2 27	1·59	160	0·00994				
2 42	1·55	200	0·00775	2 45	1·50	180	0·00833				
2 52	1·36	225	0·00529	2 53	1·25	160	0·00781				
3 5	1·56	400	0·00390	3 8	1·45	250	0·00580				
3 25 P.M.	1·53	450	0·00340	3 21 P.M.	1·72	500	0·00344				

At the close of this communication we may remark that, by help of the pendulum-apparatus described, we have constructed a portable instrument by which a large number of measurements can be made on a few square inches of paper. We reserve the description of this instrument for a future occasion.

VIII. *On the Immunity enjoyed by the Stomach from being digested by its own Secretion during Life.* *By* F. W. PAVY, *M.D.*

Received April 29,—Read May 7, 1863.

IN a communication, bearing the above title, that was read before the Royal Society, January 8, 1863, I brought forward experimental evidence which had conducted me to view the immunity enjoyed by the stomach from being digested by its own secretion during life, as resulting from the neutralizing influence on the acidity of the gastric juice exerted by the stream of alkaline blood flowing through its parietes. The opposition that this view received on the evening of its announcement induced me to extend my experiments, and as from the additional results obtained some important confirmatory evidence can be adduced, I have deemed it desirable to present this further communication, in which the whole subject is concisely reviewed with the aid of the new matter that has been brought to light.

JOHN HUNTER directed attention to the point under consideration in a paper entitled "On the Digestion of the Stomach after death," which is contained in the Philosophical Transactions for 1772. After adverting to the fact that in occasional instances, especially in persons who have died of sudden and violent deaths, the stomach is found so dissolved at its greater extremity as to have allowed of the escape of its contents into the abdominal cavity, and, without an actual perforation occurring, that there are very few dead bodies in which some degree of digestion of the coats of the organ may not be observed, HUNTER gives reasons for concluding that the condition described must be owing to the action of the digestive fluid after the occurrence of death, and not the result of disease in the living subject. The stomach being thus affirmed to be susceptible of digestion by its own secretion after death, it became necessary to account for its not undergoing a similar process of digestion during life. According to HUNTER's view it was the "living principle" that afforded the required protection to the living organ.

Post-mortem examinations of the human body supply constantly recurring examples of the gastric solution that HUNTER has described. Experimentally, however, the effect may be rendered much more strikingly manifest. If, for instance, an animal, as a rabbit, be killed at a period of digestion, and afterwards exposed to artificial warmth to prevent its temperature from falling, not only the stomach but many of the surrounding parts will be found to have been dissolved. With a rabbit killed in the evening and placed in a warm situation (100° to 110° Fahr.) during the night, I have seen in the morning the stomach, diaphragm, part of the liver and lungs, and the intercostal muscles of the side upon which the animal was laid, all digested away; with the muscles and skin of the neck and upper extremity on the same side also in a semi-digested state.

Submitted to examination, HUNTER'S idea about the protecting influence of the "living principle" does not stand the test of experiment. To CLAUDE BERNARD, of Paris, science is indebted for suggesting an ingenious mode of experimenting with reference to this point. Through an artificial opening into the stomach of a dog, BERNARD introduced the hind legs of a living frog whilst digestion was going on. As the result, the parts were digested and dissolved away notwithstanding the frog continued alive. My own experience enables me to offer corroborative evidence as regards this experiment upon the frog; but further, I have found that tissues belonging to a warm-blooded mammal have likewise shown themselves susceptible of attack under subjection to the influence of the gastric digestive menstruum.

Taking for experiment a vigorous rabbit, I carefully introduced one of its ears through a fistulous opening into the stomach of a dog at a period of full digestion. Precautions were used to avoid inflicting mechanical injury upon the ear in placing and retaining it in position; and, at the same time, to avoid, as far as possible, obstructing the flow of blood through its vessels. At the end of two hours, the ear was withdrawn, and several spots of erosion, some as large as a sixpenny piece, were observed on its surface; but nowhere was it eaten completely through. On being replaced for another two hours and a half, the tip, to the extent of rather more than half an inch, was almost completely removed, a small fragment only being left attached by a narrow shred to the remainder of the ear; a considerable escape of blood took place, especially towards the latter part of the experiment.

To replace the refuted influence of the "living principle," it has been suggested, that it is the epithelial lining which gives to the stomach the immunity from destruction it enjoys during life. The stomach, it has been said, is lined with an epithelial layer, and this, with the mucus secreted, acts as a kind of varnish in protecting the deeper parts. Whilst digestion is proceeding, the epithelium and mucus are constantly being dissolved like the food contained in the stomach; but, a fresh supply being as constantly produced, the organ is thereby maintained intact. Death taking place, and the epithelial layer being no longer produced, the gastric juice, after acting upon and dissolving it, reaches the deeper coats, and then, continuing to exert its influence, may ultimately, the temperature being maintained sufficiently favourable for the purpose, occasion a perforation of the organ.

Such is the view that has been propounded, but, like the "living principle," it fails to stand the test of experiment.

As regards the mucus, an exaggerated notion may be formed, respecting its amount and importance, if an examination of the stomach be made when even a short time only has been allowed to elapse after death. With the rabbit, for example, under such circumstances, on opening the stomach a more or less thick, pulpy, white pellicle is found to adhere to the mass of food which the organ contains. This, however, consists of digested mucous membrane; for, examined immediately after death, the stomach lifts off from the food, leaving the latter uncovered by anything that is visible. The mucous membrane itself, also, is firm in structure throughout.

That the stomach cannot derive its protection from the epithelial layer, as suggested, is proved by the fact, that a patch of mucous membrane may be removed, and food will afterwards be digested without the slightest sign of attack being made upon the deeper coats of the organ. I have several times performed the experiment, to enable me to speak safely on the point, and never have I had the slightest evidence, that depriving the stomach of a portion of its mucous membrane has left the denuded part in a position of greater insecurity than the rest, on the score of liability to digestive attack. It is upon the dog that these experiments have been made; and, upon one occasion, after removing the mucous membrane, and exposing the muscular fibres over a space of about an inch and a half in diameter, the animal was allowed to live for ten days. It ate food every day, and seemed scarcely affected by the operation. Life was destroyed whilst digestion was being carried on, and the lesion in the stomach was found very nearly repaired: new matter had been deposited in the place of what had been removed, and the denuded spot had contracted to much less than its original dimensions. In other experiments, I have examined the stomach at earlier periods after the operation. Life has always been destroyed whilst digestion has been going on. The day after the operation, I have found the denuded spot irregular and raw. Then lymph is deposited upon its surface, and, apparently through the organization of this, the walls are gradually thickened, and the process of reparation carried out.

In addition to the evidence afforded by experiment, it may be assumed, upon reflection, that something more constant—some condition presenting less exposure to the chance of being influenced by external circumstances than that supplied by the existence of an epithelial layer, would be required to account for that unfailing security from *ante-mortem* solution which the stomach appears to enjoy. From the articles swallowed, abrasion of the mucous membrane may be presumed to have been not unfrequently produced, and ulceration is not so uncommon an occurrence; yet, perforation has not been observed as the necessary result. Perforation, it is true, does sometimes occur as a consequence of ulceration, but the same is the case in other parts of the alimentary tract, and there is reason to regard it here, as elsewhere, as resulting from a gradual advance of the ulcerative process, and not from a special digestive action exerted by the gastric juice.

The notion, then, that the stomach is prevented from being digested during life, because it is a living structure, is disproved by the consideration that the parts of living animals that have been introduced into the digesting stomach have not shown themselves capable of resisting its digestive influence. That the epithelial layer, also, with its capacity for constant renewal, does not afford the explanation needed, is proved by the absence of any solvent action being exerted by the digestive fluid upon the deeper coats when the part has been completely denuded of its mucous membrane. The question, therefore (and an exceedingly important one it must be admitted by all to be), still remains open for solution, Why does the stomach, composed as it is of digestible materials, escape being digested itself, whilst digestion is being carried on in its interior? It is evident, whatever explanation, with any pretence to sufficiency, is given, must comprise some broad principle of action capable of providing against all contingencies—capable of affording, in fact, that uninterrupted security during life which upon looking around us we observe the stomach to enjoy.

The view that I have to offer refers the immunity observed to the circulation within the walls of the organ of an alkaline current; and this agrees with the principle I have laid down as indispensable, for the circulation of blood forms with us an essential condition of life. It will not be disputed, that the presence of acidity is one of the necessary circumstances for the accomplishment of gastric digestion. Now, alkalinity is a constant character of the blood, and as during life the walls of the stomach are everywhere permeated by a current of this alkaline blood, we have here an opposing influence, the effect of which would be to destroy, by neutralizing its acidity, the solvent properties of the digestive fluid tending to penetrate and act upon the texture of the organ.

The following point is also worthy of note in passing. In the arrangement of the vascular supply, a doubly effective barrier is, as it were, provided. The vessels pass from below upwards towards the surface: capillaries having this direction ramify between the tubules by which the acid of the gastric juice is secreted. Acid being separated by secretion below must leave the blood that is proceeding upwards correspondingly increased in alkalinity; and thus, at the period when the largest amount of acid is flowing into the stomach, and the greatest protection is required, then is the provision afforded in its highest state of efficiency. Looking to nature's secretion alone, the act creating a demand for protection enhances the character of the protection provided.

The blood being stagnant after death, the opposing influence is lost that is offered by the circulating current. Should life happen to be cut short at a period of digestion, there is only the neutralizing power of the blood actually contained in the vessels of the stomach, to impede the progress of attack upon the organ itself; and the consequence is, that digestion of its parietes proceeds, as long as the temperature remains favourable for the process, and the solvent power of the digestive liquid is unexhausted. There is, therefore, no want of harmony between the effect that occurs after death, and the explanation that refers the protection afforded during life to the neutralizing influence of the circulation.

Having thus stated the nature of the view propounded, I next proceed to show in what manner it answers to the test of experiment.

It occurred to me, that, if the circulation really fulfilled the office I have alleged, the act of arresting the flow of blood through the walls of the stomach during life ought to lead to the same, or about the same, effect on the organ, other circumstances being equal, as would occur after death. The experiment being performed upon dogs and rabbits, I observed, as the result, digestion proceed to the extent of perforation in the rabbit, whilst in the dog I did not witness a greater effect than some amount of solution of the mucous layer.

Having before me the effect I have described as ensuing when a rabbit is killed, and its temperature is afterwards maintained artificially, and taking this as an index of the effect to be looked for in these experiments, I had to account for the absence of perforation occurring in the dog. I conceived, at first, that the circulation in the surrounding parts which would exist during life, and not after death, might produce a modifying influence on the result. To what extent this is true is shown by the following experiments.

A couple of rabbits that had been fed alike were killed at a period of digestion. The stomachs were immediately removed, and the one immersed in some freshly-drawn, defibrinated sheep's blood; the other, in a solution of gum and sugar made to correspond to the blood in density. The gum was introduced to take the place of the albumen, and the sugar, the salines, so as to have a fluid that would behave about like blood as regards osmosis. The liquids were placed side by side in an oven, and the temperature maintained at about 100° Fahr. At the end of 4½ hours, the stomach immersed in the solution of gum and sugar had undergone perforation, and allowed of the escape of its contents. The other was still entire, but digested in its interior so as to be reduced to only a thin layer. In another experiment the effect was not allowed to proceed so far. Both stomachs remained externally entire, but that immersed in the solution of gum and sugar presented, in a distinctly marked manner, evidence of more extensive attack than the other.

Through much subsequent experience I learned that I had in reality been labouring under an exaggerated notion, and that the standard I had taken from the rabbit was unjust in its application to the dog. The result of actual experiment on this animal shows a marked difference in degree of effect produced by the digestive action of the contents of the stomach after death to that which occurs in the rabbit. In the experiments thus performed, the animals have been killed about four or five hours after a meal of animal food. The temperature of the body has then been maintained for five and six hours closely to that belonging to life. Now, at the end of this time, the stomach has only shown signs of more or less digestion of its mucous membrane, a condition that has been about equalled in some of my experiments, where the flow of blood through the stomach has been arrested at a period of digestion, and the animal allowed to live for about five or six hours afterwards. By means of ligatures applied around the pylorus and the end of the œsophagus, and also around the vessels passing between the spleen and greater curvature of the organ, its circulation is with security and facility arrested, and its contents at the same time prevented from escaping. It is in this way that the experiments during life have been all conducted.

In the case of the rabbit, as I have said, I have witnessed digestion of the stomach proceed to the extent of perforation, as the result of stopping the flow of blood through the vessels. The process of digestion, however, being so much influenced by the temperature, it is necessary to observe, with a small animal like the rabbit, that it is not placed in a cold situation, for the heat to decline after the operation has been performed. Without artificial warmth and during cool weather I have seen the cardiac extremity of the stomach digested away in less than eight hours. In an experiment, however, where artificial warmth was employed, perforation was observed at the end of four hours. The operation was performed four hours after food had been given. The animal was then placed in an atmosphere with the thermometer standing at 92° Fahr. In four hours' time it was killed, and the parts were examined immediately. The stomach throughout was in an advanced state of digestion, and was perforated in one spot of about the size of a shilling.

The contents of the stomach in the rabbit are always observed most powerfully acid, much more so, according to what I have seen, than in the case of the dog. From the nature of the food some acid may be generated in addition to that derived from the blood by secretion. Now, upon the quantity of acid, amongst other circumstances, the energy of the digestive menstruum depends, and, in harmony with this, it can be shown, that if an acid (an acid that is not of a nature to exercise of itself any direct erosive effect) be introduced into the stomach of a dog at a period either of digestion or fasting, and the circulation through the walls of the organ be afterwards stopped, the effect which occurs is even considerably stronger than what has been hitherto referred to in the rabbit. I may mention three experiments in proof of this assertion. The acids employed were purposely selected on account of their non-corrosive properties. In the first, the animal was taken six hours after a full meal of animal food. One fluid ounce of the dilute phosphoric acid of the London Pharmacopœia, mixed with an equal quantity of water, was introduced into the stomach, and the circulation through the organ afterwards, in the usual way, arrested. Death took place during the night, and a large perforation was found in the cardiac extremity of the stomach. In the second, six drachms of the same acid, diluted with an equal quantity of water, were employed, and this time upon an empty stomach. Perforation took place in 2½ hours' time. In the third, 60 grains of citric acid, dissolved in two ounces of water, constituted the acid used, and this time also it was at a period of fasting that the experiment was performed. In four hours death occurred from perforation.

It is thus rendered evident, that all that is wanted in the dog to produce digestive destruction of the stomach when its circulation is arrested, is the presence of a sufficient amount of acid in its interior. With a limited amount of acid the power of the gastric juice soon becomes exhausted, and there being food as well as the stomach to act upon, this exhaustion may occur before any marked attack upon the organ has taken place. With a larger amount of acid, however, the exhaustion does not at this early period arrive, and the stomach continues to be acted upon until a perforation of its coats may be effected.

In striking contrast to the effect above narrated, of introducing a mild acid into the stomach and ligaturing the vessels, are the results I have obtained from introducing the acid *without the operation* on the vessels. Three experiments were performed, using the same acids, and the same quantities of them, that had been employed before. Ligatures were placed around the end of the œsophagus and the pylorus to secure the retention of the acid in the stomach, care being taken, however, to avoid including the vessels. The circulation was thus left free to exercise its neutralizing influence, whilst, in other respects, the circumstances of the experiment were the same as before. Where one ounce of the dilute phosphoric acid, mixed with an equal quantity of water, was employed, the animal was alive on the following day, and when killed, the stomach was found free from unnatural appearance, with the exception of a number of small ulcerated spots strewed over the internal surface. These did not extend through the mucous membrane, and looked like what might be supposed to result from the action of an irritant. Where the six drachms of dilute phosphoric acid, and the same of water, were used, the animal was also alive on the following day. The mucous membrane of the stomach presented here and there an appearance of congestion, and a few small spots of superficial ulceration towards the pyloric end. The organ was otherwise found in a natural condition. With the 60 grains of citric acid, dissolved in two ounces of water, the animal, from some cause, died during the night. The stomach was found everywhere perfect; and, in this case, was without the slightest appearance of ulceration of its surface. The last dog had been taken at a period of digestion, the other two of fasting.

It will be seen how strongly the above results stand in support of the view I have brought forward. The stomach yields to the digestive influence of its contents in one set of experiments, and not in the other, the only difference in the experiments being, that the flow of blood through the vessels of the organ is arrested in the former and allowed to continue in the latter. The circulation being allowed to continue, a check is offered to the penetration of the walls of the stomach by its contents in an acid state, and thus the freedom from attack that occurred. The circulation, on the other hand, being arrested, there is no such neutralizing influence in operation; the acid menstruum, therefore, is able to attack the stomach just as it does the food in its interior.

A mode of experimenting which I am indebted to Dr. SHARPEY for suggesting, likewise gives confirmation to the view I have propounded. If an incision be made into the stomach and a portion of the opposite wall be drawn forward; and then, if a ligature be placed around this so as to stop the circulation of blood through the part, the constricted portion, on being returned and left projecting in the interior of the stomach, will undergo digestion just as if it consisted of a morsel of food. In one experiment the operation was performed on a dog at a period of fasting. Food was given on the following morning, and 7½ hours afterwards the animal was killed. In the act of removing the stomach the parts surrounded by the ligature fell asunder, leaving a large circular opening from 1½ to 2 inches in diameter. There was not a vestige of the constricted mass to be discovered; it had all been digested away. In another experiment some food was in the stomach when the ligature was applied. Although vomiting twice occurred soon after the operation was completed, and death took place in twenty time without any more food being given, more than half the projecting mass was digested away, as though it had been cleanly sliced off transversely; so that, when the ligature was removed and the stomach spread out, a hole fully an inch in diameter presented itself. As such an effect could not have taken place without the presence of gastric juice, it is to be inferred, that the whole of the contents of the stomach had not been ejected by the vomiting, and that the appearance observed was produced during the first few hours after the operation. At death there was nothing whatever contained in the organ. The special attack upon the most projecting part of the ligatured mass is probably to be explained by the contracted state of the stomach allowing only this portion to fairly present itself as a part of the surface in contact with the contents, as long as any remained. I might bring forward more evidence than the above from my laboratory experience; but these experiments I think suffice to show, that a portion of the stomach, to the exclusion of the rest, may be rendered susceptible of digestion by the removal of the protecting influence I conceive to be afforded by the circulation.

It will naturally be required of me to reconcile the view I have advanced with the effect that was noticed in an early part of this communication as occurring where the living frog's legs and rabbit's ear were introduced through a fistulous opening into the stomach, and submitted to the influence of the digestive menstruum. If the circulation, through its neutralizing power, protect the stomach, why should it not afford equal protection to the tissues of living animals, introduced through a fistulous opening into the digesting organ? I thus state the question openly, because it is one that requires to be openly met.

According to the proposition I have offered the stomach is protected, because the neutralizing power of its circulation is sufficient to overcome the acidity of the gastric juice which is tending to penetrate and attack its texture. Now, this consideration, it will be seen, involves the result in a question of degree of power between two opposing influences. Diminish the neutralizing power of the circulation beyond a certain point, and allow the strength of the digestive liquid to remain the same; theoretically, the result should be in favour of digestion instead of protection; practically, this may be regarded as what happens in the experiments with the frog's legs and rabbit's ear. Allow, on the other hand, the neutralizing power belonging to the circulation of the stomach to remain the same, but increase beyond a certain point the strength of acidity of the digestive liquid; theoretically, digestion of the stomach's parietes would be looked for as the result; practically, it can be shown that this is really what occurs, as will be seen by an experiment to which I shall presently refer.

With the living frog's legs introduced into the digesting dog's stomach, it may be fairly taken that the amount of blood possessed by the frog would be totally inadequate to furnish the required means of resistance to the influence of the acidity of the dog's gastric juice. With the rabbit's ear the vascularity is so much less than that of the parietes of the dog's stomach, that there is nothing, to my own mind, incomprehensible in the fact of the one yielding to, and the other resisting attack. No comparison can be drawn between the position of the stomach, and that of the rabbit's ear. The stomach is not only in itself exceedingly vascular, but is entirely surrounded by equally vascular parts. The rabbit's ear is only supplied with blood that reaches it at its base, and, immersed in the stomach, it would lie completely bathed all around by gastric juice.

From the experiments I have mentioned it has been seen, that the introduction of a moderately strong acid liquid into the stomach leads to the production of a solvent effect on the organ when its circulation is stopped, which does not occur when the circulation is allowed to remain free. Now, if the strength of the acid be increased, the stomach shows itself to be susceptible of attack, although the circulation may have been left undisturbed. In an experiment upon a dog whilst fasting, I introduced 3 ounces of a liquid, consisting of 3 drachms of muriatic acid and the remainder of water, into the stomach, and afterwards ligatured the end of the œsophagus and pylorus without including the vessels. In one hour and forty minutes death took place, and, on the parts being examined immediately, perforation was found, with an escape of the contents of the stomach into the peritoneal cavity. The interior of the stomach throughout had undergone an extensive dissolution; and in the neighbourhood of the perforation, which was at the cardiac extremity, the texture presented quite a gelatinized appearance. This result was evidently the effect of digestion; for the acid, at the strength it was employed (one part in eight), does not possess such physical corrosive properties. A considerable escape of blood had taken place from the stomach during its attack; and it may be reverted to, in connexion with this, that there was also a considerable amount of hemorrhage observed from the rabbit's ear, whilst being attacked in the stomach of the dog. I take it, in the above experiment, that the height of acidity in the stomach was very much too great for the neutralizing capacity of the circulation: and thus, the rapid progress of digestive solution.

There is one more point that remains to receive consideration. It would be incompatible with my view, that a living organism could exist *in a free state* in the stomach, whilst digestion is going on, without being attacked; unless this organism should consist of, or be protected by, an indigestible material. It is well known, however, that larvæ of the *Œstrus* inhabit the stomach of the horse, but it will be found that they live with their heads firmly attached to, and buried in, the mucous membrane; indeed, there is sometimes quite a honeycomb arrangement in which the greater portion of the animal can be lodged. Living upon the juices of the animal these larvæ infest, they become more or less, as it were, a part and parcel of the stomach's parietes. It is further to be remarked, that the principle (chitine), which forms the basis of the tunic of insects, is of an exceedingly indigestible character. By Professor SIMONDS I have been informed of an entozoon (a species of *Filaria*) which he has found in the last stomach of the sheep; but this parasite also lives firmly attached to the mucous membrane, and in connexion with the juices of the animal it infests. I have not been able to learn, that any example can be brought forward of life being carried on, under isolated circumstances, in a digestible organism placed in the interior of an actively digesting stomach. The older physiologists found, in their early experiments on digestion, that such animals as leeches and earthworms, placed in perforated metal spheres, and introduced in a living state into the digesting stomach, were attacked, just as if they had consisted of ordinary food.

The following *résumé* may be taken as representating the main points of what has been adduced in this communication:—

That Hunter's suggestion of the "living principle" forming the source of protection to the stomach from being digested by its own secretion during life, is negatived by Claude Bernard's discovery, that parts of living animals introduced through a fistulous opening into the digesting stomach, are observed to undergo digestion like its other contents.

That the epithelial layer, also, with its capacity for constant renewal, does not furnish the explanation required, is proved by the experimental evidence brought forward, showing, that a patch of mucous membrane may be removed, and digestion still be carried on, without the denuded part being digested.

That in default of the sufficiency of these suggestions, the view I have been led to entertain refers the immunity enjoyed by the stomach from being digested during life, to the influence of an alkaline circulation. Acidity is necessary for digestion, and alkalinity is a constant character of the blood. Whilst the walls of the stomach, therefore, are permeated by a current of blood, an opposing influence is offered to digestive attack. Death taking place, there is no longer a circulation of alkaline fluid to exert a neutralizing effect on the acidity of the gastric juice tending to penetrate and attack the organ: the consequence is, that digestion now proceeds, according to the nature of the circumstances that prevail.

That this view is supported by experimental evidence of the following description:—

> By ligaturing the vessels of the stomach so as to arrest the flow of blood through the organ, it is rendered susceptible of attack by its contents during life in like manner as after death.
>
> In the rabbit, digestion of the stomach has been thus observed to proceed to the extent of perforation.
>
> In the dog, the action has not been witnessed to proceed beyond a solution of the mucous layer.
>
> Upon introducing, however, into the stomach of the dog, previous to ligaturing the vessels, moderate quantity of a dilute acid—mineral or vegetable—a perforation of the organ in each of the three experiments performed has ensued.
>
> The introduction of the same acids, similarly diluted, and in like amounts, without the operation of ligaturing the vessels, so that the circulation has been left free, has not occasioned any digestive attack.
>
> By pinching up and ligaturing a portion of the walls of the stomach so as to

leave a constricted mass projecting into the cavity of the organ, this has been found to undergo digestion like a morsel of food.

That the attack upon the living frog's legs and rabbit's ear introduced into the digesting stomach of a dog need not be looked upon as forming any valid objection to the view propounded. The explanation is one that involves the result in a question of degree of power between two opposing influences. Because, through degree of vascularity, the neutralizing power of the circulation is sufficient to hold in check the solvent action of the gastric juice in the case of the walls of the stomach, it does not follow that it should similarly be sufficient to do so in the case of the legs of a frog or the ear of a rabbit. The circumstances are far from identical in the two cases; and, in support of what has been stated, it can be shown by experiment that even with the stomach itself, by increasing the acidity of its contents beyond a certain point, *its* circulation is no longer adequate to enable it to resist digestion.

That the capacity of a living and digestible organism to exist in an isolated state in the interior of the digesting stomach would be incompatible with the view that has been announced. Instances can be brought forward of animals inhabiting the stomach, but they do not form examples of the above description.

IX. *On Thallium.* *By* WILLIAM CROOKES, *Esq.*

Communicated by Professor G. G. STOKES, *Sec. R.S.*

Received February 5,—Read February 19, 1863.

Occurrence, Distribution, and extraction from the Ore.

1. SINCE the date of the last paper on Thallium which I had the honour of communicating to the Royal Society[*], I have been unremittingly engaged in attempting to find a source from which this metal could be extracted in quantity. Having first discovered thallium in the deposit from the chambers of a sulphuric-acid manufactory, I naturally turned my attention towards similar deposits from English oil-of-vitriol works where pyrites was burnt. Applications were accordingly made to several large manufacturers for specimens of the pyrites which they used, and also for some of the deposit from their leaden chambers. These requests, with scarcely an exception, were readily responded to, and in a short time I was in possession of specimens from nearly thirty different establishments. In many instances thallium was detected in the pyrites, but I was disappointed to find that the deposits of sulphate of lead from the chambers contained no thallium whatever. I then applied to manufacturers who I had ascertained were constantly burning thalliferous pyrites, and obtained from them specimens of the products in different stages of their manufacture, but in no instance did I find an accumulation of thallium in any part of the operations.

2. In the operation of burning the pyrites, the thallium oxidizes with the sulphur and volatilizes into the leaden chambers; it there meets with aqueous vapour, sulphurous and sulphuric acids, and becomes converted into sulphate of the protoxide of thallium. This being readily soluble both in water and dilute sulphuric acid, and not being reduced by contact with the leaden sides, remains in solution and accompanies the sulphuric acid in its subsequent stages of concentration, &c. It is not probable therefore that any thallium can accumulate in the insoluble deposit, but it will remain dissolved in the liquid, where indeed I have found it—not however in quantities sufficient to be worth extracting, as it is present in scarcely a larger proportion than in the original pyrites. That this view of the path followed by thallium is correct, I am satisfied both from careful analyses of products from various manufactories, and also by experiments tried on a small scale in my own laboratory. M. LAMY states that he extracts thallium from similar deposits to those which I have examined; but as I have experimented on residues from English manufactories in which they burn pyrites almost, if not quite, as rich in thallium as that used in M. KUHLMANN'S works, there

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[*] Proceedings of the Royal Society, June 19, 1862.

must be some cause or local arrangement in their manufactory, different from what is usually adopted in this country, to occasion so large an accumulation of thallium at one particular stage of the operations*.

3. Having failed in my endeavours to find a residue from a manufactory which would yield thallium, I turned my attention to the ores in which it was likely to occur.

The unique collection of minerals brought together at the International Exhibition of 1862 enabled me, through the kindness of the various commissioners and class-superintendents, to verify the opinion which I had formerly expressed as to thallium being a very widely distributed element. It most frequently, indeed almost invariably, occurs in iron pyrites containing more or less copper; but I have also detected it in native sulphur, in zinc, cadmium, bismuth, mercury, and antimony ores, as well as in the manufactured products from these minerals. Thallium is confined to no particular country, but is at the same time by no means uniformly distributed in mineral veins from the same locality, or even in adjacent rocks in the same mine. Hitherto I have detected no approach to law in its distribution. In every country mineralogically represented in the late Exhibition I have detected the presence of thallium, when the minerals examined were at all numerous and could be regarded as fair average samples of the different deposits. Doubtless much of this apparent abundance is to be attributed to the extreme delicacy of the test employed, as ores in which thallium is present, only in the proportion of 1 to 100,000, give evident traces of it in the spectroscope. Many pyrites, however, contain more than a mere trace of thallium: it is present, in sufficient quantity to be readily extracted by direct treatment, in pyrites from various parts of North and South America, France, Belgium, Spain, as well as Cornwall, Cumberland, and many parts of Ireland.

4. The optical process of detecting thallium in a mineral is very simple. A few grains only of the ore have to be crushed to a fine powder in an agate mortar, and a portion taken up on a moistened loop of platinum wire. Upon gradually introducing this into the outer edge of the flame of a Bunsen's gas-burner, the characteristic green line will appear as a continuous glow, lasting from a few seconds to half a minute or more, according to the richness of the specimen. By employing an opake screen in the eye-piece of the spectroscope to protect the eye from the glare of the sodium line, I have in half a grain of mineral detected thallium when it was only present in the proportion of 1 to 500,000. After a few experiments of this kind, and having a thalliferous pyrites of known richness for comparison, it is easy to give a rough estimate as to the quantity of thallium present

5. One of the richest thalliferous minerals in the Exhibition was the prominent block

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* This anomaly has just been cleared up by a paper communicated to the French Academy, on the 26th ultimo, by M. Fréd. Kuhlmann, in which he explains that, in order to prevent the passage of arsenic from the pyrites into the sulphuric acid, he interposes, between the pyrites-kilns and the ordinary leaden chambers, a large supplementary chamber, in which the products of combustion are lowered in temperature and deposit the more easily condensible volatile matters. It is in this deposit that thallium is found.

of pyrites, weighing nearly two tons, which formed so conspicuous an object in the Belgian department; it was exhibited by the Société Anonyme de Rocheux et d'Oneux of Theux, near Spa (Belgium, Class I., No. 18 in the Catalogue). Accompanying the principal block were several smaller specimens from different parts of the same locality. Thallium was detected in all, but in very varying proportions. By the great kindness of Professor CHANDELON of Liège, one of the jurors of Class II., I was enabled to examine a series of examples from four separate chambers, and eight different rocks from the mine whence these specimens were brought, he having most courteously taken the trouble to visit the locality and select the specimens for me. In some, as in rocks C, D, and G, no appreciable quantity of thallium was present; in chamber No. 1 there was a trace only, whilst in chamber No. 2, and in rock H, it was present in comparative abundance. Professor CHANDELON, to whom I desire to offer my hearty thanks for his uniform courtesy and valuable assistance, informs me that those portions of the mineral vein which I have found most rich in thallium are close to a vein of blende and calamine, which is worked in the neighbourhood for zinc. I am promised some of this blende and calamine for analysis; and it will be of some interest to know the relative quantity of thallium they contain in comparison with that in the adjacent pyrites. I have already found considerable quantities of thallium in metallic zinc and cadmium manufactured from this ore.

Professor CHANDELON not only troubled himself to make several journeys from Liège to the mines on my account, but likewise induced M. RÉNARD (the manager) and the proprietors to present me with upwards of two tons of the mineral carefully selected from those parts of the mine which I had found to be richest. This arrived in September last.

6. Numerous experiments have led me to adopt the following process for extracting the new metal from its ore. The pyrites is first broken up into pieces about the size of a walnut, and placed in cast-iron retorts, capable of holding 20 lbs. each; five of these are arranged in a reverberatory furnace, so that the flame may lick round and heat them uniformly; iron condensers being luted on, the temperature is raised until the retorts become of a bright red heat, at which they are kept for about four hours. They are then allowed to cool, and the product removed from the condenser; 100 lbs. of pyrites usually give from 13 to 17 lbs. of sulphur. The sulphur is of various colours, according to the temperature attained, sometimes being orange, sometimes purple-brown, sometimes bright grass-green, and is always highly crystalline. Experiments tried at various times during the distillation, show that thallium is present in the first portions of sulphur which distil over, as well as in the last, although the proportion somewhat increases towards the end. When the heat has been sufficient, and the yield of sulphur above 15 per cent., scarcely a trace of thallium remains behind in the retorts; but with a low heat, the sulphur is almost pure yellow in colour, and contains very little thallium. Each operation lasts about five hours; the iron retorts frequently serve for two, and sometimes for as many as five or six heatings. They gradually,

however, become converted into sulphide, and then melt away. This sulphide of iron contains no thallium. Within the last three months I have distilled upwards of 16 cwt. of iron pyrites, yielding me 2¼ cwt. of thalliferous sulphur. The results are appended:—

						lbs.	ozs.	
November	8,	1862,	14 lbs.	of pyrites	yielded	2	6	of sulphur.
„	14,	„	24	„	„	3	8	„
„	29,	„	120	„	„	12	7	„
December	11,	„	70	„	„	8	12	„
„	13,	„	100	„	„	12	5	„
„	15,	„	100	„	„	12	8	„
„	16,	„	100	„	„	14	5	„
January	3,	1863,	100	„	„	13	5	„
„	5,		100	„	„	12	9	„
„	7,		100	„	„	11	4	„
„	8,		100	„	„	15	4	„
„	9,		100	„	„	14	7	„
„	10,		100	„	„	14	13	„
„	12,		100	„	„	16	3	„
„	14,		100	„	„	15	2	„
„	24,		100	„	„	16	4	„
„	26,		100	„	„	13	13	„
„	27,		100	„	„	17	2	„
„	29,		100	„	„	14	5	„
„	30,		100	„	„	11	9	„
			1828			252	3	

It may be of some interest to state that, although the mineral is put in in its ordinary air-dry state, I invariably get from each 100 lbs. of pyrites about half a pint of liquid distilled over before the sulphur comes. Upon evaporating this down, water, having a peculiar empyreumatic odour, goes off, oxide of iron is precipitated, and sulphite of the protoxide of iron separates in large, nearly colourless crystals: no thallium is present.

7. I have met with great difficulty in extracting economically, and without loss, the whole of the thallium from this sulphur. On the small scale, nothing is easier than to boil it, finely powdered, in fuming nitric acid or *aqua regia*, until the residuary sulphur is of a pure yellow colour, and then to extract the thallium from the solution by processes to be hereafter described; but such operations are impracticable when working by the hundredweight.

An attempt was made to separate the thallium by converting the thalliferous sulphur into chloride. Half a pound was treated in this manner; but upon rectifying the product, thallium was found in the distillate as well as in the residue. An immense number of experiments were tried to extract the thallium from the chloride of sulphur, but with no success, unless indeed means were employed which would have been equally easy and economical upon the original sulphur. The simplest plan would doubtless be to dissolve out the sulphur with bisulphide of carbon, in which sulphide of thallium is insoluble; and this is a plan that I hope before long to have in full work; hitherto, however, I have found the following the readiest method:—In a large cast-iron caldron dissolve 12 lbs. of good caustic soda in 1½ gallon of water, heat it to the boiling-point, and then add 18 lbs. of the sulphur in large lumps, just as it comes from the receivers. A few pounds should be added at a time, as dissolved, and the mixture kept gently boiling, water being added from time to time to replace that lost by evaporation. When no more sulphur can be dissolved, dilute the mixture with four or five times its bulk of water, and allow it to cool. A voluminous black precipitate will separate, which must be collected on a calico filter. The greater portion of the thallium remains in this precipitate in the form of sulphide, together with iron, copper, &c.; but some passes through dissolved in the alkaline liquid. I have hitherto been unable to recover the thallium from this solution without an incommensurate expenditure of both acids and time. After a slight washing, the black precipitate is transferred to a large dish, and boiled in sulphuric acid until sulphuretted hydrogen ceases to be evolved; nitric acid is then added in small quantities, and the mixture is boiled until all solvent action has ceased; it is then diluted with water, and filtered. Evaporate down until all excess of nitric acid is removed, and then add hydrochloric acid and sulphite of soda to the liquid; this produces an immediate precipitation of the thallium in the state of protochloride as a white crystalline precipitate, only slightly soluble in water; as, however, a certain quantity of this chloride remains dissolved, it is advisable to add, after the sulphite of soda, iodide of potassium, which precipitates the whole of the thallium in the form of an insoluble yellow iodide. If the solution turns almost black upon adding the iodide of potassium, or the precipitate comes down of a dirty grey colour, it is a sign that an insufficient quantity of sulphite of soda has been added; a further addition will remedy this. A little copper, which is generally present in the sulphur, will likewise be precipitated in the form of subiodide. The iodides are to be filtered off, and washed until they are free from iron. They may now be decomposed by heating with oil of vitriol, which converts them into sulphates; but the temperature required for this being very high, and the decomposition difficult to effect perfectly without loss of thallium by volatilization, I prefer the following, though somewhat longer process:—Boil the iodides in excess of sulphide of ammonium until they are entirely converted into sulphides; filter, and wash with weak sulphuretted water until no iodine can be detected in the washings; then boil the precipitate with strong sulphuric acid, adding a little nitric acid from time to time; evaporate until the sulphuric acid begins to go off in white vapours, and then, after dilution with water, add an excess of ammonia. Now add cyanide of potassium until the blue ammoniacal solution of copper is decolorized, and then a slight excess of sulphide of ammonium, and gently warm. Filter and wash with dilute sulphuretted water until all the cyanide of potassium, &c. is removed, and boil the precipitated sulphides with strong sulphuric and nitric acids until the mass becomes perfectly white. Drive off the whole of the nitric acid and the greater portion of excess of sulphuric acid by heat, and then boil the residue for some time in water. Filter, and wash well. A white insoluble residue will generally be left, containing a little thallium; but the greater portion of this metal will be in the solution in the form of sulphate.

8. From this solution the metal is readily obtained in the metallic state by voltaic precipitation. Two or three cells of a Grove's battery, with platinum terminals dipping into the aqueous solution of the sulphate (either acid or rendered alkaline with ammonia), produce an immediate reduction, oxygen being evolved at one pole, and the metal coming down at the other. The appearance presented when a tolerably strong solution of thallium is undergoing reduction is very beautiful. If the energy of the current bears a proper proportion to the strength and acidity of the liquid, no hydrogen is evolved at the negative electrode, but the metal grows from it in large crystalline fern-like branches spreading out into brilliant metallic plates, and darting long needle-shaped crystals, sometimes upwards of an inch in length, towards the positive pole, the appearance strikingly resembling that known as the tin tree. Some of the tabular crystals, as seen in the liquid, are beautifully sharp and well defined, their angles being temptingly measurable; considerable difficulty is, however, met with in disengaging them from the electrode, and removing them in a perfect state from the liquid. So long as thallium is present in the solution, no hydrogen is evolved with a moderate current; as soon as bubbles of gas begin to form, the reduction may be considered complete. The crystalline metallic sponge may now be squeezed into a compact mass round the platinum terminal, and, being disconnected from the battery, quickly removed from the acid liquid, rinsed with a jet from a wash bottle, and transferred to a basin of pure water. The metal is then carefully removed from the platinum, and kneaded with the fingers into as solid a lump as possible. It will be found to retain its metallic lustre perfectly under water, and coheres together readily by pressure. The lump may now, after having been dried with blotting-paper, be put into a steel-crushing mortar and strongly hammered until it assumes the form of a solid ingot. To obtain this in a fused mass, the best plan is to break it up into small pieces, and drop them one at a time into a crucible containing fused cyanide of potassium, at a low red heat; they melt at once and run together at the bottom of the crucible into a brilliant metallic button: allow the crucible to cool, and dissolve out the flux with water, when the thallium will be left in the form of an irregular lump, owing to its remaining liquid and contracting after the cyanide has solidified. As long as the surface is wet, either with the solution of cyanide or with pure water, the metal presents a highly crystalline appearance, resembling tin when washed with acids; this disappears upon exposure to the air, owing to the formation of a pellicle of oxide.

Physical Characteristics of Thallium.

9. Thallium has a distinct colour of its own, not being absolutely identical with any

other metal. The true colour can be seen by scraping a piece of the metal under water, or fusing it in hydrogen and allowing the melted globule to flow away from the dross. In appearance it most resembles tin and cadmium, not being so brilliantly white as silver, but without the blue tinge belonging to lead. It is susceptible of taking a very high polish: by rubbing it under water with a fine polishing stone, the surface can be made smooth and bright, reflecting as perfectly as a mirror.

It oxidizes in the air with almost the rapidity of an alkali-metal. When freshly cut with a knife, if the eye follows the blade, the proper colour of the metal will be seen to assume, in a few seconds, a yellowish cast, caused by a thin coating of the protoxide, which continues to increase until the metallic lustre is obscured by a grey film, scarcely distinguishable from the superficial tarnish of metallic cadmium. At this stage the oxidation appears to be almost arrested, and the metal may be freely handled and exposed to the air, with scarcely any further change. After having remained in the air for some weeks, the surface becomes covered with a white powder, which easily rubs off, and has a strong biting taste.

When rubbed between the fingers, a faint peculiar smell may be observed, unlike that produced by any other metal under the same circumstances. If a perfectly bright surface of thallium is applied to the tongue, no taste whatever is observed; but a tarnished surface tastes strongly alkaline, and somewhat sweet like oxide of lead; whilst if the surface is more oxidized than usual, from the metal having been long exposed to the air, or previously raised to a high temperature, the taste is very caustic and biting, remaining on the tongue for some hours, and resembling that observed when the tongue is applied to the terminals of a voltaic pile.

10. Thallium is the softest known metal admitting of free exposure to the atmosphere: it can be cut, pressed out, and moulded with the finger-nail with the utmost ease; and whilst it is incapable of abrading the surface of a piece of lead, this latter metal scratches thallium with great facility. Thallium marks paper as easily as lead: the streak is blue at first, but almost instantly turns yellowish, and in the course of a few hours nearly fades out, from oxidation. The writing can, however, be restored at any time to more than its original blackness by exposure to sulphuretted hydrogen or sulphide of ammonium. Thallium is too soft to file well or be cut with a saw, as it clogs up the teeth directly, and it does not become brittle when exposed to a low temperature. It has very little tenacity, being inferior to lead in this respect. It is very malleable, and may be hammered out into leaves as thin as paper; it may also be moulded and pressed in a die, taking a very sharp impression. When repeatedly hammered, it does not appear to get sensibly harder or require annealing. I have succeeded in drawing it into wire; but, owing to its want of tenacity, this is a matter of some difficulty; the wire may, however, be obtained in a very easy way by pressure. Dr. MATTHIESSEN has been good enough to prepare for me several specimens of wire by pressing the metal with a powerful vice through a fine hole in a steel box, receiving the wire as it issued forth in tubes filled with dry carbonic acid, and various liquids: I have also prepared it since in similar

apparatus of my own. It is squeezed into wire more readily than lead, and when received into dry carbonic acid or petroleum, without contact with air, and instantly sealed up, its true metallic lustre and colour are very apparent; if, however, a very minute trace of air obtains access, it assumes a deep blue appearance. This is remarkable, as the first superficial coat of oxide which forms on a freshly cut surface, freely exposed to the air, is distinctly yellow.

11. I have carefully determined the specific gravity of thallium. It varies according to the treatment the metal has previously undergone. A lump melted and allowed to cool slowly under cyanide of potassium, was found to have a specific gravity of 11·81. The same lump, after being strongly hammered in a steel mortar, had its density increased to 11·88. Another portion, experimented on in the form of thick wire, had a density of 11·91 immediately it had been squeezed through the die. I believe that thallium is capable of undergoing still greater condensation. If in the process of squeezing it into wire the vice is screwed up until the thallium is just making its appearance through the fine aperture, and the pressure is then kept stationary, the issue of metal proceeds for a few seconds and then stops. Upon now applying the flame of a spirit-lamp to the die, a piece of wire from an eighth to half an inch long quickly shoots out. This cannot be due to ordinary expansion by heat, as it commences and terminates abruptly at an apparently definite temperature. Were it expansion by heat, the formation of wire would proceed nearly uniformly as the temperature increased. The most probable explanation is, that the application of heat gives the metal power suddenly to release itself from an abnormal state of condensation into which the enormous pressure had forced it.

Thallium wire is almost devoid of elasticity; it retains any form into which it is bent with scarcely a tendency to spring towards its former position.

When freshly prepared, thallium wire is perfectly amorphous, and remains so if kept at the ordinary temperature in petroleum or carbonic acid; in water, however, it gradually assumes a superficial crystalline appearance, resembling the *moire métallique* of tin plate: this effect is immediately produced when thallium in wire, ingot, or plate, tarnished or clean, is boiled in water.

12. In fusibility thallium stands between bismuth and lead, its melting-point being 550° Fahr.: it does not appear to become soft and pasty before undergoing complete fusion. Two pieces of the metal, when perfectly clean, are capable of welding together in the cold by strong pressure. I have repeatedly filled the steel die with small scraps and cuttings of thallium, and squeezed them out into one continuous length. Wire so made is apparently as tenacious as that obtained from one lump.

Thallium volatilizes easily: when heated out of contact with air, it evolves vapours at a red heat, and boils below a white heat. In a current of hydrogen gas it may be easily distilled at a red heat; it does not, however, condense very perfectly; for if the hydrogen be ignited, even after traversing four or five feet of cold glass tube, it burns with a bright green flame.

13. Thallium is a pretty good conductor of heat and electricity. My friend Dr. MATTHIESSEN is at present engaged in the quantitative determination of these constants of the metal, upon specimens with which I have supplied him. Its electro-chemical position is very near cadmium, being precipitated from the sulphate by zinc and iron, but not by cadmium or copper. Professor WHEATSTONE is examining in this respect specimens of the metal specially purified for the purpose.

Thallium is strongly diamagnetic. A small permanent horseshoe magnet was suspended vertically, poles downwards, to one arm of a very delicate balance, capable of turning to 0·0005 of a grain. After being accurately counterpoised, a lump of metallic thallium, weighing about 200 grains, was placed beneath the magnet, very close, but not touching. Upon now observing the weight of the magnet, it was seen to be decidedly repelled by the metal, losing in weight 0·003 of a grain.

By the kindness of Professor FARADAY, who himself tried most of the experiments, the above observation was verified with the large electro-magnet belonging to the Royal Institution. Upon suspending a long cylindrical helix of thallium wire horizontally in the magnetic field, and making contact with the battery, it was strongly driven to an equatorial position. A lump of the metal, suspended almost in contact with one of the magnetic poles, was repelled from it, being permanently deflected nearly a tenth of an inch from the perpendicular. From a comparison with bismuth under somewhat similar circumstances, I am inclined to believe that thallium is one of the most diamagnetic bodies known.

14. Thallium readily alloys with other metals. With 95 per cent. of copper it forms a hard button, flattening somewhat under the hammer, but soon cracking at the edges; the further addition of thallium produces a very hard and brittle gold-coloured alloy; and when the proportion of thallium is further increased, the colour of the copper is entirely lost. A very minute quantity of thallium, less than half a per cent., melted with copper, greatly diminishes its malleability and ductility, acting in this respect like arsenic. I believe that the variation in the physical properties of different specimens of commercial copper (a variation which has never yet received satisfactory explanation) is to be attributed to the presence of more or less minute traces of thallium, as I have found it present in many samples of *bad* copper, as well as in some specimens of crystallized sulphate (29). This subject is still under investigation.

Five per cent. of thallium alloy readily with tin, when they are melted together under cyanide of potassium; the resulting compound is perfectly malleable.

With mercury, thallium forms a solid crystalline amalgam.

Thallium melts readily with platinum: if a portion of the metal is placed on the end of a platinum wire and heated to redness, a fusible alloy is obtained, which is crystalline, very hard, and almost as brittle as glass under the hammer. If this alloy is heated before the blowpipe, the characteristic green colour is vividly communicated to the outer flame; before the oxyhydrogen blowpipe the green light is of extraordinary splendour. A similar alloy is left when the platinochloride of thallium is heated to redness in a crucible, chlorine going off together with a little thallium.

15. The remarkable simplicity of the spectrum of thallium has given rise to repeated experiments with a view to ascertain whether, and under what circumstances, it could be obtained compound. I have already stated* that "a flame of sufficient temperature to bring the orange line of lithium into view produces no addition to the one thallium-line; and an application of telescopic power strong enough to separate the two sodium lines a considerable distance apart still shows the thallium-line single." The former observation has lately been verified and extended by Dr. W. ALLEN MILLER†, who has also noticed that at the high temperature of the electric spark several new lines, especially two green and a blue line, make their appearance. The latter observation, as to the unresolubility of the green line, was recently put to a crucial experiment. My friend Mr. BROWNING, the well-known philosophical-instrument maker, kindly allowed me to examine the thallium-spectrum in a spectroscope which he is making for Mr. GASSIOT. This instrument is furnished with nine flint-glass prisms of large size; and although as yet unfitted with the elaborate adjustments with which it will be ultimately furnished, it is even now capable of producing, with a moderate magnifying power, an apparent separation of the eighth of an inch in the two sodium-lines, at the standard microscopic distance of ten inches. Under this enormous amplifying power the thallium-line was still seen single, being as fine and sharply defined upon the black ground as either of the constituents of the double sodium-line. I have stated that in a spectroscope of the ordinary size the line appears to be identical in refrangibility with the line δ in the barium-spectrum; a comparison of the two spectra in this large instrument showed me that these lines do not coincide in position.

The delicacy of this optical test for thallium is very great. I prepared a standard solution of sulphate of thallium, and diluted it until it was in the proportion of one grain dissolved in fifteen gallons (or about 1,000,000 grains) of water. Upon dipping a platinum wire loop into this solution, and holding the moistened end in the flame of the spectroscope, the green line was distinctly visible. The quantity of liquid taken up by the platinum wire was about the fifth of a grain, containing, therefore, no more than $\frac{1}{5,000,000}$th of a grain of sulphate of thallium.

A flame strongly coloured with thallium can be obtained by passing hydrogen over chloride of thallium at a high temperature and then igniting it. The absolute monochromatic character of the light renders everything illuminated by it either green or jet black: coloured sealing-wax, ribbons, a bouquet of flowers, as well as coloured precipitates, are entirely altered in appearance, whilst the human face assumes a horrible, corpse-like green hue.

The green thallium-line can be reversed in the spectrum; but this is an experiment of some difficulty. The effect can, however, be well seen by adopting the plan which I have used to show the same phenomenon in the case of sodium—by holding a small thallium-flame in front of a larger one coloured with the same metal, the mantle of the front flame being projected as a black line on the hinder flame. If a trace of lithia is added to the larger flame, insufficient to destroy the green colour, the effect of contrast is very striking; the border of the front flame being opake to thallium, whilst it is transparent to lithium, the flame appears of a beautiful green colour with a crimson edge.

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* Proceedings of the Royal Society, June 19, 1862.

† Ibid. January 15, 1863.

16. The atomic weight of thallium has been a subject to which my attention has been directed for some months, in fact ever since I commenced to obtain the metal in sufficient quantities to enable me to purify it to the requisite degree without too much diminishing my stock. The investigation is far from concluded as yet, and the numbers which I have obtained must be regarded as only approximate M. LAMY* gives the equivalent as 204. As, however, he gives no statement respecting the processes adopted to arrive at this figure, and we are not even informed of the number of experiments of which this is the mean result, of the quantities of material operated upon, or the divergence of each result from the mean 204, it is impossible to know what value is to be attached to it. Below I give the results of five separate determinations by different methods: I admit they do not agree so closely as one would like in experiments of this sort; indeed the discrepancies are beyond the probable error of analysis, and seem to point to some other disturbing cause not yet ascertained. I give, however, all the necessary figures, and the results may be taken for what they are worth.

About 200 grains of thallium, prepared as already described, were specially tested for, and purified from, metals with which it was likely to be contaminated, by processes appended in the *analytical notes*, and were obtained in the form of sulphate. This was recrystallized twice, and the metal precipitated from its aqueous solution by two Grove's batteries, platinum electrodes being employed. The metal was then fused under cyanide of potassium, and, after being cleaned in dilute acid, preserved for use in a dry bottle filled with coal-gas. Some of this purified metal was then dissolved in dilute sulphuric acid, the solution was evaporated down, and the residue heated until sulphuric acid ceased to come off; it was then redissolved in water, and the sulphate of thallium allowed to crystallize. The salt was then considered to be sufficiently pure for analysis.

I. Some of the crystals were heated to incipient fusion and weighed, they were then dissolved in water, and iodide of potassium was added until no further precipitation of iodide of thallium took place. The precipitate was then warmed and allowed to settle, collected on a tared filter, washed with water, dried in a water-bath and weighed.

II. Another portion of sulphate of thallium was heated and weighed as above described, and the aqueous solution precipitated with nitrate of baryta. The precipitated sulphate of baryta was then collected as usual, well washed, and weighed.

III. A third portion of sulphate of thallium was weighed as before, dissolved in a small quantity of warm water, and mixed with a slight excess of pure hydrochloric acid; alcohol was then added, and the precipitated chloride of thallium collected on a tared filter, washed with alcohol, and weighed.

IV. A piece of metallic thallium was weighed and converted into sulphate. The excess of sulphuric acid being driven off by heat, the remaining sulphate of thallium was heated to its fusing-point and then weighed.

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* Comptes Rendus, December 8, 1862.

V. The sulphate of thallium obtained in experiment IV. was dissolved in water and mixed with an excess of bichloride of platinum. The precipitated platinochloride of thallium, which is more insoluble than the potassium salt, was then collected on a tared filter, washed, and weighed.

The following Table shows the experimental results obtained:—

Experiment I.

7·342 grains of sulphate of thallium yielded 9·655 grains of iodide of thallium. Calling x the equivalent of thallium, we have the following proportion,

$$7{\cdot}342 : x+48 :: 9{\cdot}655 : x+127, \therefore x=202{\cdot}73.$$

Experiment II.

9·883 grains of sulphate of thallium gave 4·577 grains of sulphate of baryta.

$$9{\cdot}883 : x+48 :: 4{\cdot}577 : x+116{\cdot}5, \therefore x=203{\cdot}55.$$

Experiment III.

8·555 grains of sulphate of thallium yielded 8·127 grains of chloride of thallium.

$$8{\cdot}555 : x+48 :: 8{\cdot}127 : x+35{\cdot}5, \therefore x=201{\cdot}85.$$

Experiment IV.

10·113 grains of thallium yielded 12·503 grains of sulphate of thallium.

$$10{\cdot}113 : x :: 12{\cdot}503 : x+48, \therefore x=203{\cdot}1.$$

Experiment V.

12·503 grains of sulphate of thallium yielded 20·312 grains of platinochloride of thallium.

$$12{\cdot}503 : x+48 :: 20{\cdot}312 : x+205{\cdot}2, \therefore x=203{\cdot}56.$$

I have therefore adopted the mean result 202·96 or 203 as the equivalent of thallium, writing the protoxide TlO and the sulphate $TlO.SO_3$, unless, indeed, as appears probable from theoretical considerations, these compounds have to be expressed Tl_2O and $Tl_2O.SO_3$, in which case the atomic weight would be half this number.

Chemical Properties of Thallium.

17. Thallium does not decompose pure water, either at the common temperature or when boiling. If, however, steam be passed over the metal at a red heat, it is decomposed, with formation of oxide of thallium and separation of hydrogen, the gas evolved burning with a decided green flame. The oxide which forms superficially when thallium

is exposed to the air is the protoxide, a powerful base, soluble in water, forming a liquid which is strongly alkaline to test-paper. If a lump of thallium, weighing 50 or 100 grains, is placed, after exposure to the air for a few days, in an ounce of water and boiled for a few seconds, the solution will be found to possess alkaline characters. It turns litmus paper strongly blue, browns turmeric paper, has a metallic alkaline taste, and perfectly neutralizes acids. It also precipitates alumina from a solution of alum, evolves ammonia from chloride of ammonium, and reacts with hydrochloric acid, iodide of potassium, sulphide of ammonium, &c., in the characteristic manner of a thallium-salt. As might be imagined, thallium is readily acted on by air and water jointly; and by shaking up pure thallium wire in a bottle with an insufficient quantity of water to cover it, allowing fresh air to have access from time to time, a strong solution of oxide of thallium can be obtained. When thallium is melted in the air, it behaves very similarly to lead, rapidly oxidizing and becoming coated with a fusible oxide resembling litharge. Upon continuing the heat, this increases, whilst the bright globule in its centre diminishes in size. The fused oxide is absorbed by bone-ash, and I have ascertained that a silver-thallium alloy can be cupelled like silver-lead. When the metal is strongly heated on charcoal before the blowpipe, it volatilizes in brownish fumes, which are without odour. Upon removing the heat, the red-hot globule of metal continues to burn and give off vapours for some time afterwards, like pure antimony under similar circumstances. On cooling, the adjacent parts of the charcoal are coated with globules of sublimed metal. The oxide resembling litharge is the same as that formed by the superficial action of air on the metal, or steam at a high temperature; it may also be prepared in strong solution by decomposing sulphate of thallium with baryta water and filtering. The oxide may be obtained in the crystalline and anhydrous state by evaporating this solution to dryness *in vacuo*. Its physical characters having been fully described by M. LAMY, I have not further experimented with it.

Alcohol exerts a curious action upon thallium. A coil of pure thallium wire was placed in a tube with some absolute alcohol, just sufficient to cover it. At first no action was apparent, except a slight opalescence of the spirit. In the course of a few hours this had disappeared, and upon close examination needle-shaped crystals, together with a few drops of a colourless heavy liquid, were observed adhering to the sides of the tube and sinking in the alcohol. In three days the wire was nearly eaten away, whilst the oily drops had considerably increased in bulk. The alcoholic liquid was carefully decanted from the heavy oil and tested: dilution with water produced no change in it; it was strongly alkaline to test-paper, and reacted in other respects like a strong solution of protoxide of thallium. Upon the addition of a drop of hydrochloric acid, a thick curdy precipitate of protochloride of thallium was produced.

The oily liquid was decomposed upon the addition of water, solidifying to a yellow crystalline mass of protoxide of thallium, which dissolved on further addition of water and heating.

Upon evaporating the alcoholic liquid over a water-bath, a further formation of the heavy oil took place. The production of this oily liquid, by dissolving oxide of thallium in alcohol and evaporating, has been previously observed by M. LAMY, who calls it Thallic Alcohol.

18. Thallium dissolves in sulphuric acid with ease, evolving hydrogen. The gas given off burns with a flame in which thallium can frequently be detected with the spectroscope, although I have hitherto failed in proving the existence of a gaseous compound of hydrogen and thallium. Upon evaporating the solution, sulphate of thallium crystallizes out. I have little to add to M. LAMY's description of this salt. It forms large, well-defined colourless crystals; when heated to a little above the boiling-point of sulphuric acid, they fuse to a clear liquid, which on cooling appears glassy and slightly crystalline. The salt is soluble in twenty or thirty times its weight of cold water, and in much less when boiling, crystallizing out with facility upon cooling.

Thallium dissolves with the utmost rapidity in nitric acid. A piece of the metal thrown into this acid mixed with half its bulk of water, runs about on the surface like sodium on water, rapidly dissolving, and evolving nitric oxide mixed with nitrous oxide. I have not found any ammonia produced in this reaction. When the liquid cools, it becomes almost solid, from the crystallization of nitrate of thallium, which is nearly insoluble in nitric acid, although it is very soluble in water.

Hydrochloric acid attacks thallium but slowly, the action soon ceasing, owing to the formation of a layer of difficultly soluble chloride of thallium. When hydrochloric acid or a soluble chloride is added to a solution of the protoxide of thallium or one of its soluble salts, a white curdy precipitate of protochloride of thallium, $Tl\,Cl$, is thrown down, scarcely differing at first sight from chloride of silver. It has, however, a crystalline appearance, is slightly soluble in cold water, moderately so in boiling water, from which it crystallizes out on cooling like chloride of lead, and is insoluble in alcohol. When boiled in nitric acid or *aqua regia* it is converted into a higher chloride, crystallizing out in large spangles of a yellow colour. The same chloride is formed by the action of nitrohydrochloric acid upon the metal or its sulphide. It is more soluble in water and acids than the protochloride, and is precipitated in the latter form upon the addition of sulphite of soda to its solution.

I have already described * the properties of the sulphide of thallium and some other of its insoluble salts. Having since worked upon purer as well as larger quantities of the metal, I have an addition or two to make to my previous descriptions. Thus the protiodide of thallium is of a bright yellow colour, the red tinge which I formerly noticed in it being due to the presence of a persalt of thallium. It is insoluble in excess of dilute solution of iodide of potassium, being soluble only when the latter is concentrated. The protocarbonate is soluble in water.

Concentrated acetic acid dissolves thallium slowly when heated, forming a soluble acetate; very dilute cold acid has no action upon thallium. Owing to the solubility of the oxide of thallium, no precipitate is produced in the protosalts of this metal by potash, soda, or ammonia.

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* Proceedings of the Royal Society, June 19, 1862.

Neutral or slightly acid protosalts of thallium are incompletely precipitated by sulphuretted hydrogen, and not at all when a large excess of acid is present. Sulphide of ammonium, as I have already stated, precipitates them perfectly, reducing the metal to the state of protosulphide when in a higher state of oxidation.

19. The compounds of thallium are not only volatile when heated in the dry state, but many of its salts volatilize when their aqueous solutions are boiled. The chlorides are especially volatile, insomuch that loss is experienced in evaporating them down. Ten grains of pure metallic thallium were dissolved in a considerable excess of nitrohydrochloric acid, and the solution was gently boiled down in a retort. Upon testing the acid distillate by supersaturation with ammonia and addition of sulphide of ammonium, a considerable precipitate of sulphide of thallium was formed. Nitrohydrochloric acid was then added to the residue of sesquichloride of thallium remaining in the retort, and the distillation was repeated over a water-bath, care being taken that the evaporation in this case was conducted below the boiling-point of the liquid. Upon now testing the distillate, traces of thallium were still found in it: the metal in this case could not have been carried over mechanically, as the liquid in the retort had not once entered into ebullition.

Having for upwards of a year had considerable quantities of liquids containing thallium evaporated in open dishes in my laboratory, it was natural to anticipate, after the above experiment, that some quantities of the metal had been thus volatilized along with the aqueous vapour, and would be found adhering to the walls and deposited with the dust on the upper shelves of the room; a small portion of dust was accordingly removed from a shelf at a height of above 10 feet from the ground, and tested for thallium. A brilliant green line in the spectroscope showed me that this metal was present in more than minute traces.

20. Thallium may be determined quantitatively by precipitation, either as protochloride, iodide, or double chloride of platinum and thallium. The chloride must be washed with alcohol, as it is slightly soluble in water. The iodide and platinochloride are practically insoluble.

Position of Thallium amongst elementary bodies.

21. When I discovered thallium two years ago, owing to the excessively minute portion of substance which I had under examination, misled by its constant occurrence with sulphur and selenium, and basing my conjectures upon some of the properties first noticed—namely, its complete volatility below a red heat, its precipitation in the elementary form by zinc, its non-precipitation from an acid solution by alkalies, and its solubility in water when fused with nitre and carbonate of soda—reasoning upon these observations, I ventured to suggest that it was *probably* a metalloid belonging to

the sulphur group, although, I added, "I hesitate to assert this very positively"*. In speaking of a metalloid of the sulphur group, I should explain that I had in view, not a decidedly non-metallic body like sulphur, but one of the connecting links between metals and non-metals—a metalloid in the strict meaning of the word, like tellurium or arsenic. If I had formed any particular view upon the matter, knowing so little of the properties of the new body, it was that it might possibly prove to be a higher link in the sulphur, selenium, and tellurium chain. It was not long before further research showed me that the body under examination had, in addition to the characters already mentioned, others which gave it strictly metallic characters; and although no formal publication of this isolated fact was immediately made, the element was commonly spoken of in scientific circles as a new metal, and was so described by me at the Exhibition on the 1st of May last. I enter thus into details on so trifling a subject because French chemists are inclined to attach undue importance to the term, and misinterpret the meaning of metalloid.

Even with our present knowledge of the chemical and physical characteristics of thallium, it is not easy to assign its true position in the scale of elements. I cannot admit, with the French chemists, that it is an alkali-metal. Almost the only property which thallium possesses in common with the alkali-metals is the solubility of its oxide, and perhaps its forming an insoluble platinum-salt. But oxides of lead, silver, and mercury are also soluble in water, reacting in many respects like alkaline solutions; and oxide of thallium is far more analogous to these than to potash and soda, inasmuch as it has scarcely any affinity for water, becoming anhydrous, in a vacuum, even in the cold. In opposition to these reasons for classing it with the alkalies, we have numerous facts to prove that its true position is by the side of mercury, lead, or silver. The ready dehydration of its basic oxide—the insolubility of its sulphide, iodide, chloride, bromide, chromate, phosphate, sulphocyanide, and ferrocyanide—its great atomic weight—its ready reduction by zinc to the metallic state—and, according to Dr. MILLER, the complexity of its photographic spectrum—all prove that thallium cannot consistently be classed anywhere but amongst the heavy metals, mercury, silver, lead, &c.

22. Those who remember how readily figures can be moulded to suit any theory, will attach slight importance to the argument adduced by M. DUMAS in favour of thallium being related to potassium and sodium because its equivalent is rather near a figure obtained by adding twice the atomic weight of one metal to four times the atomic weight of the other. By similar processes of addition, multiplication, or subtraction it would not be difficult to prove a relationship between thallium and any desired group. Thus twice the equivalent of tellurium added to that of arsenic would make one equivalent of thallium, an argument in favour of its being a metalloid; one equivalent of mercury and one equivalent of lead added together make one equivalent of thallium, as also do one equivalent of silver and two equivalents of molybdenum—each proving it to be a heavy metal of the silver and lead group. Were it worth while

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* Chemical News, March 30, 1861, p. 193, and Phil. Mag., April 1861.

to pursue these relationships further, it would not be difficult to find many coincidences less strained than the one brought forward by M. DUMAS.

Analytical Notes on Thallium.

23. The chemical identification of thallium when associated with other metals is not difficult. I have already discovered very exact methods of detecting the presence of thallium in, and separating it from, most of its associated metals, and further experiment will doubtless still more increase the accuracy of its analytical detection and estimation.

In starting with the analysis of a thalliferous mineral by the ordinary analytical tables, in which Group I. is precipitated by hydrochloric acid, Group II. by hydrosulphuric acid, Group III. by ammonia, and Group IV. by sulphide of ammonium, a slight analytical difficulty will be at first met with, as, unless special precautions are taken, thallium will appear in all four groups. Thus if the thallium be in the state of a sesquisalt, no precipitate will be produced in a moderately dilute solution upon addition of hydrochloric acid; if, on the contrary, it be as a protosalt, the great bulk will come down in this group. In either case it will be advisable to reduce the remainder of the metal to the state of a protosalt, by passing sulphurous acid through the filtrate and heating. If, upon allowing the solution to cool after this treatment, a white crystalline precipitate of protochloride of thallium is produced, it will show that the metal originally existed in the state of a sesquisalt. This precipitate may be filtered off and examined separately. In the filtrate from this, even were there sufficient acid present to prevent the sulphide of thallium by itself from being precipitated by sulphuretted hydrogen, it will be partially carried down by other metals of this group which may be present. What escapes this precipitant will in a similar manner be partially carried down with the oxides of the third group, whilst the remaining thallium escaping the first three group-tests will be precipitated by sulphide of ammonium.

Thallium may be very accurately separated from most metals. Some of the analytical methods which I have employed for many months are very delicate; others, on the contrary, still require elucidation.

24. *Thallium from Zinc.*—(I will assume that a piece of commercial zinc has to be tested for thallium.) Dissolve the metal in sulphuric acid, adding a little nitric acid towards the end to effect the perfect solution of the black residue. Evaporate to drive off nitric acid; dissolve in a small quantity of water; filter from sulphate of lead, if any be present, and heat the moderately acid solution with excess of sulphite of soda. Allow the liquid to cool, and add a few drops of solution of iodide of potassium. A yellow precipitate of iodide of thallium will be thrown down. Many specimens of commercial zinc, tested in this manner, will be found to contain thallium.

25. *Thallium from Iron* (thalliferous iron pyrites).—Dissolve 30 or 40 grains of the finely powdered mineral in nitrohydrochloric acid; evaporate with excess of hydrochloric acid to drive off the nitric acid; redissolve in water; add sulphite of soda in

excess, and heat until all the iron is reduced to the proto-state; and then add iodide of potassium. Iodide of thallium will be precipitated. This test is sufficiently delicate to show thallium in a pyrites which does not contain more than 1 in 10,000.

Thallium from Manganese may be separated as thallium from iron.

26. *Thallium from Mercury.*—I have not yet ascertained a delicate and reliable method of separating thallium from salts of the suboxide of mercury. It is, however, very readily separated from persalts of mercury; and therefore the best plan is to peroxidize both metals by boiling with nitrohydrochloric acid; then reduce the thallium with sulphite of soda, and add iodide of potassium to the almost neutral solution. If much mercury be present, the precipitate will be almost pure scarlet; but on further addition of iodide of potassium, drop by drop, the iodide of mercury will dissolve, and will leave the iodide of thallium as an insoluble yellow powder. Upon warming the liquid the precipitate collects together and readily settles to the bottom. This is a very delicate test.

27. *Thallium from Lead.*—Evaporate the solutions to dryness with excess of sulphuric acid, and extract with hot water. Sulphate of lead will be left behind, whilst sulphate of thallium will be dissolved. This is a very ready process, but is not quite so accurate as the succeeding one.

28. *Thallium from Bismuth or Lead.*—Dilute the solution and add a *slight* excess of carbonate of soda; add solution of cyanide of potassium (free from sulphide), and allow the mixture to stand for an hour at the temperature of about 100° F.; then filter and wash: the residue will contain all the bismuth or lead. To the clear filtrate add sulphide of ammonium, and warm gently for some time; the deep-brown sulphide of thallium will gradually collect together in flakes at the bottom of the vessel. It must be washed with water containing a little sulphide of ammonium, as it readily oxidizes when moist, and might pass through the filter as sulphate of thallium.

This is an exceedingly delicate test for thallium in bismuth; and by its means it can be detected in most specimens of commercial bismuth and its salts, even when sold as pure. The presence of thallium in some samples of bismuth has been suspected by Dr. W. Bird Herapath[*]. The analytical method which he gives is not calculated to detect it except perhaps when present in comparatively large quantities. By the above process it will be found to be a very frequent constituent of bismuth compounds, even when working upon no more than 50 grains of material.

29. *Thallium from Copper.*—To the acid solution add ammonia in excess, and then cyanide of potassium until the blue colour has entirely disappeared; then add sulphide of ammonium, and gently warm for some time. Sulphide of thallium will gradually collect together in the liquid. By this test I have detected the presence of thallium in many specimens of copper as met with in commerce, as well as in crystallized sulphate of copper. It is extremely delicate.

Through the kindness of Dr. Matthiessen I have been enabled to examine for thallium a specimen of copper prepared in Spain by a process called "cementation." This consists in allowing copper pyrites to oxidize slowly, washing out the resulting sulphate of copper, and precipitating the solution with metallic iron. The pulverulent copper is then heated until it coheres, and the metal sent into the market in the form of pigs, no further purification being attempted.

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[*] Pharmaceutical Journal, Jan. 1, 1863.

This metal was found by Dr. Matthiessen to have a conducting-power for electricity of about 15, pure copper being 100. The exact metallic impurity which rendered it so preeminently bad had not been ascertained.. It was tested for thallium in the manner above described, and found to contain a large quantity. It is evident, from the way the copper was extracted, that any thallium which might have been present in the pyrites would accompany the copper.

30. *Thallium from Cadmium.*—To the acid solution of these metals add bichromate of potash, then excess of ammonia, and boil; insoluble chromate of thallium will be precipitated. This is not so delicate a test as some of the above, although by its means I have frequently detected thallium in metallic cadmium and its salts. Commercial sulphide of cadmium, as sold for artists' use, varies considerably in tint. Dark-coloured samples frequently contain thallium. I may especially instance, as being highly thalliferous, a beautiful specimen of this sulphide from Nouvelle Montagne, near Liège, which formed a prominent object in the Belgian department of the late Exhibition.

31. *Thallium from Gold.*—The gold may be separated by the usual process of reduction to the metallic state with oxalic acid, all the free nitric acid having been previously removed by evaporation with hydrochloric acid.

32. *Thallium from Antimony, Tin, and Arsenic.*—A very good method of separating these metals is to add excess of sulphide of ammonium to the alkaline solution. Sulphide of thallium will be precipitated, whilst the other sulphides will remain dissolved.

Most of the above processes have been tried with weighed quantities of the different metals, seldom taking more than 1 of thallium to 1000 parts of the other metal. They can therefore be relied upon to that extent; whilst some of them are much more delicate, as, for instance, the separation of thallium from iron, copper, bismuth, and lead.

It is advisable, in testing for small quantities of thallium, to appeal to the spectroscope for confirmatory evidence of the presence of this element in any precipitate suspected to contain it.

33. *Thallium from Sulphuric and Hydrochloric Acids.*—I have frequently met with specimens of commercial hydrochloric and sulphuric acids which contained almost enough thallium to be worth extracting. I may especially mention a very crude yellow hydrochloric acid now to be met with at about 8 shillings the hundredweight. Two ounces of this was neutralized with soda, and a few drops of sulphide of ammonium were added. A black precipitate was obtained, which in the spectroscope showed evident presence of thallium. At my request, Messrs. Hopkin and Williams, the well-known manufacturing chemists, treated 112 lbs. of this acid in the above manner, and forwarded me

the black precipitate obtained. It was worked up in the manner described in the first part of this paper (7.), and yielded a little over four grains of metallic thallium.

From sulphuric acid it may be separated in a similar way.

It is not difficult to understand how thallium gets into these acids. Messrs. CHANCE Brothers and Co., of Birmingham, makers of the thalliferous hydrochloric acid, have obligingly informed me that the process by which it is produced is the ordinary one of decomposing common salt in cast-iron pans and fire-brick furnaces. The acid is condensed in high stone towers or chambers filled with coke, and is afterwards collected in gutta-percha cisterns, and bottled or drawn off. The sulphuric acid used in the manufacture is obtained from iron pyrites burnt in kilns. Upon examining specimens of pyrites and lead-chamber-deposit forwarded by Messrs. CHANCE and Co., I find that the former contains thallium, but scarcely any appreciable traces are in the deposit, thus agreeing with the results of my previous investigations on this subject.

X. On the Distribution of Surfaces of the Third Order into Species, in reference to the absence or presence of Singular Points, and the reality of their Lines. By Dr. SCHLÄFLI, *Professor of Mathematics in the University of Berne. Communicated by* ARTHUR CAYLEY, *F.R.S.**

Received December 18,—Read December 18, 1862.

THE theory of the 27 lines on a surface of the third order is due to Mr. CAYLEY and Dr. SALMON; and the effect, as regards the 27 lines, of a singular point or points on the surface was first considered by Dr. SALMON in the paper "On the triple tangent planes of a surface of the third order," Camb. and Dub. Math. Journ. vol. iv. pp. 252–260 (1849). The theory as regards the reality or non-reality of the lines on a general surface of the third order, is discussed in Dr. SCHLÄFLI's paper, "An attempt to determine the 27 lines &c.," Quart. Math. Journ. vol. ii. pp. 56–65, and 110–120. This theory is reproduced and developed in the present memoir under the heading, I. General cubic surface of the third order and twelfth class; but the greater part of the memoir relates to the singular forms which are here first completely enumerated, and are considered under the headings II., III. &c. to XXII., viz. II. Cubic surface with a proper node, and therefore of the tenth class, &c., down to XXII. Ruled surface of the third order. Each of these families is discussed generally (that is, without regard to reality or non-reality), by means of a properly selected canonical form of equation; and for the most part, or in many instances, the reciprocal equation (or equation of the surface in plane-coordinates) is given, as also the equation of the Hessian surface and those of the Spinode curve; and it is further discussed and divided into species according to the reality or non-reality of its lines and planes. The following synopsis may be convenient:—

I. General cubic surface, or surface of the third order and twelfth class. Species I. 1, 2, 3, 4, 5.
II. Cubic surface with a proper node, and therefore of the tenth class. Species II. 1, 2, 3, 4, 5.
III. Cubic surface of the ninth class with a biplanar node. Species III. 1, 2, 3, 4.
IV. Cubic surface of the eighth class with two proper nodes. Species IV. 1, 2, 3, 4, 5, 6.
V. Cubic surface of the eighth class with a biplanar node. Species V. 1, 2, 3, 4.

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* Dr. SCHLÄFLI authorized me to make any alterations in the phraseology of his memoir, and to add remarks which might appear to me desirable. Passages in [], or distinguished by my initials, are by me, but I have not thought it necessary to distinguish alterations which are merely verbal or of trifling importance.—A. C.

VI. Cubic surface of the seventh class with a biplanar and a proper node. Species VI. 1, 2.

VII. Cubic surface of the seventh class with a biplanar node.. Species VII. 1, 2.

VIII. Cubic surface of the sixth class with three proper nodes. Species VIII. 1, 2, 3, 4.

IX. Cubic surface of the sixth class with two biplanar nodes. Species IX. 1, 2, 3, 4.

X. Cubic surface of the sixth class with a biplanar and a proper node. Species X. 1, 2.

XI. Cubic surface of the sixth class with a biplanar node. Species XI. 1, 2.

XII. Cubic surface of the sixth class with a uniplanar node. Species XII. 1, 2.

XIII. Cubic surface of the fifth class with a biplanar and two proper nodes. Species XIII. 1, 2.

XIV. Cubic surface of the fifth class with a biplanar node and a proper node. Species XIV. 1.

XV. Cubic surface of the fifth class with a uniplanar node. Species XV. 1.

XVI. Cubic surface of the fourth class with four proper nodes. Species XVI. 1, 2, 3.

XVII. Cubic surface of the fourth class with two biplanar and one proper node. Species XVII. 1, 2, 3.

XVIII. Cubic surface of the fourth class with one biplanar and two proper nodes. Species XVIII. 1.

XIX. Cubic surface of the fourth class with a biplanar and a proper node. Species XIX. 1.

XX. Cubic surface of the fourth class with a uniplanar node. Species XX. 1.

XXI. Cubic surface of the third class with three biplanar nodes. Species XXI. 1, 2.

XXII. Ruled surface of the third order and the third class. Species XXII. 1, 2, 3.—A.C.

I. *General cubic surface, or surface of the third order and twelfth class.*

Art. 1. As the system of coordinates undergoes various transformations (sometimes imaginary ones), it becomes necessary to adhere to an invariable system of a real meaning, for instance the usual one of three rectangular coordinates. We shall call this the system of *fundamental coordinates*, and define it by the condition that the coordinates of every real point (or the ratios of them, if they be four in number) shall be real. Consequently any system of rational and integral equations, expressed in variables of a real meaning, and where all the coefficients are real, will be termed a *real system* (of equations), whether there be real solutions or none, provided that the number of equations do not exceed that of the variables, or of the quantities to be determined. The degree of the system will be the number of solutions of it when augmented by a sufficient number of arbitrary linear equations; and such degree will generally be the product of the degrees of the single equations. It is obvious that the system, whenever its degree is *odd*, represents a *real* continuum of as many dimensions as there are independent variables; for instance, every real quaternary cubic represents a real surface.

It is known* that on the surface of the third order there are 27 lines which form 45 triangles in such manner that through each line there pass five planes meeting the surface in this line and two other lines, or say five triangle-planes. Lines not intersecting each other may be termed *independent lines*, as far as a surface of the third order is capable of containing all of them; the greatest number of such lines is four; that is to say, in whatever manner we may choose two, three, or four not intersecting lines on the surface, the system has always the same properties. Let two independent lines I. and II. on the surface be given, and imagine any one of the five triangle-planes passing through I.; then II. must intersect one of the two other sides of this triangle; in other words, this triangle affords a line cutting both I. and II., and a line cutting I. alone. Hence it appears that there are five lines cutting both I. and II., five lines cutting I. only, five lines cutting II. only, and ten lines cutting neither I. nor II.

[The theory of the 27 lines depends on the expression of the equation of the surface in the form $P-Q=0$, where P and Q are real or imaginary cubics breaking up into linear factors; in fact, if the equation be so expressed, it is at once seen that each of the planes $P=0$ meets each of the planes $Q=0$ in a line on the surface, so that the form gives at once 9 out of the 27 lines. The three planes represented by the equation $P=0$ (or $Q=0$) are termed a *Trihedral* of the surface.]

Art. 2. Prop. *It is always possible to find a trihedral represented by a real quaternary cubic.*

The truth of this proposition is evident when all the 27 lines are real. But when some of them are imaginary, these are conjugate by pairs. As the case when two conjugate lines intersect one another is fitter for our purpose, we begin with the other case when two conjugate lines do not intersect each other.

The problem, then, of finding the five lines intersecting such pair of conjugate lines depends on a real system. Hence among the five lines there will be an odd number of real ones; and imaginary ones, when existing, will be conjugate by pairs. Call the given two independent and conjugate lines I. and II., and the five lines intersecting each of them a, b, c, d, e. If d and e be imaginary and conjugate, the plane containing I. and d will be conjugate to that containing II. and e, and these two planes will not intersect in a line of the surface (for if they did, a line of the surface would unite the intersection of II. and d with that of I. and e; and it is obviously a great loss of generality if *three* lines of the surface meet in a point). But if all the five lines a, b, c, d, e be real, then—because they can be intersected simultaneously only by the lines I. and II., and because through each of the five lines there passes at least one real triangle-plane—*it must be possible to choose* among all the real planes each passing through any one of the real lines a, b, c, d, e, *two real triangle-planes not intersecting in a line of the surface.*

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* See Cambridge and Dublin Math. Journ. vol. IV. p. 118, the original memoirs of Messrs. Cayley and Salmon on the triple tangent-planes of the cubic surface.

As to the easier case first mentioned, when there are on the surface two conjugate lines intersecting each other, it is plain at first sight that they afford us four pairs of conjugate triangle-planes not intersecting in a line of the surface.

Now whether we have two conjugate planes, or two real planes not intersecting in a line of the surface, the third plane completing them to a trihedral is singly determined by a real system and is therefore real; and hence the trihedral is represented by a real cubic.

Art. 3. PROP. *A real cubic surface of the twelfth class* (or, what is the same thing, without nodes) *can always be represented by* $uvw+xyz=0$, *where both* uvw *and* xyz *are real cubics breaking up into linear factors.*

Let $\lambda A+B=0$ be a cubic equation expressed in the fundamental coordinates with real coefficients, λ a numerical factor imaginary if possible, A B cubics each decomposible into linear factors, but A real and B imaginary if possible, and let λ', B' be respectively conjugate to λ, B. Then $(\lambda-\lambda')A+B-B'=0$ must be an identical equation, and each solution satisfying the system $A=0$, $B=0$ will therefore also satisfy $B'=0$. But it would be a loss of generality if, through the nine lines in which the two trihedrals A and B intersect each other, there should pass a third trihedral B'. Therefore we must have $\lambda=\lambda'$, $B=B'$; in other words, if one trihedral of a pair is represented by a real cubic, its associate is also so, and the trihedral-pair equation does not imply any imaginary numerical factor. We are therefore justified in asserting that a real cubic surface (without nodes) can always be exhibited in one of these three trihedral-pair forms $uvw+xyz=0$; 1. u, v, w, x, y, z are all real; 2. u, v, w, x are real, y is conjugate to z; 3. u, x are real, v is conjugate to w, and y to z.

Art. 4. To save the reader the trouble of consulting my paper in vol. ii. of the Quarterly Mathematical Journal, I will give here a scheme which serves to determine and denote the twenty-seven lines. In space, only four linear functions can be independent; any fifth one will be a linear and homogeneous function of these four linear functions. Hence it is plain that in the identical equation

$$Au+Bv+Cw+Dx+Ey+Fz=0$$

the coefficients are linear and homogeneous functions of two arbitrary constants; and of course only their ratio is here of importance. The identical equation

$$Au(Bv+Dx)(Cw+Dx)+Dx(Au+Ey)(Au+Fz)=ABCuvw+DEFxyz$$

then suggests the propriety of making the six coefficients subject to the condition $ABC=DEF$, because we have then a transformation of the original trihedral-pair form into another like form. But the condition (being a cubic equation) has three roots, according to which we may put

$$\Sigma au=au+bv+cw+dx+ey+fz=0,\quad abc=def;$$
$$\Sigma a'u=0,\quad a'b'c'=d'e'f';\quad \Sigma a''u=0,\quad a''b''c''=d''e''f''.$$

We denote the line $(u=0,\ x=0)$ by $\overline{ux}$, and so on for all the nine lines arising from the intersection of the two trihedrals uvw, xyz. Again, since there are twenty-seven forms of the equation of the surface such as

$$au(bv+dx)(cw+dx)+dx(au+ey)(au+fz)=0,$$

the equations $au+dx=0$, $bv+ey=0$, $cw+fz=0$ belong to a line of the surface which we denote by l, while (ux), $(ux)'$, $(ux)''$ respectively represent the triangle-planes

$$au+dx=0,\ a'u+d'x=0,\ a''u+d''x=0,$$

and so on. Now in the scheme of the nine initial lines

$$\begin{array}{c|ccc} & x & y & z \\ \hline u & \overline{ux} & \overline{uy} & \overline{uz} \\ v & \overline{vx} & \overline{vy} & \overline{vz} \\ w & \overline{wx} & \overline{wy} & \overline{wz} \end{array}$$

we may first perform all the positive permutations of the columns, and then deduce from these the negative ones by permuting only y and z. In each permutation we keep in view only the lines placed in the principal diagonal. We thus obtain the following easily intelligible scheme

$$\begin{array}{llllllll|llllllll} \text{through} & \overline{ux}, & \overline{vy}, & \overline{wz} & \text{pass} & l, & l', & l'', & \text{through} & \overline{ux}, & \overline{vz}, & \overline{wy} & \text{pass} & p, & p', & p'', \\ ,, & \overline{uy}, & \overline{vz}, & \overline{wx} & ,, & m, & m', & m'', & ,, & \overline{uz}, & \overline{vy}, & \overline{wx} & ,, & q, & q', & q'', \\ ,, & \overline{uz}, & \overline{vx}, & \overline{wy} & ,, & n, & n', & n'', & ,, & \overline{uy}, & \overline{vx}, & \overline{wz} & ,, & r, & r', & r''. \end{array}$$

The plane (ux) contains the lines $\overline{ux}$, l, p, and so on; and the plane containing l, m', n'' may be represented by (lmn), and so on.

I do not think it worth while to show that the equation ABC=DEF, when explicitly written, always has real coefficients, and that each of the cases hereafter coming into consideration can be *constructed*, and that it therefore *exists*.

Art. 5. *First case.*—u, v, w, x, y, z are all of them real.

A. The cubic condition (ABC=DEF) has three real roots. It is then at once plain that all the twenty-seven lines and all the forty-five triangle-planes are real. *First species*, I., 1.

B. The cubic condition has but one real root, to which let belong the coefficients a, b, ... Each geometrical form then changes into its conjugate by merely permuting the two accents $'$ and $''$. So the nine initial lines and the six lines l, m, n, p, q, r (together fifteen) are real, and the remaining lines are imaginary and form a double-six

$$\begin{pmatrix} l', & m', & n'; & p'', & q'', & r'' \\ l'', & m'', & n'', & p', & q', & r' \end{pmatrix}$$

where any two corresponding lines are also conjugate. Fifteen lines and fifteen planes are real. *Second species*, I., 2.

Art. 6. *Second case.*—y and z only are imaginary, and therefore conjugate.

A. The cubic condition has three real roots. Each form changes into its conjugate by merely permuting y and z. Therefore, in the trihedral-pair scheme, only the first column contains real lines, the two other columns are conjugate; and as to the eighteen remaining lines, their two schemes are conjugate in the above-mentioned order. Three lines and thirteen planes are real; for there is one real triangle through each side of which there pass, besides the plane of the triangle, four other real planes. *Fourth species*, I., 4.

B. The cubic condition has but one real root to which let belong the coefficients a, b, ... Each form changes into its conjugate one by permuting at once y, z and the two accents $'$ and $''$. Three lines and seven planes are real. The real lines form a triangle, through each side of which there pass, besides the plane of the triangle, two other real planes. *Fifth species*, same as third case B, *infra.*

Third case.—v is conjugate to w; y to z; and u, x are real.

A. The cubic condition has three real roots. Each form changes into its conjugate by permuting at once v, w, and y, z. The three above-mentioned schemes (each of nine lines) change hereby respectively into

$$\begin{array}{ccc|ccc|ccc}
\overline{ux} & \overline{uz} & \overline{uy} & l & l' & l'' & p & p' & p'' \\
\overline{wx} & \overline{wz} & \overline{wy} & n & n' & n'' & r & r' & r'' \\
\overline{vx} & \overline{vz} & \overline{vy} & m & m' & m'' & q & q' & q''
\end{array}.$$

The comparison shows that only $\overline{ux}$, l, l', l'', p, p', p'' keep their places, and are therefore real. Of the planes, only u, x, (ux), $(ux)'$, $(ux)''$ are real. Seven lines and five planes are real; namely, through a real line there pass five real planes, three of which, (ux), $(ux)'$, $(ux)''$, contain real triangles. *Third species*, I., 3.

B. The cubic condition has but one real root. To find the form conjugate to a given one, we must at once permute v, w, also y, z, and lastly the two accents $'$ and $''$. The three schemes of lines by this process become

$$\begin{array}{ccc|ccc|ccc}
\overline{ux} & \overline{uz} & \overline{uy} & l & l'' & l' & p & p'' & p' \\
\overline{wx} & \overline{wz} & \overline{wy} & n & n'' & n' & r & r'' & r' \\
\overline{vx} & \overline{vz} & \overline{vy} & m & m'' & m' & q & q'' & q'
\end{array}$$

Only $\overline{ux}$, l, p keep their places, and therefore are real. Besides the planes u, x (ux), also the planes (lmn), (lnm), (pqr), (prq) are real. The three real lines form a triangle, through each side of which there pass two more real planes. *Fifth species*, I., 5.

Art. 7. *How many kinds of nodes can exist on a cubic surface?*

Considering in the first instance the theory of an ordinary node or conical point, let us imagine a surface of the nth order with a node, at which we are allowed to place the point of reference $\frac{\partial}{\partial w}$*. Let then an arbitrary line be given, through which tangent planes to the surface are to pass, and through this line draw the planes of reference $z=0$ (through the node) and $w=0$ (not passing through the node). The equation of the surface will then take the form

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* As to this mode of expression, see foot-note to art. 8.—A. C.

$$F=Pw^{n-2}+Qw^{n-3}+Rw^{n-4}+\&c.=0,$$

where

$$P=(x, y, z)^2,\ Q=(x, y, z)^3,\ R=(x, y, z)^4,\ \&c.,$$

and the points of contact of tangent planes passing through the given line ($z=0$, $w=0$) must satisfy the conditions $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$. In the proximity of the node the system of the three equations reduces itself to $P=0$, $\frac{\partial P}{\partial x}=0$, $\frac{\partial P}{\partial y}=0$ $\left(\text{or, what is the same thing, } \frac{\partial P}{\partial x}=0,\ \frac{\partial P}{\partial y}=0,\ z\frac{\partial P}{\partial z}=0\right)$, if none of these equations be a necessary consequence of the other two. The node $\frac{\partial}{\partial w}$ then represents two solutions, because the equations are respectively of the degrees 2, 1, 1 [or, what is the same thing, among the tangent planes through the line the plane passing through the node counts for two tangent planes; that is, the class of the surface is diminished by 2]. The exception $\left(\frac{\partial P}{\partial x}=0,\ \frac{\partial P}{\partial y}=0,\ z=0\right)$ is inadmissible; for should the plane $z=0$ touch the cone $P=0$, the line ($z=0$, $w=0$) would not be arbitrarily chosen. The only possible exception is when the three equations

$$\frac{\partial P}{\partial x}=0,\ \frac{\partial P}{\partial y}=0,\ \frac{\partial P}{\partial z}=0$$

can be simultaneously satisfied. Consequently so long as the nodal cone $P=0$ does not break up into a pair of planes, there are two solutions, or the class is diminished by 2.

In the excepted case, where the nodal cone $P=0$ breaks up into a pair of planes, we may assume $P=xy$ (or $P=x^2$, to be discussed in the sequel); and since now the equations $xy=0$, $x=0$, $y=0$, are no longer independent, we must go on to consider also

$$Q=az^3+Lz^2+Mz+N,$$

where

$$L=(x, y),\quad M=(x, y)^2,\quad N=(x, y)^3.$$

For the sake of shortness, let $w=1$. We then have

$$xy+az^3+Lz^2+Mz+N+\&c.=0,$$

$$y\ +\frac{\partial L}{\partial x}z^2+\frac{\partial M}{\partial x}z+\frac{\partial N}{\partial x}+\&c.=0,$$

$$x\ +\frac{\partial L}{\partial y}z^2+\frac{\partial M}{\partial y}z+\frac{\partial N}{\partial y}+\&c.=0,$$

and unless the constant a vanish, the system (in the proximity of the node) reduces itself to $x=0$, $y=0$, $z^3=0$; that is to say, a biplanar node, in general, counts for three solutions, or diminishes the class by 3.

Next it remains to put $a=0$, $L=ax+by$, when the system becomes

$$x+bz^2+\ldots.=0,\quad y+az^2+\ldots.=0,\quad xy+(ax+by)z^2+\ldots+Kz^4+\ldots=0,$$

where Kz^4 is borrowed from R; and the last equation of the system reduces itself by means of the others to $(K-ab)z^4+\ldots=0$. The node here unites four solutions, unless $K-ab$ should vanish; that is to say, if the nodal edge $(x=0,\ y=0)$ lie on the cone $Q=0$, the biplanar node lowers the class of the surface by 4, unless the portion of the surface surrounding the node be, in the first approximation, represented by the form $(x+bz^2)(y+az^2)+$ terms of the fifth order in regard to z, $=0$.

The further supposition would be $K-ab=0$; but let us now assume a cubic surface, that we may have $K=0$, and therefore $ab=0$. Selecting the case $b=0$, we put

$$Q=axz^2+(bx^2+cxy+dy^2)z+N,$$

whence

$$x+(cx+2dy)z+\ldots=0,\quad y+az^2+\ldots=0,$$

or neglecting higher orders than here come into consideration, $y=-az^2$, $x=2adz^3$, whereby $F=0$ becomes $a^2dz^5+\ldots=0$, so that the system is reduced to $x=0$, $y=0$, $a^2dz^5=0$. That is to say, if one of the nodal planes touch the surface along the nodal edge, the biplanar node lowers the class of the surface by 5, unless the cone $Q=0$ have that line of contact either for a double line (if $a=0$), or for a line of inflexion (if $d=0$).

The exceptional supposition then to be made separates itself into $a=0$ and $d=0$. But $a=0$ would cause all the terms of F to be of the second degree, at least in respect to x, y, so that the surface would have $(x=0,\ y=0)$ for a double line. Assuming then $d=0$, we may put

$$Q=xz^2+(ax^2+bxy)z+cx^3+dx^2y+exy^2+fy^3,$$

when the system reduces itself to $x=0$, $y=0$, $-fz^6=0$. That is to say, if one of the nodal planes osculate the surface along the nodal edge, the biplanar node lowers the class by 6. Here we must stop; for if we suppose $f=0$, the cubic F becomes divisible by x.

We go on to the case where the nodal cone becomes a pair of coincident planes, or say where we have a uniplanar node. The equation of the surface is

$$F=x^2w+ay^3+3by^2z+3cyz^2+dz^3+x(ey^2+fyz+gz^2)+x^2(hy+jz)+Kx^3=0.$$

For indefinitely small values of x, y, z, the equation $\frac{\partial F}{\partial x}=0$ causes x to be of the second order in respect to y, z. The system of conditions for the point of contact (in the proximity of the node) of a tangent-plane passing through the line $(z=0,\ w=0)$ reduces itself therefore to

$$x=0,\quad ay^2+2byz+cz^2=0,\quad ay^3+3by^2z+3cyz^2+dz^3=0,$$

unless the discriminant of the last-mentioned cubic should vanish. Except in this case, the system shows that the node counts for six solutions of

$$\left(F=0,\ \frac{\partial F}{\partial x}=0,\ \frac{\partial F}{\partial y}=0\right),$$

or, what is the same thing, that a uniplanar node lowers in general the class by 6.

But if the binary cubic $ay^3+3by^2z+3cyz^2+dy^3$ contain a squared factor, we may denote this by y^3, and then write

$$F=x^2w+ay^3+by^2z+(cy^2+dyz+ez^2)x=0$$

for the equation of the surface; for it is plain that we are allowed to disregard the subsequent terms divisible by x^2. On forming the equation in plane-coordinates, it is immediately seen that this surface is of the fifth class, unless $b=0$; that is, in the general case, the class is diminished by 7.

Lastly, if $b=0$, then we have

$$F=x^2w+ay^3+(cy^2+dyz+ez^2)x=0;$$

and by forming the equation in plane-coordinates, the surface would be found to be of the fourth class, that is, the class of the surface is diminished by 8.

A closer discussion of the last two cases is reserved for a fit occasion.

In the whole we are to distinguish eight kinds of nodes on the cubic surface: 1, the proper node, which lowers the class by *two*; 2, the biplanar node, where the nodal edge does not belong to the surface and which lowers the class by *three*; 3, the biplanar node, where a plane different from both nodal planes touches the surface along the nodal edge and which lowers the class by *four*; 4, the biplanar node, where one of the two nodal planes touches the surface along the nodal edge and which lowers the class by *five*; 5, the biplanar node, where one nodal plane osculates the surface along the nodal edge and which lowers the class by *six*; 6, the uniplanar node, where the nodal plane intersects the surface in three distinct lines and which lowers the class by six; 7, the uniplanar node, where the nodal plane touches the surface along a line and which lowers the class by *seven*; 8, the uniplanar node, where the nodal plane osculates the surface along a line and which lowers the class by *eight*.

Art. 8. *On the case of two nodes on the cubic surface.*

Let f be the quaternary cubic of the surface, P, Q the symbols* of two different nodes on it; then P^2f, Q^2f will identically vanish. If now R be the symbol of any third point, the symbol $\alpha P+\beta Q+\gamma R$, where α, β, γ denote arbitrary multipliers, will belong to a point in the same plane with the points P, Q, R, and the equation

$$(\alpha P+\beta Q+\gamma R)^3f=6\alpha\beta\gamma PQRf+3\gamma^2(\alpha P+\beta Q)R^2f+\gamma^3R^3f=0$$

will represent the section of the surface made by the plane. Then if the point R satisfy the condition $PQRf=0$, the equation will become divisible by γ^2, that is to say, the equation $PQRf=0$, in respect to the elements of R, represents a plane touching the surface along the line joining the nodes P and Q, and besides intersecting it in a line represented by

$$3\alpha PR^2f+3\beta QR^2f+\gamma R^3f=0,$$

if here α, β, γ are regarded as planimetrical coordinates, and the point R as fixed. In the sequel I shall sometimes term the former line *axis* and the latter *transversal.*

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* If x', y', z', w' are the coordinates of a node, x', y', z', w' current coordinates, then the symbol P of the node is $=x'\partial_x+y'\partial_y+z'\partial_z+w'\partial_w$ and P^2f is $=P'f'=(x\partial_{x'}+y\partial_{y'}+z\partial_{z'}+w\partial_{w'})f'$, which vanishes identically, that is independently of x, y, z, w, in virtue of the equations $\partial_{x'}f'=0$, &c. satisfied at the node.—A. C.

If $Pw+Q=0$, where $P=(x, y, z)^2$, $Q=(x, y, z)^3$, be the equation of a cubic surface with a node, I shall call the six lines represented by the system $P=0$, $Q=0$ *nodal rays.* They belong to the surface, and it is plain that two of them at least must coincide in order that the surface may have another node, and this will lie on the line uniting two or more rays of the first node.

II. *Cubic surface with a proper node, and therefore of the tenth class.*

Art. 9. The equation of this surface can always be thrown into the form $Pw+Q=0$, where $P=(x, y, z)^2$, $Q=(x, y, z)^3$.

Let l be a linear and homogeneous function of x, y, z, then

$$P(w-l)+Q+lP=0$$

is the same equation. But we may in fifteen different ways dispose of the three coefficients in l, so that $Q+lP$ breaks up into three linear factors, and are therefore allowed to write

$$\varphi=(ax^2+by^2+cz^2+2dyz+2ezx+2fxy)w+2xyz=0$$

as an equation of the surface, where, for the sake of shortness, the ternary quadric ax^2+&c. of the nodal cone may be denoted by χ, and the derivatives of $\frac{1}{2}\chi$ by X, Y, Z. Again, let

$$\Delta=abc-ad^2-be^2-cf^2+2def,\quad A=bc-d^2,\quad B=ca-e^2,\quad C=ab-f^2,$$
$$D=ef-ad,\quad E=fd-be,\quad F=de-cf,$$

and determine the constant λ by the quadratic equation $(a\lambda-D)^2-BC=0$, then $\chi+2\lambda yz$ will break up into two linear factors, and $\varphi=(\chi+2\lambda yz)w+2(x-\lambda w)yz$ will be a trihedral-pair form of the surface. Its particularity is sufficiently determined by the condition that an edge of one trihedral intersects an edge of the other trihedral, the point of intersection being the node. I wished only to notice the connexion of such form with the presence of a proper node, yet will no longer dwell upon it, because I prefer to select hereafter one of those ten trihedral-pairs in which no plane passes through the node, for investigating by its help the position of the 27 lines.

Let p, q, r, s denote plane-coordinates such that $px'+qy'+rz'+sw'=0$ shall be the equation in point-coordinates x', y', z', w' of a tangent plane to the surface $\varphi=0$. To find then the *reciprocal equation* of the surface, we are concerned with the system

$$\varphi=0,\ \frac{\partial\varphi}{\partial x}:\frac{\partial\varphi}{\partial y}:\frac{\partial\varphi}{\partial z}:\frac{\partial\varphi}{\partial w}=p:q:r:s,$$

where the first equation may also be replaced by $l+sw=px+qy+rz+sw=0$. The equations

$$\tfrac{1}{2}\frac{\partial\varphi}{\partial x}=Xw+yz,\quad \tfrac{1}{2}\frac{\partial\varphi}{\partial y}=Yw+xz,\quad \tfrac{1}{2}\frac{\partial\varphi}{\partial z}=Zw+xy,\quad \frac{\partial\varphi}{\partial w}=\chi$$

lead to the system

$$p\chi+2lX-2syz=0,\quad q\chi+2lY-2szx=0,\quad r\chi+2lZ-2sxy=0,$$

the equations whereof are the derivatives of

$$l\chi-2sxyz\big(=(px+qy+rz)\chi-2sxyz\big)=0$$

with respect to x, y, z. The reciprocal equation of the surface therefore is of the form $\Omega=0$, where Ω is a decimic function of (p, q, r, s), which multiplied by s^2 is the discriminant of the ternary cubic

$$\begin{aligned}&(3ap)x^3+3.(aq+2fp)x^2y+3.(ar+2ep)x^2z+3.(bp+2fq)xy^2\\&+6.(dp+eq+fr-s)xyz+3.(cp+2er)xz^2+(3bq)y^3+3.(br+2dq)y^2z\\&+3.(cq+2dr)yz^2+(3cr)z^3.\end{aligned}$$

Hence to work out the decimic in question we may use the invariants of the fourth and sixth order which Dr. SALMON* denotes by S and T, only that we replace the latter notation by $-8\mathrm{T}$. Putting, then,

$$\Phi=\mathrm{A}p^2+\mathrm{B}q^2+\mathrm{C}r^2+2\mathrm{D}qr+2\mathrm{E}rp+2\mathrm{F}pq,\quad \tfrac{1}{2}d\Phi=\mathrm{P}dp+\mathrm{Q}dq+\mathrm{R}dr,$$

$$t=dp+eq+fr,\quad \mathrm{U}=adqr+berp+cfpq,\quad \mathrm{V}=2\Delta pqr-aqr\mathrm{P}-brp\mathrm{Q}-cpq\mathrm{R},$$

$$\mathrm{W}=a^2\mathrm{A}q^2r^2+b^2\mathrm{B}r^2p^2+c^2\mathrm{C}p^2q^2+2pqr(bc\mathrm{D}p+ca\mathrm{E}q+ab\mathrm{F}r),$$

$$\mathrm{L}=s^2-2ts-\Phi,\quad \mathrm{M}=\mathrm{U}s+\mathrm{V},\quad \mathrm{N}=2abcpqrs+\mathrm{W},\quad \mathrm{S}^3-\mathrm{T}^2=108s^2\Omega,$$

we find

$$\mathrm{S}=\mathrm{L}^2-12s\mathrm{M},\quad \mathrm{T}=\mathrm{L}^3-18s\mathrm{LM}-54s^2\mathrm{N},$$

$$\begin{aligned}\Omega&=\mathrm{L}^3\mathrm{N}+\mathrm{L}^2\mathrm{M}^2-18s\mathrm{LMN}-16s\mathrm{M}^3-27s^2\mathrm{N}^2\\&=2abcpqrs^7+\{abc\Sigma aq^2r^2+2pqr\Sigma bc(2ef-7ad)p\}s^6\\&+2\{\Sigma bc^2(ef-3ad)p^3q^2+pqr\Sigma bc(-3abc+21ad^2+be^2+cf^2-12def)p^2\\&+2pqr\Sigma a(-8abcd+16bcef-6d(be^2+cf^2)+2d^2ef)qr\}s^5\\&+(\quad)s^4+(\quad)s^3+(\quad)s^2+(\quad)s\\&-\Delta\Phi^2(cq^2-2dqr+br^2)(ar^2-2erp+cp^2)(bp^2-2fpq+aq^2),\end{aligned}$$

and $\Omega=0$ is the reciprocal equation of the surface.

If $-16\mathrm{H}$ denote the Hessian of the cubic φ, then

$$\mathrm{H}=\Delta\chi w^2+2(\chi\Sigma\mathrm{D}x-3\Delta xyz)w-\Sigma a^2x^4+2\Sigma bcy^2z^2+4xyz\Sigma(ad+ef)x;$$

the spinode curve therefore is represented by the system

$$\varphi=0,\quad 8\Delta xyzw-8xyz\Sigma adx+\Sigma a^2x^4-2\Sigma bcy^2z^2=0;$$

hence it is a complete curve of the twelfth degree, and has the node of the cubic surface for a sixfold point, where the six nodal rays are tangents to the curve.

Art. 10. Starting from a trihedral-pair form $uvw+xyz=0$, where no four of the six planes have a point in common, and letting $\mathfrak{D}$ be a linear differentiation symbol signifying that the differentials of the four fundamental coordinates may be replaced by arbitrary quantities ($\mathfrak{D}=\alpha\partial_x+\beta\partial_y+\gamma\partial_z+\delta\partial_w$, if for the moment (x, y, z, w) are the fundamental coordinates), we see that at the node the differential equation $\mathfrak{D}(uvw+xyz)=0$ ought not to be different from the general identical equation

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* Higher Plane Curves, pp. 184 and 186.

$$A\mathfrak{D}u+B\mathfrak{D}v+C\mathfrak{D}w+D\mathfrak{D}x+E\mathfrak{D}y+F\mathfrak{D}z=0;$$

hence the coefficients of the differentials in both equations must be proportional. But since in the former the coefficients vw, uw, uv, yz, xz, xy satisfy the equation

$$vw.uw.uv=yz.xz.xy,$$

or, which is the same thing,

$$(uvw+xyz)(uvw-xyz)=0,$$

the coefficients in the latter differential equation belong to one of the roots of the well-known cubic condition. Let them, for instance, be a', b', c', d', e', f'; then in consequence of the equation of the surface the proportions in question become

$$a'u=b'v=c'w=-d'x=-e'y=-f'z;$$

or, because without any loss of generality (since the linear functions $u, v, \ldots$ imply arbitrary numerical factors) we may replace a', b', c', d', e', f' by 1, 1, 1, 1, 1, 1, more simply

$$u=v=w=-x=-y=-z$$

at the node. Hence, and from the first and third identical relations, we get

$$a+b+c=d+e+f,\quad abc=def,\quad a''+b''+c''=d''+e''+f'',\quad a''b''c''=d''e''f''.$$

But we may put

$$a''=\lambda a+\mu,\quad b''=\lambda b+\mu, \text{ \&c.}$$

and we then obtain

$$(bc+ca+ab-ef-fd-de)\lambda^2\mu=0.$$

The factor within the brackets, if vanishing, would require one of the six cases such as $a=d$, $b=e$, $c=f$, and leave λ, μ indeterminate. Avoiding so great a restriction, and keeping to the proper meaning of the auxiliary cubic condition, we find that it has two equal roots $\lambda=0$, and a single root $\mu=0$. Consequently the constants corresponding to the single root are a, b, c, d, e, f, and satisfy the equations

$$a+b+c=d+e+f,\quad abc=def;$$

the constants in the accented sets are all of them equal to unity. Hence the line l' coincides with l'', m' with m'', and so on, and all these six pairs of coincident lines pass through the node. It may also be observed that they formed in the general case a double-six, and that now the corresponding lines (in both sixes) also coincide. Moreover, since the three independent lines l', m', n' (in the general case) are intersected by each of the three independent lines p'', q'', r'', all these six lines lie (in the general case) upon a quadratic surface; and now that all the six lines meet in a point, the quadratic surface must degenerate into a cone. Let

$$P=(v+x)(w+x)-(u+y)(u+z),\quad Q=(u+x)(u+y)(u+z),$$

then

$$uP+Q=ux(u+v+w+x+y+z)+(uvw+xyz);$$

and because $u+v+w+x+y+z=0$ is the second (or third) identical relation, and $uvw+xyz=0$ the equation of the surface, the latter is changed into $uP+Q=0$, which form shows the nodal cone $P=0$, the equation of which may also be exhibited under the symmetrical form

$$vw+wu+uv-yz-zx-xy=0.$$

Art. 11. *Distribution into species.*—It is plain that a single node of a real cubic surface cannot but be a real point. We may therefore draw through it three (real) fundamental planes (which call x, y, z) and take the fourth plane at pleasure (call it w); the equation of the surface then is $wP+Q=0$, where the functions P, Q contain only x, y, z, and therefore represent cones respectively of the second and third orders; and it is obvious that as well in P as in Q all the coefficients will be real. Hence as to the six nodal rays ($P=0$, $Q=0$), all of them may be real, or four, or two, or none. So we might distinguish four species of the cubic surface with a single proper node; but in the last of the mentioned cases (when the node is an isolated point of the surface) the cone $P=0$ may be real or imaginary. Let us therefore distinguish five species.

First species, II. 1. *All six nodal rays are real.*—The surface is constructed, when we assume six constants and six linear functions of the fundamental coordinates, all of them real, and satisfy the equations

$$a+b+c=d+e+f,\quad abc=def,\quad u+v+w+x+y+z=0,\quad au+bv+cw+dx+ey+fz=0,$$

where $bc+ca+ab-ef-fd-de$ must not vanish. Then $uvw+xyz=0$ is the equation of the surface. Not passing through the node, there are fifteen simple real lines, which form fifteen triangles, each line being common to three simple triangle-planes. Of the fifteen planes to be twice counted, each contains one of the simple lines and two nodal rays. This species constitutes the transition from the first to the second species of the general surface*.

Second species, II. 2. *Only four nodal rays are real.*—While we keep to the same system of equations as before, it is possible to dispose of the constants and linear functions in such manner that a, b, c are respectively conjugate to d, e, f, and u, v, w to x, y, z. Then by permuting i and $-i$, the three schemes

$$\begin{array}{ccc|ccc|ccc}
\overline{ux} & \overline{uy} & \overline{uz} & l & (l' & l'') & p & (p' & p'') \\
\overline{vx} & \overline{vy} & \overline{vz} & m & (m' & m'') & q & (q' & q'') \\
\overline{wx} & \overline{wy} & \overline{wz} & n & (n' & n'') & r & (r' & r'')
\end{array}$$

change into

$$\begin{array}{ccc|ccc|ccc}
\overline{ux} & \overline{vx} & \overline{wx} & l & (l' & l'') & p & (p' & p'') \\
\overline{uy} & \overline{vy} & \overline{wy} & n & (n' & n'') & q & (q' & q'') \\
\overline{uz} & \overline{vz} & \overline{wz} & m & (m' & m'') & r & (r' & r'')
\end{array}$$

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* Viz. from I. 1 to I. 2, and so in other cases where the species of the general surface are referred to.—A. C.

Hence the four nodal rays (l', l''), (p', p''), (q', q''), (r', r'') and the remaining ones (m', m''), (n', n'') are conjugate. Of the simple lines seven only, viz. $\overline{ux}$, $\overline{wy}$, $\overline{wz}$, l, p, q, r are real and form three real triangles which have the line l in common. Besides these three simple planes there are seven real planes to be twice counted, each of which passes through the node and one of the seven real simple lines. When the two equal roots of the cubic condition separate themselves into real roots, the four real nodal rays become eight real lines, and the surface changes into the general one of the second species. In the other case, only the plane passing through the two conjugate nodal rays resolves itself into two real planes (in the former case into two conjugate planes), so that there arises a general surface of the third species.

Third species, II. 3. *Only two nodal rays are real.*—It is possible to satisfy the above system in such manner that the constants a, d are real, b conjugate to c, and e to f; again, that the planes u, x are real, v conjugate to w, and y to z. By the change of i into $-i$ the three original schemes then change into

$$\begin{array}{ccc|ccc|ccc}
\overline{ux} & \overline{uz} & \overline{uy} & l & (l' & l'') & p & (p' & p'') \\
\overline{wx} & \overline{wz} & \overline{wy} & n & (n' & n'') & r & (r' & r'') \\
\overline{vx} & \overline{vz} & \overline{vy} & m & (m' & m'') & q & (q' & q'')
\end{array}$$

The two nodal rays (l', l''), (p', p'') alone are real; and (not passing through the node) only the lines $\overline{ux}$, l, p, forming a triangle, are real. Besides the three simple planes u, x, (ux) the only real planes are the three planes (to be twice counted), which pass through the node and through one of the real simple lines. This case forms the transition from the third to the fifth species of the general surface.

Fourth and fifth species, II. 4, and II. 5. *Three pairs of conjugate nodal rays.*—The above system is compatible with the condition that e shall be conjugate to f, and the plane y to z, while all the other constants and planes are real. Then in the first of the three original schemes of lines the second and third columns interchange, and the second and third schemes interchange. Hence the nodal rays (l', l''), (m', m''), (n', n'') are respectively conjugate to (p', p''), (q', q''), (r', r''), and of the simple lines only $\overline{ux}$, $\overline{vx}$, $\overline{wx}$, forming a triangle, are real. Of simple planes only the seven x; u, (ux); v, (vx); w, (wx) are real, and of planes to be twice counted only those joining two conjugate nodal rays, therefore three in number. The case is intermediate between the fourth and fifth species of the general surface.

To decide the question, when is the nodal cone real or not? We throw its quadric $\mathrm{P}=(v+x)(w+x)-(u+y)(u+z)$ into the form

$$-(d-b)(d-c)\mathrm{P}=\{(d-c)(v+x)+(a-f)(u+y)\}\{(d-c)(v+x)+(a-e)(u+z)\}$$
$$+[(d-b)(d-c)-(a-e)(a-f)](u+y)(u+z).$$

On the right-hand side the first term is positive as a product of two conjugate factors, and in the second term $(u+y)(u+z)$ is positive for the same reason. Hence the cone is real when $(d-b)(d-c)-(a-e)(a-f)$ is negative; in the opposite case it is imaginary.

But if we eliminate a and d by the help of the equations

$$a+b+c=d+e+f,\quad abc=def,$$

the expression becomes

$$(b-e)(b-f)(c-e)(c-f):(bc-ef),$$

where the numerator is positive, since its factors are conjugate by pairs. The nodal cone is therefore real when the denominator $bc-ef$ is *negative* (*fourth species*, II. 4), but imaginary when $bc-ef$ is *positive* (*fifth species*, II. 5).

III. *Cubic surface of the ninth class with a biplanar node.*

Art. 12. The equation $xyw+z^3=0$, where, in the proximity of the node, only w remains finite, when discussed under both suppositions of x, y being real or conjugate, gives a preliminary view of the biplanar node at the point $\frac{\partial}{\partial w}$. A plane turning about its edge ($x=0$, $y=0$) cuts the surface in a curve with a cusp, which changes its direction into the opposite one whenever the turning plane has passed one of the two real nodal planes; or always keeps its direction if the nodal planes be conjugate, so that in the latter case the surface here terminates in the form of a thorn [viz. in such a form as is generated by the revolution of a semicubical parabola about the cuspidal tangent].

The equation of the surface is $uvw+Q=0$, where u, v are linear functions and Q a cubic one of x, y, z. Denote the three nodal rays ($u=0$, $Q=0$ by 1, 2, 3, and the three ($v=0$, $Q=0$) by 4, 5, 6. Then each combination such as (14, 25, 36) gives a determinate position of the plane $w=0$, in virtue of which the cone Q breaks up into three planes. Keeping to the order of 1, 2, 3 and permuting only 4, 5, 6, we see there are six such transformations. But whenever $Q=xyz$, the surface contains a simple triangle ($w=0$, $xyz=0$); and it is also easy to see that the three positive permutations give one trihedral, and that three negative ones give the other trihedral of a trihedral-pair where no four of the six planes meet in a point, the only possible trihedral-pair of such kind.

If in art. 9 we put $\chi=2(lx+my+nz)(l'x+m'y+n'z)$,

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ p, & q, & r \end{vmatrix} =\lambda p+\mu q+\nu r=\sigma,\quad \Sigma ll'(mn'+m'n)qr=\upsilon,\quad \Sigma ll'\lambda qr=\psi,$$

then we have

$$A=-\lambda^2,\ B=-\mu^2,\ C=-\nu^2,\ D=-\mu\nu,\ E=-\nu\lambda,\quad F=-\lambda\mu,$$

$$\Delta=0,\quad t=\Sigma(mn'+m'n)p,\quad U=2\upsilon,\quad V=2\sigma\psi,\quad W=-4\psi^2,\ L=s^2-2ts+\sigma^2,$$

$$M=2(\upsilon s+\sigma\psi),\quad N=4(4lmnl'm'n'pqrs-\psi^2),$$

$$\frac{\Omega}{4s}=4lmnl'm'n'pqr\{L^3-36sL(\upsilon s+\sigma\psi)+216s^2\psi^2-432lmnl'm'n'pqrs^3\}$$

$$+L^2\{s(\upsilon^2-\psi^2)+2\sigma\upsilon\psi+2t\psi^2\}+36L\psi^2(\upsilon s+\sigma\psi)-32(\upsilon s+\sigma\psi)^3-108s\psi^4.$$

The first term of the expression according to the descending powers of s is

$$4lmnl'm'n'pqrs^6,$$

and the last is

$$-4\lambda\mu\nu\sigma^3(nq-mr)(lr-np)(mp-lq)(n'q-m'r)(l'r-n'p)(m'p-l'q).$$

The system

$$(lx+my+nz)(l'x+m'y+n'z)w+xyz=0,$$
$$\Sigma l^2l'^2x^4-2\Sigma mm'nn'y^2z^2-4xyz\Sigma ll'(mn'+m'n)x=0$$

represents the spinode curve, which is therefore a complete curve of the twelfth degree and has the node for an eightfold point, where the tangents are determined by the system

$$(\Sigma lx)(\Sigma l'x)=0,\ \ \Sigma\lambda^2y^2z^2-2xyz\Sigma\mu\nu x=0,$$

since the cone drawn from the node through the spinode curve may also be thrown into the form

$$\Sigma lx\,.\,\Sigma l'x\,.\,\{\Sigma ll'x^2-\Sigma(mn'+m'n)yz\}+\Sigma\lambda^2y^2z^2-2xyz\Sigma\mu\nu x=0.$$

Art. 13. Let us represent by $uvw+xyz=0$ the only possible trihedral-pair no plane of which passes through the node, and considering this as a particular case of art. 10, let

$$u+v+w+x+y+z=0$$

be that identical relation which answers to the two equal roots which we know must exist of the cubic condition, and

$$\mathrm{A}u+\mathrm{B}v+\mathrm{C}w+\mathrm{D}x+\mathrm{E}y+\mathrm{F}z=0$$

any other identical relation. Then the coefficients in the relation corresponding to the single root of the cubic condition will be

$$a=\lambda\mathrm{A}+\mu,\ \ b=\lambda\mathrm{B}+\mu,\ \&\text{c.};$$

and since this condition

$$(\lambda\mathrm{A}+\mu)(\lambda\mathrm{B}+\mu)(\lambda\mathrm{C}+\mu)-(\lambda\mathrm{D}+\mu)(\lambda\mathrm{E}+\mu)(\lambda\mathrm{F}+\mu)=0$$

must be divisible by λ^2, it follows

$$\mathrm{A}+\mathrm{B}+\mathrm{C}=\mathrm{D}+\mathrm{E}+\mathrm{F},\ \ a=(\mathrm{A}-\mathrm{D})(\mathrm{A}-\mathrm{E})(\mathrm{A}-\mathrm{F}),\ \&\text{c.},\ \ d=(\mathrm{A}-\mathrm{D})(\mathrm{B}-\mathrm{D})(\mathrm{C}-\mathrm{D}),\ \&\text{c.}$$

Again, at the end of art. 11 we had a form of the nodal cone P containing only the three variables $v+x$, $u+y$, $u+z$, in respect to which the discriminant of P is

$$\frac{(b-d)(c-d)-(a-e)(a-f)}{(b-d)^2}=\frac{\Sigma\mathrm{BC}-\Sigma\mathrm{EF}}{(\mathrm{B}-\mathrm{D})^2}.$$

Now in order that the nodal cone may break up into two planes, we must have

$$\mathrm{BC}+\mathrm{CA}+\mathrm{AB}=\mathrm{EF}+\mathrm{FD}+\mathrm{DE},$$

which reduces the cubic condition to

$$(\mathrm{ABC}-\mathrm{DEF})\lambda^3=0.$$

Rejecting the solution

$$\mathrm{ABC}=\mathrm{DEF}$$

as giving rise to

$$\mathrm{A}=\mathrm{D},\ \ \mathrm{B}=\mathrm{E},\ \ \mathrm{C}=\mathrm{F}$$

for instance, and thus bringing

$$u+x=0,\ v+y=0,\ w+z=0$$

into one and the same plane, we infer that if a trihedral-pair form, explicitly not singular, belong to a cubic surface of the ninth class, the cubic condition inherent to such a trihedral-pair must have three equal roots.

Reciprocally, let $uvw+xyz=0$ be the equation of the surface, and

$$u+v+w+x+y+z=0,\ \mathrm{A}u+\mathrm{B}v+\mathrm{C}w+\mathrm{D}x+\mathrm{E}y+\mathrm{F}z=0$$

identical relations, where

$$\mathrm{A+B+C=D+E+F,\ BC+CA+AB=EF+FD+DE},$$

but where ABC−DEF is different from zero, then we have a set of proportions such as

$$\frac{\mathrm{A-E}}{\mathrm{B-D}}=\frac{\mathrm{C-D}}{\mathrm{A-F}};$$

and since at the node $u=v=w=-x=-y=-z$, the nodal cone is represented by

$$\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}+\frac{\partial}{\partial w}-\frac{\partial}{\partial x}-\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)(uvw-xyz)=vw+uw+uv-yz-zx-xy=0.$$

But because the equation

$$\begin{aligned}&\{(\mathrm{B-E})(u+x)-(\mathrm{A-D})(v+y)\}\,\{(\mathrm{B-F})(u+x)-(\mathrm{A-D})(v+z)\}\\&\quad=[(\mathrm{B-E})(\mathrm{B-F})-(\mathrm{A-D})(\mathrm{C-D})](u+x)^2\\&\quad-(\mathrm{A+B+C-D-E-F})(\mathrm{A-D})(u+x)(v+y+z)\\&\quad+(\mathrm{A-D})[(\mathrm{A-D})(u+v)+\mathrm{C}(u+x)](u+v+w+x+y+z)\\&\quad-(\mathrm{A-D})(u+x)(\mathrm{A}u+\mathrm{B}v+\mathrm{C}w+\mathrm{D}x+\mathrm{E}y+\mathrm{F}z)\\&\quad-(\mathrm{A-D})^2(vw+uw+uv-yz-xz-xy)\end{aligned}$$

is *explicitly* identical, therefore the equation

$$\begin{aligned}&-(\mathrm{A-D})^2(vw+wu+uv-yz-zx-xy)\\&\quad=\{(\mathrm{B-E})(u+x)-(\mathrm{A-D})(v+y)\}\,\{(\mathrm{B-F})(u+x)-(\mathrm{A-D})(v+z)\}\end{aligned}$$

is identical in respect to the fundamental coordinates; in other words, the nodal cone breaks up into a pair of planes. The nodal edge may be represented by

$$u=s+\mathrm{A}t,\ v=s+\mathrm{B}t,\ w=s+\mathrm{C}t,\ x=-s-\mathrm{D}t,\ y=-s-\mathrm{E}t,\ z=-s-\mathrm{F}t,$$

where s, t denote independent variables.

Now it is plain that the equation $u+x=0$, for instance, represents at once the three planes previously denoted by (ux), $(ux)'$, $(ux)''$, wherefore now the three lines l, l', l'' coincide, and so on. Each of the six nodal rays thus unites three (independent) lines of the surface; only the lines $uvw=0$, $xyz=0$ are simple lines. We have in all $6\cdot3+9\cdot1=27$ lines. One nodal plane unites all the six planes such as (lmn), and the other all the six planes such as (pqr). Of the nine planes joining any ray of the one nodal plane with any ray of the other, each unites three planes such as (ux), $(ux)'$, $(ux)''$;

only the six planes of the trihedral-pair here chosen are simple triangle-planes. There are in all $2 \cdot 6 + 9 \cdot 3 + 6 \cdot 1 = 45$ triangle-planes.

Art. 14. There are four species.

First species, III. 1.—u, v, w, x, y, z are real. Everything is then real.

Second species, III. 2.—u is conjugate to x, v to y, and w to z. Both nodal planes are real; one of them contains the real ray l and the two conjugate rays m, n; the other nodal plane contains the three real rays p, q, r. Of the nine simple lines three only, $\overline{ux}$, $\overline{vy}$, $\overline{wz}$, are real.

Third species, III. 3.—u, x are real, v is conjugate to w, y to z. Both nodal planes are real, and each of them contains a real and two conjugate rays; for l and p are real, and m is conjugate to n, q to r. Of the nine simple lines one only, $\overline{ux}$, is real.

Fourth species, III. 4.—u, v, w, x are real, y is conjugate to z. The two nodal planes are conjugate; for l, m, n are respectively conjugate to p, q, r. Of the nine simple lines three only, forming the triangle ($x=0$, $uvw=0$), are real.

The enumeration is complete, because all cases that can happen in respect to the nodal rays are exhausted.

IV. *Cubic surface of the eighth class with two proper nodes.*

Art. 15. From art. 8 we already know that the line joining the two nodes, or *axis*, unites two and the same rays of each node, and that there is a singular tangent plane which touches the surface, and therefore also each nodal cone along the *axis*, and besides intersects the surface in a single line which we have termed the *transversal*. Since then, besides the axis, each nodal cone has four rays not passing through the other node, there are in all ten nodal rays which represent twenty lines of the surface (considered as though it were general), so that there remain only seven simple lines, one of which is the transversal above mentioned. Because this transversal is not intersected by the eight disengaged nodal rays, but only by the axis, that is by four lines, it must meet all the six other lines, and will therefore form with them three triangles. Besides such triangle, there pass through each of the six lines four other planes, which are of course those passing through one or the other node, each of them counting for two triangle-planes. Again, a plane through the axis and a disengaged ray of one node must intersect the surface in a third line, which cannot but be a disengaged ray of the other node. Such plane counts for four triangle-planes; for any one of the four disengaged rays of one node, since it determines with each of the three remaining rays three triangle-planes, must determine with the axis two such planes; and because it is made up of two independent lines of the surface, the two planes must be twice counted. As to the singular tangent plane, it counts twice, because through the transversal there already pass three simple triangle-planes. The surface thus has a line representing four lines, viz. the axis; eight other nodal rays, each representing two lines; and seven simple lines, viz. the transversal and the remaining sides of the three simple triangles standing upon it; in all $1 \cdot 4 + 8 \cdot 2 + 7 \cdot 1 = 27$ lines. Again, the surface has four planes each representing four triangle-planes of the surface, viz. those passing through the axis and one ray of either node; thirteen planes each representing two triangle-planes, viz. the singular tangent plane and the twice six other planes each of them through two disengaged rays of the same node; lastly, the three simple triangle-planes passing through the transversal; in all $4\cdot4+13\cdot2+3\cdot1=45$ triangle-planes.

We proceed now to reduce the equation of the surface in question to its simplest form. Let $x=0$ be the equation of the singular tangent-plane, and let the plane $y=0$ pass through the axis, while the planes $z=0$ and $w=0$ touch respectively the nodal cones in lines belonging to the plane $y=0$, then the term yzw and those divisible by z^2, w^2, xz, xw will disappear, and we may therefore write

$$xzw+y^2(z+w)+ax^3+4bx^2y+6cxy^2+4dy^3=0.$$

But this cubic if multiplied by x becomes

$$(xz+y^2)(xw+y^2)-(y-dx)^4+6(c+d^2)x^2y^2+4(b-d^3)x^3y+(a+d^4)x^4,$$

while

$$xz+y^2=x(z+2dy-d^2x)+(y-dx)^2,\quad xw+y^2=x(w+2dy-d^2x)+(y-dx)^2.$$

Now it will be readily seen that the equation of the surface can in but one way be reduced to the form

$$xzw+y^2(z+w)+ax^3+bx^2y+cxy^2=0,$$

where we might also put unity instead of one of the three constants a, b, c. In respect to the fundamental coordinates, the equation implies seventeen constant elements, as it should do, since the two nodes take away two disposable constants from the full number 19.

Let us attempt to form the equation reciprocal to this. We have

$$\theta p=zw+3ax^2+2bxy+cy^2,\quad \theta q=2y(z+w)+bx^2+2cxy,\quad \theta r=xw+y^2,\quad \theta s=xz+y^2.$$

Putting then

$$\phi=px^2+qxy-(r+s)y^2,\quad \chi=ax^4+bx^3y+cx^2y^2-y^4,$$

regarding p, q, r, s in respect to $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ as constants, and eliminating z, w by the help of the original equation of the surface, we find

$$\theta^2rs=-\chi,\quad \theta\frac{\partial\phi}{\partial x}=\frac{\partial\chi}{\partial x},\quad \theta\frac{\partial\phi}{\partial y}=\frac{\partial\chi}{\partial y},$$

whence $\theta\phi=2\chi$, $2\theta rs=-\phi$; and lastly, on eliminating θ,

$$\frac{\partial}{\partial x}(\phi^2+4rs\chi)=0,\quad \frac{\partial}{\partial y}(\phi^2+4rs\chi)=0,$$

that is to say, the discriminant of the binary quartic

$$\left(px^2+qxy-(r+s)y^2\right)^2+4rs(ax^4+bx^3y+cx^2y^2-y^4)$$

must vanish, and divided by r^2s^2 it will give the reciprocal equation required.

Denoting the Hessian of the primitive cubic by 4H, we have

$$\begin{aligned}H=xzw(z+w)+y^2(z-w)^2+cx^2zw-(z+w)(3ax^3+3bx^2y+2cxy^2)\\+(b^2-3ac)x^4+bcx^3y+(c^2+12a)x^2y^2+4bxy^3.\end{aligned}$$

Hence arises for the spinode curve the system

$$\left.\begin{aligned}xzw+y^2(z+w)+ax^3+bx^2y+cxy^2=0,\\-4y^2zw-4(z+w)(ax^3+bx^2y+cxy^2)+(b^2-4ac)x^4+12ax^2y^2+4bxy^3=0,\end{aligned}\right\}$$

where the axis ($x=0$, $y=0$) counts for two solutions; therefore the spinode curve is a partial curve of the tenth degree, and each node of the original surface is a quadruple point of the curve, the nodal rays at such point being tangents to the curve.

Art. 16. We proceed to determine the lines and triangle-planes of the surface. The transversal is ($x=0$, $z+w=0$). The nodal cones are $xz+y^2=0$, $xw+y^2=0$; besides touching one another along the axis, they intersect in a conic the plane of which is $z-w=0$. This plane and the transversal therefore cut the axis harmonically in regard to the two nodes.

Cutting the surface by the plane $y-\lambda x=0$, and omitting the solution $x=0$, we obtain the equation

$$(z+\lambda^2x)(w+\lambda^2x)+(a+b\lambda+c\lambda^2-\lambda^4)x^2=0;$$

and in order that this break up into factors, the condition $\lambda^4-c\lambda^2-b\lambda-a=0$ must be fulfilled, and the equation of the section then becomes $(z+\lambda^2x)(w+\lambda^2x)=0$. Now, as is well known, the solution of the quartic condition depends upon that of the cubic equation

$$X^3-2cX^2+(c^2+4a)X-b^2=0.$$

Accordingly, in order to avoid irrationalities, we put

$$2c=\alpha^2+\beta^2+\gamma^2,\quad c^2+4a=\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2,\quad b=\alpha\beta\gamma,$$

and, for the sake of shortness, $\sigma=\frac{1}{2}(\alpha+\beta+\gamma)$, whence

$$\alpha=\sigma(\sigma-\alpha)(\sigma-\beta)(\sigma-\gamma),\quad c=\tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2),$$

and λ has the four values σ, $\alpha-\sigma$, $\beta-\sigma$, $\gamma-\sigma$. Hence the four triangle-planes passing through the axis are

$$\sigma x-y=0,\quad (\sigma-\alpha)x+y=0,\quad (\sigma-\beta)x+y=0,\quad (\sigma-\gamma)x+y=0.$$

The plane

$$\tfrac{1}{2}(-\alpha^2+\beta^2+\gamma^2)x+z+w=0$$

passing through the transversal, cuts the surface in the trilateral

$$x\{(\sigma-\alpha)x+\alpha y+z\qquad\}\{(\sigma-\beta)(\sigma-\gamma)x+\alpha y-z\}=0,$$

or

$$x\{(\sigma-\beta)(\sigma-\gamma)x+\alpha y-w\}\{\qquad\sigma(\sigma-\alpha)x+\alpha y+w\}=0,$$

in each of which representations the consecution of the three sides is the same, while in the first all the planes (or factors) pass through the node W, and in the second through the node Z. Denoting the sides of the triangle corresponding to the constant α in the same order by *axis*, a, *a*, and the planes passing through them and the nodes by (Wa), (W*a*), (Za), (Z*a*), we see that through each nodal ray there pass three planes, as follows:—

(Wa), (Wb), (Wc) through the ray $\sigma x-y=0, \quad \sigma^2 x+z=0,$
(Wa), (W*b*), (W*c*) through the ray $(\sigma-\alpha)x+y=0, \quad (\sigma-\alpha)^2 x+z=0,$
(Wb), (W*c*), (W*a*) through the ray $(\sigma-\beta)x+y=0, \quad (\sigma-\beta)^2 x+z=0,$
(Wc), (W*a*), (W*b*) through the ray $(\sigma-\gamma)x+y=0, \quad (\sigma-\gamma)^2 x+z=0.$

If we permute the nodes W and Z, we must in this scheme also permute a with *a*, b with *b*, c with *c*, and z with w.

Art. 17. In order to get a trihedral-pair form, let

$$P=(\sigma-\beta)(\sigma-\gamma)x+\alpha y-z, \quad Q=(\sigma-\gamma)(\sigma-\alpha)x+\beta y-z,$$
$$R=(\sigma-\alpha)(\sigma-\beta)x+\gamma y-z, \quad S=w-z,$$

and

$$(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)=\theta;$$

then it will be found that

$$\Sigma\alpha(Q-R)P(P-S)+2\sigma(Q-R)(R-P)(P-Q)=\theta\{xzw+y^2(z+w)+ax^3+bx^2y+cxy^2\};$$

but the left-hand side of this identical equation is equal to

$$-\Sigma QR\{(\beta-\gamma)S+(\gamma+\alpha)Q-(\alpha+\beta)R\},$$

where

$$(\beta-\gamma)S+(\gamma+\alpha)Q-(\alpha+\beta)R=(\beta-\gamma)\{\sigma(\sigma-\alpha)x+\alpha y+w\}.$$

Put therefore

$$p=(\beta-\gamma)\{\sigma(\sigma-\alpha)x+\alpha y+w\}, \quad q=(\gamma-\alpha)\{\sigma(\sigma-\beta)r+\beta y+w\},$$
$$r=(\alpha-\beta)\{\sigma(\sigma-\gamma)x+\gamma y+w\},$$

and then

$$pQR+qRP+rPQ=0$$

will be the equation of the surface, where the six linear functions fulfil the identical relations

$$p+q+r=0, \quad \alpha p+\beta q+\gamma r+(\beta^2-\gamma^2)P+(\gamma^2-\alpha^2)Q+(\alpha^2-\beta^2)R=0.$$

If h denote a number which is ultimately made to vanish, this equation may be exhibited under the form

$$(P+hp)(Q+hq)(R+hr)-(P-hp)(Q-hq)(R-hr)=0.$$

Let $p=\lambda P$, $q=\mu Q$, $r=\nu R$; then $\lambda+\mu+\nu=0$ in virtue of the equation of the surface. Again, if for shortness we put

$$f=\alpha\lambda+\beta\mu+\gamma\nu, \quad g=\alpha^2\lambda+\beta^2\mu+\gamma^2\nu, \quad h=\sigma f-g,$$

from the foregoing relations it will next be found

$$S=g+\lambda\mu\nu, \quad P=g+(\beta-\gamma)\mu\nu, \text{ \&c.}, \quad \theta x=-\Sigma(\beta-\gamma)P,$$
$$\theta y=\Sigma(\beta-\gamma)(\sigma-\alpha)P, \quad \theta z=\Sigma(\beta-\gamma)(\sigma-\alpha)^2 P,$$

and then

$$\theta x=f^2, \quad \theta y=fh, \quad \theta z=-\theta g-h^2, \quad \theta w=\theta\lambda\mu\nu-h^2;$$

the coordinates of a point of the surface are thus expressed in terms of two independent variables; only the values $\lambda=\beta-\gamma$, $\mu=\gamma-\alpha$, $\nu=\alpha-\beta$ are inadmissible. To verify the equation of the surface we have

$$\theta(\sigma x-y)=fg, \quad (\beta-\gamma)\{(\sigma-\alpha)x+y\}=\lambda f, \text{ \&c.},$$

whence

$$\theta^2(ax^4+bx^3y+cx^2y^3-y^4)=\lambda\mu\nu f^4g;$$

and on the other side

$$\theta(xz+y^2)=-gf^2, \quad \theta(xw+y^2)=\lambda\mu\nu f^3.$$

This gives indeed

$$(xz+y^2)(xw+y^2)+ax^4+bx^3y+cx^2y^2-y^4=0;$$

but the values of the nodal cone quadratics show that three rays of the node W and one ray of the node Z cannot be expressed.

We have still to divide this sort of surface into species. Whether both nodes be real (when z, w are real) or conjugate (z, w conjugate), there are but three cases to be distinguished.

1. α, β, γ are real Then the four triangle-planes passing through the axis and the three passing through the transversal are real. IV. 1, and IV. 4.

2. α is real, β, γ are complex and conjugate. Then of the planes passing through the axis only two are real, the two others are conjugate; and of those passing through the transversal but one is real and two are conjugate. IV. 2, and IV. 5.

3 α is real, β, γ are lateral (according to the denomination proposed by GAUSS, that is to say, β^2 and γ^2 are negative). Then the planes passing through the axis are conjugate by pairs, and those passing through the transversal are all of them real IV. 3, and IV. 6.

Hence arise six species.

V. *Cubic surface of the eighth class with a biplanar node.*

Art. 18. From art. 7 it appears that the equation of this surface can be written

$$xyw+(x+y)z^2+2(ax^2+by^2)z+cx^3+dy^3=0,$$

since all the terms divisible by xy may be joined to the first term. But giving this equation the form

$$xy[w-2(a+b)z-(2ab+c)x-(2ab+d)y]$$
$$+(x+y)[(z+ax+by)^2+(c-a^2)x^2+(d-b^2)y^2]=0,$$

we see that more briefly it may also be thus written,

$$2xyw+(x+y)(z^2-ax^2-by^2)=0.$$

The equation reciprocal to this is contained in the discriminant of the binary quartic

$$r^2x^2y^2+2s(px+qy)xy(x+y)+s^2(ax^2+by^2)(x+y)^2.$$

If we put

$$\mathrm{L}=(a+b)s^2+2(p+q)s+r^2, \quad \mathrm{M}=(bp+aq)s+pq, \quad \mathrm{N}=abr^2-bp^2-aq^2,$$

and denote by S, T the same invariants as are found in Dr. SALMON's 'Higher Algebra,' p. 100, then we have

$$12S=L^2-12s^2M,\quad 216T=-L^3+18s^2LM+54s^4N,\quad 16(S^3-27T^2)=s^4\Theta,$$

and ultimately

$$\begin{aligned}\Theta=&ab(a+b)^2\{(a+b)r^2-(p-q)^2\}s^6\\&+2ab\{3(a+b)[(a-2b)p+(-2a+b)q]r^2+(p-q)^2[(-3a+5b)p+(5a-3b)q]\}s^5\\&+\{3ab(a^2-7ab+b^2)r^4+[b(9a^2+26ab-b^2)p^2-26ab(a+b)pq+a(-a^2+26ab\\&\quad+9b^2)q^2]r^2+(p-q)^2[b(-12a+b)p^2+22abpq+a(a-12b)q^2]\}s^4\\&+2\{3ab[(2a-b)p+(-a+2b)q]r^4+[b(-2a+5b)p^3+b(3a-2b)p^2q\\&\quad+a(-2a+3b)pq^2+a(5a-2b)q^3]r^2+2(p-q)^2[-2bp^3+bp^2q+apq^2-2aq^3]\}s^3\\&+\{3ab(a+b)r^6+[b(9a-2b)p^2+8abpq+a(-2a+9b)q^2]r^4\\&\quad+2[-6bp^4+bp^3q-(a+b)p^2q^2+apq^3-6aq^4]r^2+4p^2q^2(p-q)^2\}s^2\\&+2\{3ab(p+q)r^6-(3bp^3+2bp^2q+2apq^2+3aq^3)r^4+4p^2q^2(p+q)r^2\}s\\&+r^4(ar^2-p^2)(br^2-q^2)=0\end{aligned}$$

is the reciprocal equation required.

Let 16H be the Hessian of the primitive function, then

$$H=2(x+y)xyw+(x-y)^2z^2+(x+y)(3ax^3-ax^2y-bxy^2+3by^3),$$

whence the system

$$\left\|\begin{matrix}xy, & ax^2+by^2-z^2, & ax^3+by^3\\ x+y, & 2w, & z^2\end{matrix}\right\|=0$$

will represent the spinode curve, which is therefore a partial curve of the tenth degree, and in which there pass through the node six branches, in lowest approximation represented by the systems $(2xw+z^2=0,\ 2by^3w+z^4=0)$, $(2yw+z^2=0,\ 2ax^3w+z^4=0)$, and having the axis for a common tangent with a singular kind of contact. Any plane passing through the node intersects here the curve in six coincident points, any plane passing through the axis in eight, and each nodal plane in ten coincident points.

Art. 19. Let $a=\alpha^2$, $b=\beta^2$, $U=2\alpha\beta(x+y)-W$, $V=z+\alpha x+\beta y$, $W=z-\alpha x-\beta y$, $X=2\alpha\beta(x+y)+W$, $Y=-z-\alpha x+\beta y$, $Z=-z-\alpha x+\beta y$; then the original equation takes the form $UVW+XYZ$, and the six new functions fulfil the two identical relations $V+W+Y+Z=0$, $U-(\alpha+\beta)V+(\alpha+\beta)W+X-(\alpha-\beta)Y+(\alpha-\beta)Z=0$. Imagine instead of these the relations

$$hU+V+W+hX+Y+Z=0,\quad AhU+BV+CW+DhX+EY+FZ=0,$$

where

$$A=\frac{1}{h}-(\alpha+\beta),\quad B=-(\alpha+\beta)+h(\alpha+\beta)^2,\quad C=\alpha+\beta-h(\alpha-\beta)^2,$$

$$D=\frac{1}{h}-(\alpha+\beta)+4\alpha\beta h,\quad E=-(\alpha-\beta)\{1-h(\alpha-\beta)\},\quad F=-E.$$

Because the six constants fulfil the equations

$$A+B+C=D+E+F, \quad BC+CA+AB=EF+FD+DE,$$

the cubic condition inherent to the trihedral pair $hUVW+hXYZ=0$ has three equal roots. Let then h vanish, and the former system will be reproduced. At the same time such equations of triangle-planes as in art. 13 were $u+x=0$, $u+y=0$, $v+x=0$ will now become respectively $U+X=0$, $Y=0$, $V=0$, and so on; but we shall continue to denote them by (ux), (uy), (vx) as before, yet omit accents, since all three accents coincide. So we get the following survey of the twenty-seven lines on the surface, showing in what manner they coincide:—

The nodal edge (or here *axis*, since the surface is along it touched by a plane) $(x=0, y=0)$ unites the six lines l, p. The four other nodal rays unite each of them four lines such as follow, $(\overline{vy}, r)$, $(\overline{wz}, q)$; $(\overline{vz}, n)$, $(\overline{wy}, m)$. The transversal $\overline{ux}$, and the other sides of the two simple triangles standing upon it, $\overline{uy}$, $\overline{uz}$, $\overline{vx}$, $\overline{wx}$, are the only five simple lines. In all $1\cdot6+4\cdot4+5\cdot1=27$ lines.

Each nodal plane unites twelve triangle-planes, viz. $x=0$ unites (vz), (wy), (lmn); and $y=0$ unites (vy), (wz), (pqr). The four planes, joining a ray of one nodal plane with a ray of the other nodal plane, unite each of them four triangle-planes, viz.

$$V=0\ \{v, (vx)\}, \quad W=0\ \{w, (wx)\}, \quad Y=0\ \{y, (uy)\}, \quad Z=0\ \{z, (uz)\}.$$

The singular tangent plane $x+y=0$ unites the three planes (ux). Lastly, there are but two simple triangle planes, those passing through the transversal $U=0$, and $X=0$. In the whole $2\cdot12+4\cdot4+1\cdot3+2\cdot1=45$ triangle-planes.

Since the functions z, w, $x+y$, xy must always be real, there are four species.

1. All is real, and a, c are positive. V. 1.

2. x, y are real, a is positive, and b negative. The only real lines are the axis, two rays in only one nodal plane, and the transversal. V. 2.

3. x, y are real, a and b are negative. The axis and transversal are the only real lines. Each nodal plane contains two conjugate rays. V. 3.

4. x, y are conjugate, and so also are a, b. The axis and transversal are real; of the two real planes passing through the transversal, one only contains a real triangle; these four lines only are real. V 4.

VI. *Cubic surface of the seventh class with a biplanar and a proper node.*

Art. 20. If a cubic surface have two nodes, chosen for points of reference $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial w}$, its equation necessarily takes the form $lzw+mz+nw+p=0$, where $l=(x, y)$; $m, n=(x, y)^2$, $p=(x, y)^3$; and if $\frac{\partial}{\partial w}$ be a biplanar node, $lz+n$ must break up into factors, whence l must divide n, so that $lz+n$ may then be replaced by xz. And joining the terms in mz, which are divisible by xz, to the term xzw, we may write

$$xzw+xy^2z+ax^3+3bx^2y+3cxy^2+dy^3=0$$

as the equation of the surface.

The equation reciprocal to this is found by dividing the discriminant of the binary quartic $(px^2+qxy-sy^2)^2+4rsx(ax^3+3bx^2y+3cxy^2+dy^3)$ by r^2s^3 and equating the quotient to zero. Let

$$\begin{aligned}&\mathrm{L}=q^2+4(p+3cr)s,\ \mathrm{M}=-dpq+3(-2cp+bq-2bdr)s+2as^2,\\&\mathrm{N}=d^2p^2+2d(3bp-2aq+2adr)s+3(3b^2-4ac)s^2,\\&12\mathrm{S}=\mathrm{L}^2+24rs\mathrm{M},\ -216\mathrm{T}=\mathrm{L}^3+36rs\mathrm{LM}+216r^2s^2\mathrm{N},\\&\mathrm{M}^2-\mathrm{LN}=4s\mathrm{P},\ \mathrm{S}^3-27\mathrm{T}^2=r^2s^3\Theta,\end{aligned}$$

then S, T are the two invariants of the quartic in question, and

$$\Theta=\mathrm{L}^2\mathrm{P}+8r\mathrm{M}^3-9r\mathrm{LMN}-27r^2s\mathrm{N}^2=0$$

is the equation of the surface in plane-coordinates. If

$$\delta=3b\frac{\partial}{\partial a}+2c\frac{\partial}{\partial b}+d\frac{\partial}{\partial c}+q\frac{\partial}{\partial p}-2s\frac{\partial}{\partial q},$$

then $\delta\mathrm{N}=-2d\mathrm{M}$, $\delta\mathrm{M}=-d\mathrm{L}$, $\delta\mathrm{L}=12drs$, whence $\delta\mathrm{S}=0$, $\delta\mathrm{T}=0$, $\delta\Theta=0$.

The quartic function, the Hessian of the original cubic, is

$$\begin{aligned}&\{z+3(cx+dy)\}(xzw+y^2z+ax^3+3bx^2y+3cxy^2+dy^3)-4z(ax^3+3bx^2y+3cxy^2+dy^3)\\&\quad-3\{(4ac-3b^2)x^4+4adx^3y+6bdx^2y^2+4cdxy^3+d^2y^4\}.\end{aligned}$$

The spinode curve is therefore a partial curve of the ninth degree, which has the biplanar node for a quintuple and the proper node for a triple point. The tangents at the latter node are the three disengaged nodal rays; but of those at the former node one tangent is $(x=0,\ 3dy+4z=0)$, and the four remaining tangents are

$$z=0,\ (4ac-3b^2)x^4+4adx^3y+6bdx^2y^2+4cdxy^3+d^2y^4=0.$$

Art. 21. If d vanish, the edge of the biplanar node would belong to the surface, and its class would therefore sink to six, contrary to the supposition. We are therefore allowed to change z into dz and write

$$xzw+y^2z+(y+\alpha x)(y+\beta x)(y+\gamma x)=0$$

as the equation of the surface. Again, let

$$\begin{aligned}\mathrm{P}&=w-\beta\gamma x-(\beta+\gamma)y,\\\mathrm{Q}&=w-\gamma\alpha x-(\gamma+\alpha)y,\\\mathrm{R}&=w-\alpha\beta x-(\alpha+\beta)y,\\p&=(\beta-\gamma)(\alpha x+y+z),\\q&=(\gamma-\alpha)(\beta x+y+z),\\r&=(\alpha-\beta)(\gamma x+y+z);\end{aligned}$$

the six new linear functions will satisfy the identical relations

$$\begin{aligned}&p+q+r=0,\quad(\beta-\gamma)\mathrm{P}+(\gamma-\alpha)\mathrm{Q}+(\alpha-\beta)\mathrm{R}+\alpha p+\beta q+\gamma r=0,\\&\quad p\mathrm{QR}+q\mathrm{RP}+r\mathrm{PQ}=-(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\{xzw+y^2z+(y+\alpha x)(y+\beta x)(y+\gamma x)\};\end{aligned}$$

Because the six constants fulfil the equations

$$A+B+C=D+E+F, \quad BC+CA+AB=EF+FD+DE,$$

the cubic condition inherent to the trihedral pair $hUVW+hXYZ=0$ has three equal roots. Let then h vanish, and the former system will be reproduced. At the same time such equations of triangle-planes as in art. 13 were $u+x=0$, $u+y=0$, $v+x=0$ will now become respectively $U+X=0$, $Y=0$, $V=0$, and so on; but we shall continue to denote them by (ux), (uy), (vx) as before, yet omit accents, since all three accents coincide. So we get the following survey of the twenty-seven lines on the surface, showing in what manner they coincide:—

The nodal edge (or here *axis*, since the surface is along it touched by a plane) ($x=0$, $y=0$) unites the six lines l, p. The four other nodal rays unite each of them four lines such as follow, $(\overline{vy}, r)$, $(\overline{wz}, q)$; $(\overline{vz}, n)$, $(\overline{wy}, m)$. The transversal $\overline{ux}$, and the other sides of the two simple triangles standing upon it, $\overline{uy}$, $\overline{uz}$, $\overline{vx}$, $\overline{wx}$, are the only five simple lines. In all $1{\cdot}6+4{\cdot}4+5{\cdot}1=27$ lines.

Each nodal plane unites twelve triangle-planes, viz. $x=0$ unites (vz), (wy), (lmn); and $y=0$ unites (vy), (wz), (pqr). The four planes, joining a ray of one nodal plane with a ray of the other nodal plane, unite each of them four triangle-planes, viz.

$$V=0\ \{v, (vx)\}, \quad W=0\ \{w, (wx)\}, \quad Y=0\ \{y, (uy)\}, \quad Z=0\ \{z, (uz)\}.$$

The singular tangent plane $x+y=0$ unites the three planes (ux). Lastly, there are but two simple triangle planes, those passing through the transversal $U=0$, and $X=0$. In the whole $2{\cdot}12+4{\cdot}4+1{\cdot}3+2{\cdot}1=45$ triangle-planes.

Since the functions z, w, $x+y$, xy must always be real, there are four species.

1. All is real, and a, c are positive. V. 1.

2. x, y are real, a is positive, and b negative. The only real lines are the axis, two rays in only one nodal plane, and the transversal. V. 2.

3. x, y are real, a and b are negative. The axis and transversal are the only real lines. Each nodal plane contains two conjugate rays. V. 3.

4. x, y are conjugate, and so also are a, b. The axis and transversal are real; of the two real planes passing through the transversal, one only contains a real triangle; these four lines only are real. V 4.

VI. *Cubic surface of the seventh class with a biplanar and a proper node.*

Art. 20. If a cubic surface have two nodes, chosen for points of reference $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial w}$, its equation necessarily takes the form $lzw+mz+nw+p=0$, where $l=(x, y)$; $m, n=(x, y)^2$, $p=(x, y)^3$; and if $\frac{\partial}{\partial w}$ be a biplanar node, $lz+n$ must break up into factors, whence l must divide n, so that $lz+n$ may then be replaced by xz. And joining the terms in mz, which are divisible by xz, to the term xzw, we may write

$$xzw+xy^2z+ax^3+3bx^2y+3cxy^2+dy^3=0$$

as the equation of the surface.

The equation reciprocal to this is found by dividing the discriminant of the binary quartic $(px^2+qxy-sy^2)^2+4rsx(ax^3+3bx^2y+3cxy^2+dy^3)$ by r^2s^3 and equating the quotient to zero. Let

$$\begin{aligned}&L=q^2+4(p+3cr)s,\ M=-dpq+3(-2cp+bq-2bdr)s+2as^2,\\&N=d^2p^2+2d(3bp-2aq+2adr)s+3(3b^2-4ac)s^2,\\&12S=L^2+24rsM,\ -216T=L^3+36rsLM+216r^2s^2N,\\&M^2-LN=4sP,\ S^3-27T^2=r^2s^3\Theta,\end{aligned}$$

then S, T are the two invariants of the quartic in question, and

$$\Theta=L^2P+8rM^3-9rLMN-27r^2sN^2=0$$

is the equation of the surface in plane-coordinates. If

$$\delta=3b\frac{\partial}{\partial a}+2c\frac{\partial}{\partial b}+d\frac{\partial}{\partial c}+q\frac{\partial}{\partial p}-2s\frac{\partial}{\partial q},$$

then $\delta N=-2dM$, $\delta M=-dL$, $\delta L=12drs$, whence $\delta S=0$, $\delta T=0$, $\delta\Theta=0$.

The quartic function, the Hessian of the original cubic, is

$$\begin{aligned}&\{z+3(cx+dy)\}(xzw+y^2z+ax^3+3bx^2y+3cxy^2+dy^3)-4z(ax^3+3bx^2y+3cxy^2+dy^3)\\&\quad-3\{(4ac-3b^2)x^4+4adx^3y+6bdx^2y^2+4cdxy^3+d^2y^4\}.\end{aligned}$$

The spinode curve is therefore a partial curve of the ninth degree, which has the biplanar node for a quintuple and the proper node for a triple point. The tangents at the latter node are the three disengaged nodal rays; but of those at the former node one tangent is $(x=0,\ 3dy+4z=0)$, and the four remaining tangents are

$$z=0,\ (4ac-3b^2)x^4+4adx^3y+6bdx^2y^2+4cdxy^3+d^2y^4=0.$$

Art. 21. If d vanish, the edge of the biplanar node would belong to the surface, and its class would therefore sink to six, contrary to the supposition. We are therefore allowed to change z into dz and write

$$xzw+y^2z+(y+\alpha x)(y+\beta x)(y+\gamma x)=0$$

as the equation of the surface. Again, let

$$\begin{aligned}P&=w-\beta\gamma x-(\beta+\gamma)y,\\Q&=w-\gamma\alpha x-(\gamma+\alpha)y,\\R&=w-\alpha\beta x-(\alpha+\beta)y,\\p&=(\beta-\gamma)(\alpha x+y+z),\\q&=(\gamma-\alpha)(\beta x+y+z),\\r&=(\alpha-\beta)(\gamma x+y+z);\end{aligned}$$

the six new linear functions will satisfy the identical relations

$$\begin{aligned}&p+q+r=0,\quad(\beta-\gamma)P+(\gamma-\alpha)Q+(\alpha-\beta)R+\alpha p+\beta q+\gamma r=0,\\&pQR+qRP+rPQ=-(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\{xzw+y^2z+(y+\alpha x)(y+\beta x)(y+\gamma x)\};\end{aligned}$$

and the equation of the surface is now changed into

Introducing then a number h which is ultimately made to vanish, we put

$$U=P+hp,\ V=Q+hq,\ W=R+hr,\ X=-P+hp,\ Y=-Q+hq,\ Z=-R+hr,$$

whereby the equation of the surface becomes $UVW+XYZ$; and if, for the sake of shortness, we put

$$A=\alpha+(\beta-\gamma)h,\quad B=\beta+(\gamma-\alpha)h,\quad C=\gamma+(\alpha-\beta)h,$$
$$D=\alpha-(\beta-\gamma)h,\quad E=\beta-(\gamma-\alpha)h,\quad F=\gamma-(\alpha-\beta)h,$$

the above-mentioned two identical relations become

$$U+V+W+X+Y+Z=0,\quad AU+BV+CW+DX+EY+FZ=0,$$

where the six constants satisfy the relations

$$A+B+C=D+E+F,\quad BC+CA+AB=EF+FD+DE$$

strictly, while $ABC-DEF$ is different from zero. All three roots, therefore, of the cubic condition inherent to this trihedral-pair form coincide, and the corresponding relation $U+V+W+X+Y+Z=0$ counts for three such intersections.

Hence the axis $(x=0,\ y=0)$ unites six lines, viz. the lines m, n. Each of the remaining four rays of the biplanar node unites three lines; viz. $(x=0,\ y+z=0)$ unites the three lines l, $(z=0,\ \alpha x+y=0)$ unites the three lines p, $(z=0,\ \beta x+y=0)$ unites the three lines q, $(z=0,\ \gamma x+y=0)$ unites the three lines r. Each of the remaining three rays of the proper node unites two lines, viz. $(\alpha x+y=0,\ w-\alpha y=0)$ unites $\overline{vz}$, $\overline{wy}$, $(\beta x+y=0,\ w-\beta y=0)$ unites $\overline{wx}$, $\overline{uz}$; $(\gamma x+y=0,\ w-\gamma y=0)$ unites $\overline{uy}$, $\overline{vx}$. Three lines are simple viz. $(P=0,\ p=0)$ or $\overline{ux}$, $(Q=0,\ q=0)$ or $\overline{vy}$, $(R=0,\ r=0)$ or $\overline{wz}$. In the whole $1\cdot6+4\cdot3+3\cdot2+3\cdot1=27$ lines.

Of the following five triangle-planes each counts for six. The singular tangent plane $x=0$ unites all the six planes (lmn); the other plane $z=0$ of the biplanar node unites the six planes (pqr); of the three further planes which (besides $x=0$) pass through the axis, the plane $\alpha x+y=0$ unites the two triads (vz), (wy), the plane $\beta x+y=0$ unites the two triads (wx), (uz), and the plane $\gamma x+y=0$ unites the two triads (uy), (vx). The three planes which combine the single ray l of the biplanar node with any one of the rays p, q, r in the opposite nodal plane count each of them for three triangle-planes, viz. $p=0$ unites the three planes (ux), $q=0$ unites the three planes (vy), $r=0$ unites the three planes (wz). Lastly, the three planes which combine any two of the three disengaged rays of the proper node count each of them for two triangle-planes, viz. $P=0$ unites the planes u, x; $Q=0$ unites v, y; $R=0$ unites w, z; they are the planes of the two coinciding trihedrals. In all $5\cdot6+3\cdot3+3\cdot2=45$ triangle-planes.

Art. 22. As to the reality of the linear functions in the original equation, it appears that both x and z must be real, since the two planes of the biplanar node play different parts, and that it is always allowed to assume y as real, since the corresponding plane may be turned about the real axis; but w will then also be real, and of the three constants α, β, γ one at least must be real. There are therefore but two species.

1. All is real. VI. 1.

2. α is real, β and γ are conjugate. Only the axis ($x=0$, $y=0$), two rays of the biplanar node ($x=0$, $y+z=0$) and ($z=0$, $\alpha x+y=0$), one ray of the proper node ($\alpha x+y=0$, $w-\alpha y=0$), and the simple line ($P=0, p=0$) are real. VI. 2.

VII. *Cubic surface of the seventh class with a biplanar node.*

Art. 23. According to art. 7 we put

$$xyw+xz^2+(2ax^2+by^2)z+cx^3+dy^3=0$$

as an equation of the surface in question, since all the terms divisible by xy can be carried into the single term xyw. The mark of this sort of biplanar node is that one of its planes (here $x=0$) *touches* the surface along the nodal edge; if b were to vanish, it would osculate the surface, and then the class would sink to six. Since therefore b is not allowed to vanish, we may put the above equation under the form

$$xy\left(w-2\frac{d}{b}(z+ax)-\left(ab+\frac{d^2}{b^2}\right)y\right)+x\left(z+ax+\frac{d}{b}y\right)^2+by^2\left(z+ax+\frac{d}{b}y\right)+(c-a^2)x^3=0,$$

or more simply

$$xyw+xz^2+y^2z-ax^3=0.$$

We shall in the sequel retain the constant a, because its being positive or negative decides as to reality or non-reality. But now that we are concerned with the reciprocal equation of the surface, we may, on putting $a=\lambda^4$, change y, z, w respectively into λy, $\lambda^2 z$, $\lambda^3 w$, and we get

$$xyw+xz^2+y^2z-x^3=0.$$

The reciprocal septic Θ, when multiplied by s^5, is the discriminant S^3-27T^2 of the binary quartic

$$y^2(rx-sy)^2+4sx^2(sx^2+pxy+qy^2);$$

hence

$$12S=L^2+24s^2M, \quad -216T=L^3+36s^2LM+216s^4N,$$

where

$$L=r^2+4qs, \quad M=pr+2s^2, \quad N=p^2-4qs;$$

and

$$\begin{aligned}\Theta=\;&L^2.\frac{M^2-LN}{4s}+8sM^3-9sLMN-27s^3N\\ =\;&64s^7+32(3pr-4q^2)s^5+16q(5r^2+9p^2)s^4\\ &+(r^4+30p^2r^2+160pq^2r-27p^4+64q^4)s^3\\ &+4q(11pr^3+12q^2r^2-9p^3r-4p^2q^2)s^2\\ &+r^2(pr^3+12q^2r^2-p^3r-8p^2q^2)s+qr^4(r^2-p^2)=0\end{aligned}$$

is the equation of the surface in plane-coordinates.

The quartic function, the Hessian of the cubic $f=xyw+xz^2+y^2z-x^3$, is

$$xf-4xy^2z+4x^4+y^4;$$

and since the system

$$(f=0,\quad 4x^4+y^4-4xy^2z=0)$$

contains the axis $(x=0,\ y=0)$ four times, the spinode curve of the original surface is a partial curve of the *eighth* degree. An arbitrary plane passing through the node intersects the quartic cone in four lines, each of which also cuts the cubic surface in a point distinct from the node. This arbitrary plane thus intersects the spinode curve in four points distinct from the node, so that this must be a quadruple point of the curve, since it unites the remaining points of intersection. One of the four branches passing through the node is (at the lowest approximation) represented by

$$yw=-5z^2,\quad xw^2=\frac{25}{4}z^3,$$

and therefore osculates the nodal plane which is a singular tangent plane to the surface, and merely touches the other nodal plane. If t denote a very small variable number, the three other branches may be represented by

$$z=t^3w,\quad x=t^5w,\quad y=-t^6w.$$

Art. 24. The nodal plane $x=0$, which touches the surface along the nodal edge or *axis*, contains only a single disengaged ray (call it f), the other nodal plane $y=0$ contains two rays (call them g, h); and the planes combining the former ray with any one of the two latter rays are (fg) or $z+x=0$, and (fh) or $z-x=0$. It is manifest that, besides the nodal planes and these two planes, there pass no other triangle-planes through the node. The planes (fg) and (fh) intersect the surface respectively in the simple lines $(z+x=0,\ w-y=0)$ or j and $(z-x=0,\ w+y=0)$ or k. Now as a plane containing the node and any distinct and therefore simple line of the surface must be a triangle-plane and therefore combine two nodal rays, there cannot, on all such planes being exhausted, be found any other simple line of the surface. Hence these distinct lines j and k are the only simple lines of the surface, and it is obvious that they do not intersect each other. Again, since two independent lines are cut by five lines, and these lines (j and k) are simultaneously cut only by the ray f, this ray f unites five lines of the general surface; and then, because each of the simple lines j and k must besides be cut by five more lines, each of the nodal rays g and h also unites five lines. But all the lines thus far mentioned count as $3\cdot5+2\cdot1=17$ lines. Therefore the nodal edge (or *axis*) unites ten lines of the general surface, precisely those *ten* (as we know from art. 1) lines disengaged from the two independent lines j and k.

It is already proved that the two planes passing through the simple lines and the node count each of them as five. No one of the five lines united in the ray f intersects any other of them; wherefore no two of them can lie in the same triangle-plane. But two triangle-planes passing through any one of them have been already spoken of, viz. (fg) and (fh); the three remaining ones must therefore coincide with the nodal plane $x=0$;

hence this plane counts as fifteen. The ray g unites the five lines intersecting j but not k, the ray h unites the five lines intersecting k but not j, and each of the former five lines is (as may be inferred from the consideration of a simple triangle) cut by four of the latter five, which determine with it four different triangles. Therefore twenty triangle-planes coincide in the nodal plane $y=0$. In all $1·20+1·15+2·5=45$ triangle-planes.

The same consequences may be derived from a trihedral-pair form. Let

$$U=x+z-h(w-y),\quad V=-y-h(x+z),\quad W=x-z,$$
$$X=x-z+h(w+y),\quad Y=\ \ y-h(x-z),\quad Z=x+z,$$

then

$$UVW+XYZ=2h\{xyw+xz^2+y^2z-x^3+hy(z^2-x^2)\},\quad V+hW+Y+hZ=0,$$
$$U+hV-(1+h^2)W+X-hY-(1-h^2)Z=0.$$

If the number h vanish, the equation $UVW+XYZ=0$ will, at the limit, exhibit the present surface, and the former of the linear relations, by reason of the latter, counts for three relations answering to the cubic condition. Omitting the three accents as in (ux) and the permutations as in (lmn), we then get the following survey of the manner of coincidence of the 27 lines and 45 triangle-planes of the general surface.

The axis $(x=0,\ y=0)$ unites the ten lines $(\overline{vy},\ l,\ p,\ r)$. The nodal rays unite each of them five lines; viz. the ray $(x=0,\ z=0)$ unites $(\overline{ux},\ \overline{wz},\ q)$, the ray $(y=0,\ x+z=0)$ unites $(\overline{uy},\ \overline{vz},\ n)$, the ray $(y=0,\ x-z=0)$ unites $(\overline{vx},\ \overline{wy},\ m)$; and there remain but two simple lines $(x+z=0,\ w-y=0)$ or $\overline{uz}$, and $(x-z=0,\ w+y=0)$ or $\overline{wx}$.

The nodal plane $y=0$ unites the twenty triangle-planes v, y, (uy), (vx), (vz), (wy), (lmn); the nodal plane $x=0$ unites the fifteen planes (ux), (vy), (wz), (pqr); the two remaining planes unite each of them five triangle-planes, viz. $x+z=0$ unites u, [illegible] (uz); and $x-z=0$ unites w, x, (wx).

Art. 25. In the equation $xyw+xz^2+y^2z-ax^3=0$ the vanishing of the constant a would give rise to a second node $\frac{\partial}{\partial x}$. Therefore we have here only the two cases when a is positive and when it is negative. Since no two of the linear functions x, y, z, w play a like part in the equation, we are obliged to suppose them all real. So there are only two species.

1. a is positive; all is real. VII. 1.
2. a is negative; the two simple lines are conjugate, and so also the two rays in the nodal plane $y=0$. VII. 2.

VIII. *Cubic surface of the sixth class with three proper nodes.*

Art. 26. If we place the points of reference $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial w}$ at the three nodes, the equation of the surface will contain the terms x^3, x^2y, x^2z, x^2w, xzw, xwy, xyz, yzw; but the last term is capable of taking up the three next preceding terms, in other words, the three singular tangent planes which touch the surface along the *axes* (or lines joining two nodes) may be chosen for the planes $y=0$, $z=0$, $w=0$; then we are at liberty to write

$$x^3+(y+z+w)x^2+ayzw=0$$

as an equation of the surface. The term x^3, if disappearing, would not alter the class, but would merely form a particular case of the sort of surface here to be considered, which case might readily be restored from the more general form by changing a, x respectively into h^2, hx; dividing by h^3 and letting h vanish. But the term x^2y cannot disappear without bringing the class down to *five*; for the point $\frac{\partial}{\partial y}$ would then become a biplanar node. Nor is the constant a allowed to be zero or -4; for in the former case the cubic would be divisible by x^2, in the latter it would be half the expression

$$x\Sigma(x+2z)(x+2w)-(x+2y)(x+2z)(x+2w),$$

which shows a fourth proper node at the point

$$-2\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}+\frac{\partial}{\partial w}.$$

If we denote the surface by $f=0$, and let

$$\frac{\partial f}{\partial x}=pu,\ \frac{\partial f}{\partial y}=qu,\ \frac{\partial f}{\partial z}=ru,\ \frac{\partial f}{\partial w}=su,\ x^2=t,\ \chi=\tfrac{1}{4}at(t-pu)^2+(t-qu)(t-ru)(t-su),$$

where t, u are to be regarded as the independent variables, then we have not only

$$\chi=af\{x^3+x^2(y+z+w)-ayzw\},$$

but also

$$\frac{\partial\chi}{\partial t}=a(y+z+w)f;$$

whence

$$\frac{\partial\chi}{\partial t}=0,\quad \frac{\partial\chi}{\partial u}=0,$$

whenever $f=0$. That is to say, the equation reciprocal to $f=0$ is the discriminant Θ of the binary cubic χ, when equated to zero. Putting, for shortness,

$$\alpha=q+r+s,\quad \beta=rs+sq+qr,\quad \gamma=qrs,$$

we find

$$\begin{aligned}27\Theta=&\tfrac{1}{16}a^3p^3(p-q)(p-r)(p-s)\\ &+\tfrac{1}{16}a^2\{(12\beta-\alpha^2)p^4-4(2\alpha\beta+9\gamma)p^3+2(15\alpha\gamma+4\beta^2)p^2-36\beta\gamma p+27\gamma^2\}\\ &+\ \tfrac{1}{2}a\{(6\beta^2-\alpha^2\beta-9\alpha\gamma)p^2+2(6\alpha^2\gamma-\alpha\beta^2-9\beta\gamma)p+2\beta^3+27\gamma^2-9\alpha\beta\gamma\}\\ &-\ (r-s)^2(s-q)^2(q-r)^2=0,\end{aligned}$$

as the equation of the surface in plane-coordinates.

The Hessian of the primitive cubic is

$$4a^2\{ayzw(3x+y+z+w)+x^2(y^2+z^2+w^2-2zw-2wy-2yz)\}.$$

Hence the spinode curve is a complete curve of the sixth degree represented by the system

$$\left.\begin{aligned}&x^3+x^2(y+z+w)+ayzw=0,\\ &\tfrac{3}{4}x^2+x\,(y+z+w)+zw+wy+yz=0,\end{aligned}\right\}$$

which shows that the nodes are double points of the curve, and that at these points the (disengaged) nodal rays of the surface are tangents to the curve.

Art. 27. By what has been said in art. 15 we can at once judge of the disposition of the lines and triangle-planes. The three transversals are the only simple lines, and form a triangle $(x+y+z+w=0, yzw=0)$, the plane whereof is the only simple triangle-plane. The planes determined by a transversal and the opposite node intersect the surface in thrice two (disengaged) nodal rays, each of which unites two lines. Each of the three axes unites four lines. Together $3\cdot1+6\cdot2+3\cdot4=27$ lines. The singular tangent planes $y=0$, $z=0$, $w=0$ count each of them twice, and so also does each of the three planes passing through a transversal and the opposite node. Through each axis and two nodal rays there pass two planes, together six planes, each of which counts four times. Lastly, the plane $x=0$, containing the three nodes, counts eight times. Together $1\cdot1+6\cdot2+6\cdot4+1\cdot8=45$ triangle-planes.

If we assume the trihedral-pair form $UVW+XYZ=0$, where on putting $a=\frac{(\alpha-1)^2}{\alpha}$ we have

$$\begin{array}{lll} U=-(\alpha-1)(x+y+z+w), & V=-\alpha x-(\alpha-1)y, & W=x-(\alpha-1)y, \\ X=\ \ (\alpha-1)y, & Y=(\alpha-1)(x+y+w), & Z=(\alpha-1)(x+y+z), \end{array}$$

then the constants in the auxiliary relations

$a\mathrm{U}+\ldots=0$, $a'\mathrm{U}+\ldots=0$, $a''\mathrm{U}+\ldots=0$ are $a=0$, $b=1$, $c=\alpha$, $d=\alpha+1$, $e=0$, $f=0$

(therefore when e, f are imagined to be indefinitely small, a is of the order ef, whence, for instance, $a\mathrm{U}+f\mathrm{Y}=0$ reduces itself to $\mathrm{Y}=0$),

$$a'=b'=c'=d'=e'=f', \quad a''=b''=c''=d''=e''=f'',$$

and at length we get the following survey: viz., the lines are

$(x=0,\ y=0)$	$[\overline{vx}, \overline{wx}, l, p]$,
$(x=0,\ z=0)$	$[m', m'', r', r'']$,
$(x=0,\ w=0)$	$[n', n'', q', q'']$,
$(x:\ z:w=\alpha-1:-\alpha:1)$	$[l', l'']$,
$(x:\ z:w=\alpha-1:1:-\alpha)$	$[p', p'']$;
$(x:w:\ y=\alpha-1:-\alpha:1)$	$[\overline{wy}, r]$,
$(x:w:\ y=\alpha-1:1:-\alpha)$	$[\overline{vy}, m]$;
$(x\,.\ y\,.\ z=\alpha-1:-\alpha:1)$	$[\overline{vz}, q]$,
$(x:\ y\,.\ z=\alpha-1:1:-\alpha)$	$[\overline{wz}, n]$;
$(x+z+w=0,\ y=0)$	$[\overline{ux}]$,
$(x+y+w=0,\ z=0)$	$[\overline{uy}]$,
$(x+y+z=0,\ w=0)$	$[\overline{uz}]$;

and the planes are

$(x=0)$	$[(vx)', (vx)'', (wx)', (wx)'', (lmn), (lnm), (pqr), (prq)]$,
$(\ \ \alpha x+(\alpha-1)y=0)$	$[v, (vy), (vz), (wx)]$,
$(-\ x+(\alpha-1)y=0)$	$[w, (vx), (wy), (wz)]$,
$(\ \ \alpha x+(\alpha-1)z=0)$	$[(wz)', (wz)'', (nlm), (nml)]$,

$(-\ x+(\alpha-1)z=0)\quad [(vz)', (vz)'', (qpr), (qrp)],$
$(\ \ \alpha x+(\alpha-1)w=0)\quad [(wy)', (wy)'', (rpq), (rqp)],$
$(-\ x+(\alpha-1)w=0)\quad [(vy)', (vy)'', (mln), (mnl)],$
$(y=0)\quad [x, (ux)],\quad (z=0)\quad [(uy)', (uy)''],\quad (w=0)\quad [uz)', (uz)''],$
$(x+z+w=0)\quad [(ux)', (ux)''],$
$(x+y+w=0)\quad [y, (uy)],\quad (x+y+z=0)\quad [z, (uz)],$
$(x+y+z+w=0)\quad [u].$

Art. 28. One node at least must be real, for instance $\frac{\partial}{\partial y}$, and then the two others may be real or conjugate. Accordingly x is always real, and while we keep y real, z and w may be either real or conjugate. On the other hand the constant a may be between -4 and 0, or beyond these limits. From these two reasons of partition there arise four species of the surface with three proper nodes. But we prefer to distinguish five species. For if z, w be conjugate, the nodal cone $x^2+azw=0$ becomes imaginary or real, according as $a>0$ or $a<-4$.

1. z, w are real; $a(a+4)>0$, and therefore α real. All is real. VIII. 1.

2. z,w real; $-4<a<0$. Let $a=-4\sin^2\frac{\theta}{2}$, then $\alpha=\varepsilon^{i\theta}$. The real lines are the three *axes* and the three transversals. The real planes are the plane of the three nodes, the three singular tangent planes, the plane passing through a transversal and the opposite node, and the transversal plane. VIII. 2.

3. z, w conjugate, $a>0$, and therefore α positive. The two nodes $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial w}$ are conjugate, the nodal cone at the real node $\frac{\partial}{\partial y}$ is imaginary. The real lines are the axis joining the conjugate nodes, and its transversal. The real planes are the plane of the three nodes, the singular tangent plane through the real axis, two other planes which pass through the real axis, the plane passing through the real transversal and the real node, and the transversal plane. VIII. 3.

4. z, w conjugate, $a<-4$, and therefore α negative. The nodal cone at the real node is real, but its two (disengaged) rays are imaginary and conjugate. The rest as before. VIII. 4.

5. z, w conjugate, $-4<a<0$. The nodal cone at the real node is real. The real lines are the axis joining the conjugate nodes, its transversal, and the two (disengaged) rays of the real node. The real planes are that of the three nodes, the singular tangent plane through the real axis, the plane passing through the real transversal and the real node, and the transversal plane. VIII. 5.

IX. *Cubic surface of the sixth class with two biplanar nodes.*

Art. 29. From art. 20 it appears that the reduced equation of this sort of surface is

$$xzw+(y+\alpha x)(y+\beta x)(y+\gamma x)=0,$$

where $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial w}$ are the biplanar nodes. These have in common the nodal plane $x=0$, which osculates the surface along the axis ($x=0$, $y=0$). The other nodal planes are $z=0$, $w=0$, each of which intersects the surface in three nodal rays. In order however to find the reciprocal equation it is more convenient to write

$$12xzw+ax^3+3bx^2y+3cxy^2+dy^3=0$$

as the equation in point-coordinates. Then the discriminant of the binary cubic

$$rs(ax^3+3bx^2y+3cxy^2+dy^3)+3x(px+qy)^2,$$

divided by rs and equated to zero, will furnish the equation in plane-coordinates as follows; viz. this is

$$\begin{aligned}&[a^2d^2-6abcd-3b^2c^2+4ac^3+4b^3d]r^3s^3\\&+6[(ad^2-3bcd+2c^3)p^2+(4b^2d-2acd-2bc^2)pq+(2ac^2-abd-b^2c)q^2]r^2s^2\\&+3[3d^2p^4-12cdp^3q+(10bd+8c^2)p^2q^2-(4ad+8bc)pq^3+(4ac-b^2)q^4]rs\\&-4q^3(dp^3-3cp^2q+3bpq^2-aq^3)=0.\end{aligned}$$

The Hessian of the cubic $12xzw+ax^3+3bx^2y+3cxy^2+dy^3$ is

$$64\cdot81x\{4zw(cx+dy)-(ac-b^2)x^3-(ad-bc)x^2y-(bd-c^2)xy^2\}.$$

The system of the two expressions equated to zero breaks up into four times the axis ($x=0$, $y=0$) and four conics which lie in the planes

$$(4ac-3b^2)x^4+4adx^3y+6bdx^2y^2+4cdxy^3+d^2y^4=0,$$

and touch the nodal planes $z=0$, $w=0$ at the corresponding nodes. Two of these four planes are always imaginary and conjugate, the two remaining ones are real. For let

$$\mathrm{K}=a^2d^2-6abcd-3b^2c^2+4ac^3+4b^3d,\ k^3=\tfrac{1}{4}d^2\mathrm{K},$$

and take for k the single real value; again, let $l^2=k+c^2-bd$, which is positive, since

$$k^3+(c^2-bd)^3=\tfrac{1}{4}(ad^2-3bcd+2c^3)^2,$$

and determine the value of l by the condition that $l(ad^2-3bcd+2c^3)$ shall become positive; then the constants m^2, n^2 determined by

$$l(m^2+n^2)=2(ad^2-3bcd+2c^3),\quad n^2-m^2=2k+4(bd-c^2)$$

will be positive, because this system implies $mn=\sqrt{3}\,.\,k$; and the equation of the four planes breaks up into

$$\{(dy+(c+l)x)^2-m^2x^2\}\{(dy+(c-l)x)^2+n^2x^2\}=0.$$

The section made by the real plane $dy+(c+l+m)x=0$ is represented by

$$24d^2zw-m\big((2l+m)^2+n^2\big)x^2=0.$$

In the case therefore when both z and w are real, the two real planes contain also real conics; but when z and w are conjugate, one only of the two real planes intersects the surface also in a real conic, the other real plane has, besides the axis, no real point in common with the surface.

Art. 30. We now suppose $xzw+(y+\alpha x)(y+\beta x)(y+\gamma x)$ to be the equation of the surface. As this form results from that of art. 21, by changing z, w respectively into kz,

$\frac{w}{k}$, and letting k vanish, we may readily thence get a knowledge of the disposition of the twenty-seven lines and forty-five triangle-planes, and we shall in particular see that the axis here unites all the nine lines immediately afforded by a trihedral-pair. Changing then the notation for the sake of greater symmetry, we can regard the surface as though the six planes of $uvw+xyz=0$ coincided with the singular osculating plane $(x=0)$, while the nine lines $\overline{ux}$, &c. coincided with the axis. One of the two remaining nodal planes will then unite all the six planes such as (lmn), and contain the three nodal rays (l, m, n), (l', m', n'), (l'', m'', n''); the other nodal plane will unite all the six planes such as (pqr), and contain the three nodal rays (p, q, r), (p', q', r'), (p'', q'', r''). One of the remaining triangle-planes passing through the axis, for instance the plane which combines the nodal rays (l, m, n) and (p, q, r), would then unite the nine triangle-planes

$$(ux),\ (uy),\ (uz),\ (vx),\ (vy),\ (vz),\ (wx),\ (wy),\ (wz),$$

and the other two like planes would answer to the two remaining accents. In the whole $1\cdot 9+6\cdot 3=27$ lines and $3\cdot 9+3\cdot 6=45$ triangle-planes.

The singular osculating plane $x=0$, and one at least of the three other planes passing through the axis $y+dx=0$, for instance, must be real. But z, w can be either real or conjugate, and so also the constants β, γ. From this double reason of partition we get four species, IX. 1; IX. 2; IX. 3; IX. 4.

X. *Cubic surface of the sixth class with a biplanar node and a proper node.*

Art. 31. The cubic surface with a biplanar node which lowers the class by *four* can only in this way have a second node distinct from the first, when the two (disengaged) rays of one nodal plane unite themselves together apart from the nodal edge. The equation then takes the form

$$xyw+(x+y)(z^2-ax^2)=0.$$

Changing z, w respectively into $\surd a\,.\,z$, aw, we might reduce this equation to

$$xyw+(x+y)(z^2-x^2)=0.$$

But since a may be either positive or negative, in the latter case we should get z as the product of the numerical factor i $(=\sqrt{-1})$ by a real function; and to avoid this we shall retain the constant a.

If we let $a=1$ and denote the discriminant of the binary cubic

$$s^2x(x+y)^2+sy(x+y)(px+qy)+\tfrac{1}{4}r^2xy^2$$

by $\frac{1}{27}s^2\Theta$, then

$$\Theta=[r^2-(p-q)^2]s^4+[(2p-5q)r^2-2(p-2q)(p-q)^2]s^3+$$
$$[\tfrac{1}{2}r^4+\tfrac{1}{2}(p^2-pq+6q^2)r^2-p^2(p-q)^2]s^2+[\tfrac{1}{4}(2p+3q)r^4-\tfrac{1}{2}p^2(p+q)r^2]s+\tfrac{1}{16}r^4(r^2-p^2)=0$$

is the reciprocal equation of the surface.

The quartic function, the Hessian of $xyw+(x+y)(z^2-x^2)$, is

$$x(x+y)(yw+3x^2-xy)+z^2(x-y)^2.$$

Hence the spinode curve breaks up into four times the axis ($x=0$, $z=0$) joining both nodes, twice the nodal edge, or also axis, ($x=0$, $y=0$), and the complete curve

$$(yw+z^2=0,\quad x^2y+x^3-yz^2=0)$$

of the sixth degree. It has passing through the biplanar node three branches, represented in the lowest approximation by

$$yw+z^2=0,\quad x^2w+z^4=0,$$

where only w is finite, and through the proper node two branches, the tangents whereof are the two disengaged rays of this proper node, represented by

$$w=0,\quad z^2-x^2=0.$$

Art. 32. Let

$$\begin{aligned} \mathrm{U}&=-w+2h(x+y), & \mathrm{V}&=\ \ z+x+hy, & \mathrm{W}&=\ \ z-x-hy,\\ \mathrm{X}&=\ \ w+2h(x+y), & \mathrm{Y}&=-z-x-hy, & \mathrm{Z}&=-z+x-hy, \end{aligned}$$

where h denotes a constant which ultimately vanishes; then

$$\mathrm{UVW}+\mathrm{XYZ}=4h\{xyw+(x+y)(z^2-x^2-h^2y^2)\}=0$$

represents the surface in question, and

$$\mathrm{V}+\mathrm{W}+\mathrm{Y}+\mathrm{Z}=0,\quad \mathrm{U}-(1+h)\mathrm{V}+(1+h)\mathrm{W}+\mathrm{X}-(1-h)\mathrm{Y}+(1-h)\mathrm{Z}=0$$

are identical relations, the former of which, in virtue of the latter, stands for the three equations which correspond to the roots of the condition ABC=DEF. Hence we get the following survey of the manner of coincidence of some of the twenty-seven lines and forty-five triangle-planes (accents and permutations are omitted).

The axis joining both nodes ($x=0$, $z=0$) unites $\overline{vz}$, $\overline{wy}$, m, n, eight lines. The nodal edge, also an axis, ($x=0$, $y=0$) unites l, p, six lines. The two disengaged rays of the biplanar node count each of them four times, viz. ($y=0$, $x+z=0$) unites $\overline{vy}$, r, and ($y=0$, $x-z=0$) unites $\overline{wz}$, q. The two disengaged rays of the proper node count each of them twice, viz. ($w=0$, $x+z=0$) unites $\overline{uy}$, $\overline{vx}$, and ($w=0$, $x-z=0$) unites $\overline{uz}$, $\overline{wx}$. Lastly, the transversal of the nodal edge ($w=0$, $x+y=0$) is the only simple line $\overline{ux}$. Together $1\cdot8+1\cdot6+2\cdot4+2\ 2+1\cdot1=27$ lines.

The planes of the biplanar node count twelve times, viz. $x=0$ (a singular tangent plane) unites (vz), (wy), (lmn), and $y=0$ unites (vy), (wz), (pqr). The two planes combining the double ray of the biplanar node with each of its two simple rays count eight times, viz. $z+x=0$ unites v, (vx), y, (uy); and $x-z=0$ unites w, (wx), z, (uz). The plane $x+y=0$ touching the surface along the nodal edge unites (ux) three triangle-planes, and the plane $w=0$ combining the two simple rays of the proper node, unites u, x, two triangle-planes of the general surface. In all $2\cdot12+2\cdot8+1\cdot3+1\cdot2=45$ planes.

Because no two of the four linear functions x, y, z enter in a similar manner into the form $xyw+(x+y)(z^2-ax^2)=0$, all of them must be real. Only the constant a, according as it is positive or negative, gives rise to a distinction between *two* species. X. 1; X. 2.

XI. *Cubic surface of the sixth class with a biplanar node.*

Art. 33. From art. 7 (see art. 23) we know that one of the nodal planes must osculate the surface along the nodal edge, in order that the node $\frac{\partial}{\partial w}$ may lower the class by *six*, and since in the first term xyw of the equation of the surface all other terms divisible by xy may be included, we write the equation immediately in the form

$$xyw+xz^2+2ax^2z+bx^3+dy^3=0,$$

or, what is the same thing,

$$x\,.\,dy\,.\,dw+x(dz+adx)^2+d^2(b-a^2)x^3+(dy)^3=0,$$

or, to save constants,

$$xyw+xz^2+ax^3+y^3=0,$$

which is the assumed form for the equation of the cubic. It is well to observe that here all the letters are necessarily real, provided that the surface be real. Putting $a=-\mu^6$ and changing y, z, w respectively into μ^2y, μ^3z, μ^4w, we might get

$$xyw+xz^2-x^3+y^3=0,$$

where no explicit constant remains; but then z would cease to be necessarily real.

If we denote the discriminant of the binary cubic

$$(3as^2,\ -ps,\ -(qs+\tfrac{1}{4}r^2),\ 3s^2\between x,\ y)^3$$

by $\frac{3}{16}s^2\Theta$, then

$$\Theta=-64s^3p^3-(4qs+r^2)^2p-72as^3(4qs+r^2)p-a(4qs+r^2)^3+432a^2s^6=0$$

is the reciprocal equation. It is obvious that

$$27a^2\Theta=\{8p^3+9a(4qs+r^2)p-108a^2s^3\}^2-\{4p^2+3a(4qs+r^2)\}^3.$$

The Hessian of the cubic is

$$4x(xyw+xz^2-3ax^3-3y^3).$$

The spinode curve then breaks up into six times the axis ($x=0$, $y=0$), and the three distinct conics

$$(ax^3+y^3=0,\quad yw+z^2=0).$$

Art. 34. The trihedral-pair form can only be obtained by the help of two constants which ultimately vanish. Let them be h and ω, the finite constant a be $=-\varrho^2$, and

$$\begin{aligned}
\mathrm{U}&=(1+\omega+\omega h^3\varrho)y+h(1+2\omega)z+hx-\omega h^2w,\\
\mathrm{V}&=(1+\omega)y+\omega hz+\omega h\varrho x,\\
\mathrm{W}&=y+hz-h\varrho x,\\
\mathrm{X}&=-(1+\omega-\omega h^3\varrho)y-h(1+2\omega)z+h\varrho x+\omega h^2w,\\
\mathrm{Y}&=-(1+\omega)y-\omega hz+\omega h\varrho x,\\
\mathrm{Z}&=-y-hz-h\varrho x;
\end{aligned}$$

then

$$\mathrm{UVW}+\mathrm{XYZ}=2\omega h^3\varrho\{xyw+(x+\omega h^2y)(z^2-\varrho^2x^2)+(1+\omega)y^3\}$$

becomes the cubic of the surface as soon as h and ω vanish. Of the two identical relations

$$V+\omega W+Y+\omega Z=0,$$

$$U-\omega h^3\varrho V+(1+\omega^2h^3\varrho)W+X+\omega h^3\varrho Y+(1-\omega^2h^3\varrho)Z=0,$$

the former, in virtue of the latter, stands for the three equations which correspond to the condition ABC=DEF. We may therefore, in the following survey of lines and triangle-planes, omit accents and permutations.

The nodal edge ($x=0$, $y=0$) unites $\overline{ux}$, $\overline{vy}$, $\overline{wz}$, l, p, q, r, fifteen lines. The two nodal rays count six times, viz.

$$(y=0,\ z+\varrho x=0) \text{ unites } \overline{uy},\ \overline{vz},\ \overline{wx},\ n,$$

$$(y=0,\ z-\varrho x=0) \text{ unites } \overline{vx},\ \overline{wy},\ \overline{uz},\ m;$$

in the same order as they here are written, they form a double six. Together $1\cdot15+2\cdot6=27$ lines.

Each line of the one six, combined successively with the five not corresponding lines of the other six, gives rise to five triangle-planes; all the thirty planes so obtained coincide with the nodal plane $y=0$, viz. u, v, w, x, y, z, (uy), (uz), (vx), (vz), (wx), (wy), (lmn). Again, as to the fifteen lines first mentioned, which, as we know, form fifteen triangles, all their planes here coincide with the osculating nodal plane $x=0$, viz. (ux), (vy), (wz), (pqr). Together $1\cdot30+1\cdot15=45$ triangle-planes.

We can distinguish only two species, according as the constant a is negative or positive (1, the two disengaged nodal rays are real; 2, they are conjugate). XI. 1; XI. 2. The case where $a=0$ is not considered, because it would imply a proper node at the point $\frac{\partial}{\partial x}$ with the cone $yw+z^2=0$.

XII. *Cubic surface of the sixth class with a uniplanar node.*

Art. 35. The simplest form of the equation is

$$(x+y+z)^2w+xyz=0.$$

If all four letters are real, the three nodal rays ($x+y+z=0$, $xyz=0$) are all of them real, and imply the applanished proximity of the node into six angular spaces alternatively full and empty, so that there appear three flat thorns having the node for their common point*. (The surface here considered arises from III. 4, if there all the conjugate values be allowed to coincide by pairs.)

Let $sw=t$, $x+y+z=u$, and $\frac{1}{27}\Theta$ be the discriminant of the binary cubic

$$(t-pu)(t-qu)(t-ru)+\tfrac{1}{4}st^2u$$

in respect to t, u; then $\Theta=0$ will be the equation reciprocal to $u^2w+xyz=0$. Putting

$$\alpha=p+q+r,\quad \beta=qr+rp+pq,\quad \gamma=pqr,$$

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* A notion of the form of the surface may be most readily acquired by taking the equation to be

$$z^2+xy(z-mx-ny)=0.\text{—A. C.}$$

we have

$$\Theta = -(q-r)^2(r-p)^2(p-q)^2 + \tfrac{1}{2}(\alpha\beta^2 + 9\beta\gamma - 6\alpha^2\gamma)s + \tfrac{1}{16}(12\alpha\gamma - \beta^2)s^2 - \tfrac{1}{16}\gamma s^3.$$

The Hessian of the original cubic is

$$4(x+y+z)^2(x^2+y^2+z^2-2yz-2zx-2xy).$$

The spinode curve therefore breaks up into twice the nodal rays (or *axes*)

$$(x+y+z=0,\ xyz=0)$$

and a complete curve of the sixth degree, arising from the intersection of a quadratic cone, which cone is inscribed in the trihedral ($xyz=0$) of the singular tangent planes in such manner that the lines of contact are harmonical with the nodal rays in respect to the edges of the trihedral. The nodal plane does not really intersect this cone when all three planes of the trihedral are real; but it does so when one of them is real and the two others are conjugate. The node is a quadruple point on the curve of the sixth degree, and the two intersection-lines last mentioned are here a kind of cuspidal tangents.

Art. 36. In order to get a trihedral-pair form, let a, b, c be finite numbers, h a number which ultimately vanishes, and put $(b-c)(c-a)(a-b)=m$, and moreover

$$\begin{aligned} U &= (b-c)(1+ah)x + mh^2w, \\ V &= (c-a)(1+bh)y + mh^2w, \\ W &= (a-b)(1+ch)z + mh^2w, \\ X &= -mh^2w, \end{aligned}$$

$$\begin{aligned} hY &= -mh^3w + (1+ah)(1+ch)x + (1+bh)(1+ah)y + (1+ch)(1+bh)z, \\ -hZ &= mh^3w + (1+ah)(1+bh)x + (1+bh)(1+ch)y + (1+ch)(1+ah)z\,; \end{aligned}$$

then the equation

$$UVW + XYZ = m(1+ah)(1+bh)(1+ch)\{w(x+y+z)[x+y+z+h(ax+by+cz)] + xyz\}$$

is identically true, and the six functions U, V, W, X, Y, Z satisfy the identical relations

$$U+V+W+X+Y+Z=0, \quad AU+BV+CW+DX+EY+FZ=0,$$

where the numbers

$$A=(c-a)(a-b)(1+ch), \quad B=(a-b)(b-c)(1+ah), \quad C=(b-c)(c-a)(1+bh),$$
$$D=A-(b-c)^2(1+ah), \quad E=-mh, \quad F=0$$

satisfy the conditions

$$A+B+C=D+E+F, \quad BC+CA+AB=EF+FD+DE,$$

without ABC−DEF vanishing. As long therefore as h is finite, the surface UVW+XYZ=0 has a biplanar node at the point $\frac{\partial}{\partial w}$, and this becomes uniplanar when h vanishes. Omitting then accents and permutations, because the three roots of the auxiliary cubic condition are equal, we get the following survey.

The three nodal rays count eight times; for

$$(u=0,\ x=0) \text{ unites } \overline{uy},\ \overline{uz},\ l,\ p,$$
$$(u=0,\ y=0) \text{ unites } \overline{vy},\ \overline{vz},\ n,\ r,$$
$$(u=0,\ z=0) \text{ unites } \overline{wy},\ \overline{wz},\ m,\ q.$$

The sides of the triangle $(w=0,\ xyz=0)$ are simple, because they do not pass through the node; they are $\overline{ux}$, $\overline{vx}$, $\overline{wx}$ of the old notation. Together $3\cdot8+3\cdot1=27$ lines.

The nodal plane $u=0$ unites y, z, (uy), (uz), (vy), (vz), (wy), (wz), (lmn), (pqr), thirty-two triangle-planes. The three singular tangent planes count four times; for $x=0$ unites u, (ux), and so on. The transversal plane $w=0$ is the only simple triangle-plane x of the old notation. In the whole $1\cdot32+3\cdot4+1\cdot1=45$ triangle-planes.

All this might have been foreseen by the help of easy geometrical considerations.

As to reality, the function w must be real, and so must also one at least of the three functions x, y, z, for instance x. We then have only *two* species, according as y, z are real or conjugate. XII. 1; and XII 2.

XIII. *Cubic surface of the fifth class with a biplanar and two proper nodes.*

Art. 37 Such surface arises from art. 21, when there the binary cubic

$$(y+\alpha x)(y+\beta x)(y+\gamma x)$$

has two equal roots. We are then at liberty to put $\beta=\gamma=0$, $\alpha=1$, and permuting x and y we get

$$yzw+x^2(x+y+z)=0$$

as the equation of the surface, where $\frac{\partial}{\partial w}$ is the biplanar and $\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are the proper nodes. And the survey given in the same article changes into the following:—

Lines unite.			*Planes* unite.		
$(x=0,\ y=0)$	m, n,	6	$x=0$	(uz), (uy), (vx), (wx),	12
$(x=0,\ z=0)$	q, r,	6	$y=0$	(lmn),	6
$(x=0,\ w=0)$	$\overline{uy}$, $\overline{uz}$, $\overline{vx}$, $\overline{wx}$,	4	$z=0$	(pqr),	6
$(y=0,\ x+z=0)$	l,	3	$x+y=0$	(vz), (wy),	6
$(z=0,\ x+y=0)$	p,	3	$x+z=0$	(vy), (wz),	6
$(x=-z=w)$	$\overline{vy}$, $\overline{wz}$,	2	$x-w=0$	v, w, y, z,	4
$(x=-y=w)$	$\overline{vz}$, $\overline{wy}$,	2	$x+y+z=0$	(ux),	3
$(w=0,\ x+y+z=0)$	$\overline{ux}$,	1	$w=0$	u, x,	2
		27			45

The discriminant of the ternary cubic $sx^2(x+y+z)-yz(px+qy+vz)$ divided by $q^2r^2s^3$ and then equated to zero is the reciprocal equation of the surface. But this may also be derived from art. 20, and will be found to be

$$\begin{aligned}\Theta&=(s+p-q-r)\{4(q+r)s+p^2\}^2-8qr(2s+p)^3+9qr(2s+p)\{4(q+r)s+p^2\}-27q^2r^2s\\&=16(q-r)^2s^3+8\{p^2(q+r)+2p(q^2-4qr+r^2)-(q+r)(2q-r)(q-2r)\}s^2\\&+\{p^4+8p^3(q+r)-2p^2(4q^2+23qr+4r^2)+36pqr(q+r)-27q^2r^2\}s\\&+p^2(p-q)(p-r)=0.\end{aligned}$$

The Hessian of $yzw+x^2(x+y+z)$ is $4\{yzw(3x+y+z)+x^2(y-z)^2\}$. Hence the spinode curve breaks up into three times the axes joining the biplanar to the two proper nodes, twice the third axis, and a complete curve of the fourth degree formed by the intersection of the cones

$$(3x+4y+4z)^2-(6x+5y+5z)^2+9(y-z)^2=0,$$
$$(3x+4y+4z)(9w+4x+4y+4z)-16(y-z)^2=0,$$

the latter of which passes through the vertex of the former, *i. e.* through the biplanar node. This is therefore a double point of the curve, and the tangents are

$$(3x+4y+4z=0,\ yz=0).$$

There are but two species; for x, w must be real, and only y, z can either be real or conjugate.

1. All is real. XIII. 1.

2. y and z are conjugate. The two proper nodes are conjugate, and so are also the two planes of the biplanar node. The axis joining the two proper nodes, and the transversal of this axis are the only real lines. XIII. 2.

XIV. *Cubic surface of the fifth class with a biplanar node and a proper node.*

Art. 38. As we have seen above (art. 23), the presence of a biplanar node such as lowers the class by *five* reduces the equation of the surface to the form

$$xyw+xz^2+y^2z-ax^3=0.$$

Because the nodal plane $x=0$ contains but *one* disengaged ray ($x=0$, $z=0$), only the union of the *two* disengaged rays ($y=0$, $z^2-ax^2=0$) in the other nodal plane can give rise to a proper node. Hence the constant a must vanish. The surface in question is therefore represented by

$$xyw+xz^2+y^2z=0$$

in point-coordinates, and consequently by

$$\Theta=27p^2s^3+(36pqr+16q^3)s^2+(pr^3+8q^2r^2)s+qr^4=0$$

in plane-coordinates; $48s^3\Theta$ is the discriminant of the binary cubic

$$12sx^2(px+qy)+3y(rx-sy)^2;$$

and

$$108s\Theta=(54ps^2+36qrs+r^3)^2+(12qs-r^2)^3.$$

The Hessian of the original cubic is $4\{x^2yw+x^2z^2-3xy^2z+y^4\}$. The spinode curve therefore breaks up into five times the axis ($y=0$, $z=0$) joining the two nodes, four times the nodal edge ($x=0$, $y=0$), and a partial curve of the third degree, which may be represented by

$$\left\| \begin{matrix} y, & z, & w \\ 4x, & y, & -5z \end{matrix} \right\| =0,$$

or, which is the same thing, by $y=2\lambda x$, $z=\lambda^2 x$, $w=-\frac{5}{2}\lambda^3 x$, where λ denotes a variable number. Since the plane touching the original surface at this current point has the equation

$$-8\lambda^3 x+3\lambda^2 y+12\lambda z+4w=0,$$

the spinode develope is represented by the vanishing of the discriminant of the binary cubic $(-8x,\ y,\ 4z,\ 4w \between \lambda,\ 1)^3$, that is to say, by

$$V=64x^2w^2+(48xyz+y^3)w-128xz^3-3y^2z^2=0\,;$$

and we have in fact

$$xy^2V=(64x^2yw-64x^2z^2-16xy^2z+y^4)(xyw+xz^2+y^2z)+z(4xz-y^2)^3,$$

which shows that the curve is contained three times in the intersection of the original surface and the developable $V=0$. The cuspidal line of this developable is represented by

$$y=8\lambda x,\quad z=-2\lambda^2 x,\quad w=2\lambda^3 x,$$

and is therefore a partial curve of the third degree. The equation in plane-coordinates of the spinode curve is

$$675p^2s^2+16pr^3+360pqrs-320q^3s-16q^2r^2=0.$$

In point-coordinates the developable formed by the tangents of the spinode curve is represented by the vanishing of the discriminant of the binary cubic

$$(10x,\ -5y,\ 10z,\ 4w \between \lambda,\ 1)^3.$$

Art. 39. On putting

$$\begin{aligned} U&=\ \ z-hw+k(x+hy), & V&=-y-hz-hkx, & W&=-z+kx, \\ X&=-z+hw+k(x+hy), & Y&=\ \ y+hz-hkx, & Z&=\ \ z+kx, \end{aligned}$$

whence arise the identical relations

$$V+hW+Y+hZ=0 \text{ (holding three times)},$$
$$U+hkV-(1+h^2k)W+X-hkY-(1-h^2k)Z=0 \text{ (accidental)},$$

the identical equation

$$UVW+XYZ=2hk\{xyw+(x+hy)z^2+y^2z-k^2x^3-hk^2x^2y\},$$

when the constants h, k are made to vanish, enables us to perceive what arrangement is here undergone by the 27 lines and 45 triangle-planes of the general surface.

The edge	$(x=0,\ y=0)$ unites	$\overline{vy},\ l,\ p,\ r,$	10
the axis	$(y=0,\ z=0)$ unites	$\overline{uy},\ \overline{vx},\ \overline{vz},\ \overline{wy},\ m,\ n,$	10
the ray	$(x=0,\ z=0)$ unites	$\overline{ux},\ \overline{wz},\ q,$	5
the ray	$(z=0,\ w=0)$ unites	$\overline{uz},\ \overline{wx},$	2
			27 lines.

The axis joining the two nodes thus unites five rays of the proper node.

Of the nodal planes, $y=0$ unites v, y, (uy), (vx), (vz), (wy), (lmn), 20 triangle-planes; $x=0$ unites (ux), (vy), (wz), (pqr), 15 triangle-planes; and the only plane containing an actual triangle, $z=0$, unites u, w, x, z, (uz), (wx), 10 triangle-planes; $20+15+10=45$.

The only disengaged ray of the proper node unites two independent lines of the surface. The five lines intersecting both of these coincide in the disengaged ray of the biplanar node. The ten lines meeting but one of the two original lines coincide in the axis joining both nodes. And the ten remaining lines coincide in the edge of the biplanar node.

There is but *one* species, because all four linear functions x, y, z, w must be real. XIV. 1.

XV. *Cubic surface of the fifth class with a uniplanar node.*

Art. 40. We have seen above that the cubic surface with a uniplanar node can always be represented by an equation of the form $x^2w+\mathrm{P}+\mathrm{Q}x$, where $\mathrm{P}=(y,\ z)^3$, $\mathrm{Q}=(y,\ z)^2$, and that, whenever P has no two equal factors, the uniplanar node $\frac{\partial}{\partial w}$ for itself lowers the class by six; but upon considering the case where P has two equal factors, it appears that there is a further reduction of one, making the whole reduction of class to be equal *seven*. We are here allowed to write $\mathrm{P}=y^2z$; and the equation of the surface accordingly is

$$x^2w+y^2z+x(ay^2+2byz+cz^2)=0$$

or, what is the same thing,

$$x^2[cw-ac(ac+b^2)x-2abcy-c(2ac+b^2)z]+(y+bx)^2(cz+acx)+x(cz+acx)^2=0$$

or simply

$$x^2w+y^2z+xz^2=0,$$

where all the variables are necessarily real. The equation in plane-coordinates arises when the discriminant of $(-\tfrac{3}{2}q^2,\ qr,\ 2ps,\ 3qs\between x,\ y)^3$ is cleared of the factor $\tfrac{3}{4}q^2s$; hence it is

$$-64p^3s^2-16p^2r^2s+72pq^2rs+27q^4s+16q^2r^3=0.$$

The Hessian of $x^2w+y^2z+xz^2$ is $16x^2(xz-y^2)$. Hence the spinode curve breaks up into six times the double nodal ray $(x=0,\ y=0)$, twice the simple nodal ray $(x=0)$ $z=0)$, and once the complete curve $(xz-y^2=0,\ xw+2z^2=0)$, which has the node for a double point, where the double nodal ray is a tangent common to both branches of the curve.

Art. 41. Denoting by h a number which ultimately vanishes, the surface in question may also be represented by the equation

$$x^2(z+h^2w)-z(x+hy)(x-hy-h^2z)=0.$$

Hence we can see that there is but one simple line ($z=0$, $w=0$), and that all the ten lines intersecting it coincide in the simple nodal ray ($x=0$, $z=0$), while the double nodal ray ($x=0$, $y=0$) unites all the sixteen remaining lines. Again, the plane $z=0$ unites all the five triangle-planes that pass through the only simple line ($z=0$, $w=0$), and the nodal plane $x=0$ unites alone all the forty remaining triangle-planes.

There is but one species. XV. 1.

XVI. *Cubic surface of the fourth class with four proper nodes.*

Art. 42. If we choose the four nodes as points of reference, the equation of the surface necessarily takes the form $ayzw+bxzw+cxyw+dxyz=0$. None of the four constants can vanish, unless the surface break up into a plane and a quadratic surface. We are therefore at liberty to change x, y, z, w respectively into ax, by, cz, dw, when the equation of the surface becomes

$$yzw+xzw+xyw+xyz=0.$$

Since

$$p:q:r:s=\frac{1}{x^2}:\frac{1}{y^2}:\frac{1}{z^2}:\frac{1}{w^2},$$

we have

$$\sqrt{p}+\sqrt{q}+\sqrt{r}+\sqrt{s}=0,$$

or in a rational form

$$(\Sigma p^2-2\Sigma pq)^2-64pqrs=\Sigma p^4-4\Sigma p^3q+6\Sigma p^2q^2+4\Sigma p^2qr-40pqrs=0,$$

as an equation in plane-coordinates.

The Hessian of Σyzw is

$$-4\Sigma(xz+yw)(xw+yz)=-4\Sigma x^2yz=4\{4xyzw-\Sigma x\,.\,\Sigma yzw\}.$$

The spinode curve consequently breaks up into twice each axis (or edge of the tetrahedron of reference).

Art. 43. Trihedral-pair forms are, for instance,

$$(x+y)zw+(z+w)xy=0,$$

$$(x+y)(x+z)(x+w)-x^2(x+y+z+w)=0.$$

The latter shows a transversal triangle-plane $x+y+z+w=0$, which is simple as containing none of the four nodes. Its sides are the transversals of the axes; each of them belongs to two opposite axes, as for instance ($x+y=0$, $z+w=0$), being the transversal common to both singular tangent planes $x+y=0$ and $z+w=0$. The six singular tangent planes lie harmonically in regard to the point $x=y=z=w$. Let any plane $px+qy+rz+sw=0$ pass through this point, whence $p+q+r+s=0$; then to this plane will harmonically answer the point $px=qy=rz=sw$, and this will describe the surface

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w}=0,$$

while the plane turns about that fixed point.

The six axes count four times; the three transversals are simple; $6\cdot4+3\cdot1=27$ lines.

The four planes each of which contains three nodes count eight times, the six singular tangent planes count twice, the transversal plane is simple; together $4\cdot8+6\cdot2+1\cdot1=45$ triangle-planes.

There are three species; the transversal plane is always real.

1. All is real. XVI. 1.

2. x, y are real, z, w conjugate. Two nodes are real, and two are conjugate. Two axes and but one transversal are real. XVI. 2.

3. x, y are conjugate, and so also z, w. All four nodes are imaginary and conjugate by pairs. Two axes and the three transversals are real. XVI. 3.

XVII. *Cubic surface of the fourth class with two biplanar nodes and one proper node.*

Art. 44. Such surface arises from the kind IX. when there $\beta=\gamma$. With a change of letters

$$xyz+xw^2+w^3=0$$

(implying only fourteen constants) is a form to which the equation of such a surface can always be reduced.

Let

$$R^2=s^2-12qr,$$

then

$$p:(2s-R):(R-s)=(yz+w^2):3w^2:2xw,$$

whence

$$9p(R-s)+2(R-2s)^2=0,$$

or in a rational form

$$(s^2+4qr)^2-ps^3-36pqrs+27p^2qr=0$$

is the reciprocal equation of the surface.

The Hessian of the original cubic is $4x(xyz+3yzw+xw^2)$; consequently the spinode curve breaks up into four times the axis joining both biplanar nodes, three times the two other axes, and once the conic $(4x+3w=0,\ 3yz-w^2=0)$. Along this last conic the cone $(8x+9w)^2-27yz=0$ osculates the surface.

The axis $(x=0,\ w=0)$ joining both biplanar nodes counts nine times, the two other axes $(w=0,\ yz=0)$ count six times, and the two disengaged rays of the biplanar nodes $(y=0,\ x+w=0)$ and $(z=0,\ x+w=0)$ count three times; $1\cdot9+2\cdot6+2\cdot3=27$ lines.

The plane $w=0$ passing through the three nodes counts eighteen times, the plane $x+w=0$ counts nine times, and the three nodal planes count six times; in the whole $1\cdot18+1\cdot9+3\cdot6=45$ triangle-planes.

Three species may be distinguished.

1. All is real. XVII. 1.

2. y, z are conjugate. The two biplanar nodes are conjugate, and the cone of the proper node is imaginary (it has but one real point). XVII. 2.

3. y and $-z$ are conjugate. The two biplanar nodes are conjugate, and the cone of the proper node is real. XVII. 3.

XVIII. *Cubic surface of the fourth class with a biplanar node and two proper nodes.*

Art. 45. The equation of the surface in point-coordinates is

$$xyw+(x+y)z^2=0,$$

and in plane-coordinates it is

$$(p-q)^2s^2+\tfrac{1}{2}(p+q)r^2s+\tfrac{1}{16}r^4=0,$$

this last equation arising from the discriminant of the binary quadric

$$2s(x+y)(px+qy)+\tfrac{1}{2}r^2xy.$$

The Hessian of the original cubic is

$$4\{(x+y)xyw+(x-y)^2z^2\}.$$

The spinode curve therefore breaks up into four times each of the lines joining the biplanar node to the two proper nodes, twice the line joining both the proper nodes, and twice the nodal edge.

Let h, k be numbers which ultimately vanish, and write

$$\mathrm{U}=-w+2hk(x+y),\qquad \mathrm{V}=\ z+hx+ky,\qquad \mathrm{W}=\ z-hx-ky,$$
$$\mathrm{X}=\ w+2hk(x+y),\qquad \mathrm{Y}=-z-hx+ky,\qquad \mathrm{Z}=-z+hx-ky;$$

then

$$\mathrm{UVW}+\mathrm{XYZ}=4hk\{xyw+(x+y)(z^2-h^2x^2-k^2y^2)\},$$
$$\mathrm{V}+\mathrm{W}+\mathrm{Y}+\mathrm{Z}=0 \text{ (holding three times)},$$
$$\mathrm{U}-(h+k)\mathrm{V}+(h+k)\mathrm{W}+\mathrm{X}-(h-k)\mathrm{Y}+(h-k)\mathrm{Z}=0 \text{ (accidental)}.$$

Then the lines are as follows, viz.

The axis ($x=0$, $z=0$) unites	$\overline{vz}$, $\overline{wy}$, m, n,	8
the axis ($y=0$, $z=0$) unites	$\overline{vy}$, $\overline{wz}$, q, r,	8
the edge ($x=0$, $y=0$) unites	l, p,	6
the axis ($z=0$, $w=0$) unites	$\overline{uy}$, $\overline{uz}$, $\overline{vx}$, $\overline{wx}$,	4
the line ($x+y=0$, $w=0$) is	$\overline{ux}$,	1
		27

the last-mentioned line $\overline{ux}$ being the transversal common to the nodal edge and the axis joining the two proper nodes.

The plane of the three nodes $z=0$ unites v, w, y, z, (uy), (uz), (vx), (wx), sixteen triangle-planes; the nodal planes count each of them twelve times, since $x=0$ unites (vz), (wy), (lmn), and $y=0$ unites (vy), (wz), (pqr). Of the singular tangent planes, that along the nodal edge, $x+y=0$, unites (ux), three triangle-planes, and that through the two proper nodes, $w=0$, unites u, x, two triangle-planes. In all $16+12+12+3+2=45$ triangle planes.

There are two species, according as x, y are real or conjugate. As an example of the latter species, I may notice the surface generated by a variable circle the diameter whereof is parallel to the axis of a fixed parabola and intercepted between this curve and its tangent at the vertex, while the plane of the circle is perpendicular to that of the parabola.

XIX. *Cubic surface of the fourth class with a biplanar and a proper node.*

Art. 46. Such a surface is represented $xyw+xz^2+y^3=0$ in point-coordinates, and by $64ps^3+(4qs+r^2)^2=0$ in plane-coordinates. The Hessian of the original cubic is $4x(xyw+xz^2-3y^3)$, whence the spinode curve breaks up into six times the edge $(x=0,\ y=0)$ of the biplanar node and six times the axis $(y=0,\ z=0)$ joining the two nodes.

From art. 34 it appears that the axis $(y=0,\ z=0)$ joining the two nodes unites the twelve lines of a double six, and that the edge $(x=0,\ y=0)$ unites the fifteen remaining lines, $12+15=27$ lines. Moreover it is plain that the axis unites all six rays of the proper node. The nodal plane $y=0$ containing the proper node unites the thirty triangle-planes immediately arising from the double six, and the osculating nodal plane $x=0$ unites all the fifteen remaining triangle-planes, $30+15=45$ triangle-planes.

The plane $z=0$ is not fixed, for we may also write

$$xy(w-2\lambda z-\lambda^2y)+x(z+\lambda y)^2+y^3=0.$$

The equation of the surface therefore implies but thirteen disposable constants.

There is but one species, because everything must be real. XIX. 1.

XX. *Cubic surface of the fourth class with a uniplanar node.*

Art. 47. When in the form $x^2w+\mathrm{P}+\mathrm{Q}x=0$ of art. 40, P is a perfect cube, which we may denote by y^3, this equation can be reduced to $x^2w+y^3+xz^2=0$. The equation reciprocal to this is $27(4ps+r^2)^2-64q^3s=0$. Since we may also write the equation in the form

$$x^2(w+2\lambda z-\lambda^2x)+y^3+x(z-\lambda x)^2=0,$$

there is nothing to fix the positions of the planes $z=0$ and $w=0$; and the equation of the surface implies only thirteen disposable constants.

The Hessian is $48x^3y$, and the spinode curve breaks up into ten times the line $(x=0,\ y=0)$ and once the conic section $(y=0,\ xw+z^2=0)$, along which the cone $xw+z^2=0$ osculates the surface.

By the help of a constant h, which ultimately vanishes, we may represent the surface here considered in the form of art. 40,

$$x^2(-2x-h^2y-3h^3z+h^6w)+(x-h^2y)^2(x+h^2y+h^3z)+x(x+h^2y+h^3z)^2=0,$$

whence we see that here all the twenty-seven lines of the general surface coincide in the line ($x=0$, $y=0$), and all the forty-five triangle-planes in the plane $x=0$.

There is but one species. XX. 1.

XXI. *Cubic surface of the third class with three biplanar nodes.*

Art. 48. The equation is $xyz+w^3=0$ in point-coordinates, and $27pqr-s^3=0$ in plane-coordinates. The Hessian is $12xyzw$; hence the spinode curve breaks up into four times the three axes.

The three axes, as counting each for nine lines, unite all the twenty-seven lines of the surface, and this distribution of them into three groups of nine lines answers to a triad of trihedral-pairs*. The plane $w=0$ of the three nodes counts for twenty-seven triangle-planes, and each of the singular osculating planes $x=0$, $y=0$, $z=0$ counts for six triangle-planes, $27+6+6+6=45$ triangle-planes.

There are two species (if x, w be supposed to be real), according as y, z are real or conjugate. XXI. 1; XXI. 2.

XXII. *Ruled surface of the third order and third class.*

Art. 49. Let us imagine a continuous system of straight lines forming a surface of the nth order, and take at pleasure any one of these lines as an axis about which we turn an intersecting plane. The section will then consist of the axis itself and a plane curve of the $(n-1)$th order, which, of course, intersects the axis in $n-1$ points. But of these one alone can move, while the $n-2$ remaining intersections must be fixed. For the plane cuts an indefinitely near (or *consecutive*) straight line of the system in only one point, and this alone moves. Should any one of the other intersections also move, the axis would be a double line of the surface, whereas it was taken at hazard. Because then the $n-2$ remaining intersections on the axis are fixed, they must arise from a double line of the surface, such double line being met by every generating line in $n-2$ points. Again, to investigate the class of this surface we take an arbitrary line in space; it will intersect the surface in n points, and therefore meet the same number of generating lines. Each plane passing through the arbitrary line and one of these n generating lines will be a tangent plane to the surface. And since there are no other tangent planes than such as pass through a generating line, therefore the class of the surface is equal to its order.

Art. 50. For $n=3$ the double line cannot be a curve; for else an arbitrary plane section of the surface would have two double points at least, and would therefore consist of a straight line and a conic section; but this cannot be the case, unless the surface break up into a plane and a quadratic surface. The double line must therefore be a straight line. Again, since through any point of it there pass (in general) two distinct generating lines, the plane of these two lines must besides cut the surface in a third line (not belonging to the system of generating lines), and this will meet all the generating lines. For if we turn a plane about it, the section will always break up into this line itself and a quadratic curve having a double point on the double line of the surface; in other words, the section will be a triangle whereof the vertex moves along the double line, while the two sides are current generating lines, and the base rests on a straight line fixed in position*, which we shall term the *transversal*. It plays the part of the node-couple-develope, since every plane passing through it touches the surface in two points away from the double line, whereas every plane passing through a generating line touches the surface in only one point away from the double line.

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* See Quart. Math. Journal, vol. ii. p. 114.

Suppose now that there pass through the double line the planes $x=0$, $y=0$, and through the transversal the planes $z=0$, $w=0$. Then the equation of the surface will assume the form $Mz+Nw=0$, and for indefinitely small values of x, y this cubic must become of the second order. Therefore M, N cannot contain z, w, but must be of the form $(x, y)^2$, whence the equation may also be presented in the form $Ax^2+2Bxy+Cy^2=0$, where A, B, C mean homogeneous linear functions of z, w. If then we inquire for what value of the ratio $z:w$ this equation gives two equal values to the ratio $x:y$, the corresponding condition $AC-B^2$ is of the second degree in respect to the ratio required. Hence there lie on the double line only two uniplanar nodes†. We are allowed to let pass through them respectively the planes $z=0$, $w=0$. But then M, N are perfect squares, and we are also at liberty to represent them by $-y^2$, x^2, so that now the equation of the surface becomes

$$x^2w-y^2z=0.$$

Since it obviously implies only thirteen constants, the existence of a double line counts in the cubic surface for *six* conditions. The system $y=\lambda x$, $w=\lambda^2z$, where λ is an arbitrary parameter, shows the generating line in movement, and affords an easy geometrical construction of the surface, which I think it is not necessary to explain.

The equation reciprocal to $x^2w-y^2z=0$ is $p^2s+q^2r=0$; hence the surface keeps its properties, though point and plane be interchanged.

The Hessian is $-16x^2y^2$. The spinode curve therefore breaks up into eight times the double line and twice the generating lines which pass through the uniplanar nodes and along which the surface is touched by the two singular tangent planes $z=0$ and $w=0$.

There are two species, according as the two uniplanar nodes are real or conjugate. In the first species x, y, z, w are real, and whenever the ratio $z:w$ is negative, the ratio $x:y$ becomes *lateral*. In other words, when the double line between the two uniplanar nodes is contiguous with the rest of the surface, then it is isolated without them; and when isolated within, then it is contiguous without. The two planes through the transversal and one of the uniplanar nodes are singular tangent planes, and both real. XXII. 1.

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* The same thing might also be thus proved. Take any four distinct generating lines; they will in general not lie on a quadratic surface, and, because they are already intersected by the straight double line of the surface, there will be a second straight line intersecting all of them. But since this now has four points in common with the cubic surface, it must lie wholly in the surface.

The problem of drawing through a given generating line a triangle-plane is of the fifth degree, and it may be foreseen that the plane passing through it and the transversal is a single solution; the four remaining solutions must all coincide in the plane passing through the given line and the double line.

† In the language of Dr. SALMON and myself, cuspidal points.—A. C.

In the second species we may assume w conjugate to $-z$ and y to x, and write

$$(x+iy)^2(z+iw)+(x-iy)^2(z-iw)=0,$$

or, what is the same thing,

$$(x^2-y^2)z-2xyw=0,$$

whence arises the system

$$(w=\lambda z, \quad x^2-2\lambda xy-y^2=0),$$

which for all real values of λ gives also real values to the ratio $x:y$. The double line is therefore throughout contiguous to the rest of the surface, and the two singular tangent planes are conjugate. XXII. 2.

[Dr. SCHLÄFLI has omitted to notice a special form of the ruled surface of the third order which presented itself to me, and which I communicated to M. CREMONA and Dr. SALMON, and which is in fact that in which the transversal coincides with the double line. For this species, say XXII. 3, the equation may be taken to be

$$y^3+x(zx+wy)=0:$$

see SALMON'S 'Geometry of Three Dimensions,' pp. 378, 379, where however in the construction of the surface a necessary condition was (by an oversight of mine) omitted. The correct construction is as follows, viz., Given a cubic curve having a double point, and a line meeting the curve in this point (the double line of the surface); if on the line we have a series of points, and through the line a series of planes, corresponding anharmonically to each other, *and such that to the double point considered as a point of the line, there corresponds the plane through one of the tangents at the double point*, then the line drawn through a point (of the double line), and in the corresponding plane, to meet the cubic, generates the surface. The special form in question must, however, have been familiar to M. CHASLES, as I find it alluded to in the foot-note, p. 188, to a paper by him, "Description des Courbes, &c.," Comptes Rendus, 18 November 1861.—A. C.]

XI. *On the Tides of the Arctic Seas.*

By the Rev. Samuel Haughton, *M.A., F.R.S., Fellow of Trinity College, Dublin.*

Part I. *On the Diurnal Tides of Port Leopold, North Somerset.*

Received November 7, 1861,—Read January 9, 1862.

I am indebted to the courtesy of Captain Washington, R.N., Hydrographer to the Navy, for the opportunity I have had of investigating the tides of Port Leopold. Having heard that I was engaged in the discussion of the Arctic Tides, he kindly placed at my disposal the observations made on board Her Majesty's Ship 'Investigator,' during the expedition of 1848–49, under the orders of Sir James C. Ross, R.N., in search of Sir John Franklin.

The 'Investigator' was anchored, or rather fast in the ice, during the winter of 1848, in Port Leopold, North Somerset, lat. 74° N., long. 90° W., in three fathoms water; and the observations on the tides were made by Lieut. Frederick Robinson, whose care and skill in observing are highly to be commended.

By carefully laying down the daily high and low waters, I have succeeded in completely separating the Diurnal from the Semidiurnal Tide, and in resolving each tide into the portions due respectively to the action of the Sun and of the Moon.

In the following discussion of the Diurnal Tide, I shall first give the results of the actual observations, when graphically laid down, and afterwards draw the inferences which appear to follow from them, when compared with theory. The mode of reduction used by me will be evident from an inspection of the MS. diagrams which accompany this paper.

The following Table contains the Range of Diurnal Tide at High and at Low Water, and the Times of Vanishing of the Diurnal Tide at High and at Low Water.

Table I.—Range and Time of Vanishing of Diurnal Tide at Port Leopold, Prince Regent's Inlet, 1848–49.

Range of Tide.		Time of Vanishing.	
High Water.	Low Water.	High Water.	Low Water.
ft.	ft.		
2·16	1·55	November 1848.	November 1848. 4^d 12^h 30^m
2·41	1·20	21^d 9^h 15^m	17 20 0

TABLE I. (continued).

Range of Tide.		Time of Vanishing.	
High Water.	Low Water.	High Water.	Low Water.
ft.	ft.		
2·17	1·22	December 1848.	December 1848.
		6d 18h 10m	2d 21h 45m
2·40	1·57		
		19 18 50	15 18 30
2·31	1·19	January 1849.	December 1848.
		4 2 15	31 16 0
2·40	1·57		January 1849.
		17 9 30	12 23 30
2·05	1·33	February 1849.	
		1 16 55	28 13 0
2·23	1·38		February 1849.
		14 17 0	11 8 40
1·84	0·95		
		27 6 45	27 3 30
1·46*	0·87*	March 1849.	March 1849.
		13 8 30	11 17 15
1·53*	1·07*		
		25 13 15	24 10 20
1·31*	0·84*	April 1849.	April 1849.
		6 2 20	5 17 15
1·55*	0·63*		
		21 11 15	18 3 30
1·96	0·86	May 1849.	
		3 18 15	30 12 50
2·26	0·87	May 1849.	May 1849.
		19 3 15	14 10 19
2·34	1·22		
		31 14 15	27 23 20
2·03	0·97	June 1849.	June 1849.
		16 6 0	11 22 20
2·44	1·35		
		29 5 20	24 17 30
2·14	1·23	July 1849.	July 1849.
		14 12 45	10 18 50
2·27	1·28		
		27 19 30	23 23 20

To render more evident the law of range of Diurnal Tide, I here give in Plate X. figs. I. and II., a graphical representation of the first two columns of the preceding Table, by means of which the relation of the range of Diurnal Tide, at High and Low Water, to the Solstices and Equinoxes is made apparent.

There is no difficulty in understanding, as will be presently shown, why the Diurnal Tide should reach a maximum at the Solstices, and a minimum at the Equinoxes, as is shown by the curves for High and Low Water, because the Solar Diurnal Tide vanishes at the Equinoxes, and consequently the equinoctial Diurnal Tide is due solely to the Moon, while the Solstitial Diurnal Tide is due to the united action of both Sun and Moon.

The following Tables II. and III. show the interval between the vanishing of the Diurnal Tide and the time of the vanishing of the Moon's Declination.

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* Transactions of the Royal Irish Academy, vol. xxiii. pp. 133, 134, 137.

TABLE II.—Relation of the Times of Vanishing of the Diurnal Tide at High Water, to the Vanishing of the Moon's Declination.

Date.	Vanishing of Tide.	Interval from Moon's Declination Vanishing
	d h m	d h
1848. November	21 9 15	+1 6
„ December	6 18 10	+2 3
„ „	19 18 50	+2 6
1849. January	4 2 15	+2 20
„ „	17 9 30	+3 8
„ February	1 16 55	+4 8
„ „	14 17 0	+4 12
„ „	27 6 45	+2 18
„ March	13 8 30	+3 18†
„ „	25 13 15	+1 16†
„ April	6 2 20	+0 6†
„ „	21 11 15	+1 4
„ May	3 18 15	+0 12
„ „	19 3 15	+1 10
„ „	31 14 15	+1 6
„ June	16 6 0	+1 22
„ „	29 5 20	+2 12
„ July	14 12 45	+2 22
„ „	27 19 30	+3 14
	Mean ...	+2^d^ 9^h^

In this Table, the positive sign denotes that the Vanishing of the Diurnal Tide followed the Vanishing of the Moon's Declination.

TABLE III.—Relation of the Times of Vanishing of the Diurnal Tide at Low Water, to the Vanishing of the Moon's Declination.

Date	Vanishing of Tide.	Interval from Moon's Declination Vanishing.
	d h m	d h
1848. November	4 12 30	−3 3
„ „	17 20 0	−1 21
„ December	2 21 45	−1 14
„ „	15 18 30	−1 12
„ „	31 16 0	−0 21
1849. January	12 23 30	−0 18
„ „	28 13 0	+0 9
„ February	11 8 40	+0 16
„ „	27 3 30	+2 15
„ March	11 17 15	+2 4†
„ „	24 10 20	+0 14†
„ April	5 17 15	−0 4†
„ „	18 3 30	−2 3
„ „	30 12 50	−2 22
„ May...............	14 10 19	−2 15
„ „	27 23 20	−2 11
„ June	11 22 20	−2 4
„ „	24 17 30	−2 2
„ July	10 18 50	−0 18
„ „	23 23 20	−0 4
	Mean ...	−0^d^ 22^h^ 30^m^

In this Table, the positive sign denotes that the Vanishing of the Diurnal Tide follows the Vanishing of the Moon's Declination, and the negative sign denotes that it precedes it. From the mean result of the two Tables, it would seem that the Vanishing of the Diurnal Tide at Low Water *precedes* the Vanishing of the Diurnal Tide at High Water, by a mean amount of $3^d\ 7^h\ 30^m$.

The intervals at which the vanishing of the Diurnal Tide at High Water follows the Vanishing at Low Water are shown in detail in the following Table and in fig. III., Plate XI.

TABLE IV.—Intervals from the Vanishing of the Diurnal Tide at Low Water to Vanishing of Diurnal Tide at High Water.

Date.	Intervals		
	d	h	m
1848. November	3	13	15
„ December	3	20	25
„ „	4	0	20
„ „	3	10	15
1849. January	4	10	0
„ „	4	3	55
„ February	3	8	20
„ „	0	3	15
„ March	1	15	15
„ „	1	2	55
„ April	0	9	5
„ „	3	7	45
„ „	3	5	25
„ May	4	16	56
„ „	3	14	55
„ June..............	4	7	40
„ „	4	11	50
„ July	3	17	55
„ „	3	20	10
Mean ...	3^d	5^h	27^m

From the foregoing Table it is evident that the interval between the vanishing of the Diurnal Tide at the time of High and of Low Water, increases from the Equinoxes to the Solstices—an effect which is in a great degree due to the Solar Tide, which disappears at the Equinoxes and reaches a maximum at the Solstices. The regularity with which this increase of interval takes place is still better shown by the figure, which represents the Table, the abscissæ denoting time, and the ordinates the interval from the vanishing of the Diurnal Tide at Low Water to its vanishing at High Water. The minimum interval, 12 hours, occurs at the time of the Equinoxes, and the maximum interval, 4 days to $4\frac{1}{4}$ days, occurs at the time of the Solstices.

I am not aware that this feature of the Diurnal Tide has been before noticed; it is perfectly in accordance with what might be expected from Tidal Theory.

According to the best theories of the Tides, the Diurnal Tide may be represented by the expression

$$D = S \sin 2\bar{\sigma} \cos (s - i_s) + M \sin 2\bar{\mu} \cos (m - i_m). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

In this equation

D is the height of the Diurnal Tide, in feet.

S and M are the coefficients, in feet, of the Solar and Lunar Diurnal Tides.

$\bar{\sigma}$ and $\bar{\mu}$ are the Declinations of the Sun and Moon, at a period preceding the moment of observation, by an unknown interval to be determined for each luminary, and called the Age of the Solar and Lunar Diurnal Tide.

s and m are the hour-angles of the Sun and Moon, west of the meridian, at the time of observation.

i_s and i_m are the Diurnal Solitidal and Lunitidal intervals, or the times which elapse between the Sun and Moon's southing, and the time of Solar and Lunar Diurnal High Water.

At any time near the Equinox, the declination $\bar{\sigma}$ of the Sun is either zero or very small, and therefore D will vanish when $\bar{\mu}$, the Moon's declination, vanishes, and this will happen at both High and Low Water, or at any other time of the day; therefore at the equinoxes the vanishing of the diurnal tide at the time of High or of Low Water ought to be sensibly the same; but at the time of the Solstices, both members of the right-hand side of equation (1.) will have sensible values, and the Diurnal Tide will vanish when these members are equal and of opposite signs; therefore, to find the time of vanishing of the Diurnal Tide, we have

$$\cos(s-i_s)=-\frac{M}{S}\cdot\frac{\sin 2\bar{\mu}\cos(m-i_m)}{\sin 2\bar{\sigma}}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2.)$$

At the time of High Water, m, the moon's hour-angle is sensibly constant, or at least varies within narrow limits; also, since the vanishing of the Diurnal Tide at High Water occurs at intervals of about a semilunation, the moon's declination, $\bar{\mu}$, at each vanishing of the Diurnal Tide will also vary within small limits; hence in passing from the equinox to the solstice, the right-hand side of (2.) will have its change of value depending chiefly on the change of $\bar{\sigma}$; and it will therefore diminish as $\bar{\sigma}$ increases; therefore $\cos(s-i_s)$ will diminish, and $(s-i_s)$ *increase*; but s is the hour-angle of the Sun at the time of High Water, and increases day by day (48^{m} mean); therefore as we approach the Solstice, the day on which we are to expect the Diurnal Tide to vanish at the time of High Water will occur later and later.

But at the time of Low Water the angles s and m must be increased by 90° or 6 hours; and therefore (2.) becomes

$$\sin(s-i_s)=-\frac{M}{S}\cdot\frac{\sin 2\bar{\mu}\sin(m-i_m)}{\sin 2\bar{\sigma}}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.)$$

By reasoning similar to that used with respect to equation (2.), we can show that $\sin(s-i_s)$ diminishes in passing from the Equinox to the Solstice, and therefore that $(s-i_s)$ also *diminishes*; therefore the time of vanishing of the Diurnal Tide at the time of Low Water occurs earlier and earlier as we approach the Solstice. We thus see that the times of vanishing at High and Low Water move in opposite directions, and become most widely separated at the time of the Solstice. This result agrees perfectly with the facts of observation at Port Leopold recorded in Table IV. and the accompanying fig. III. Plate XI.

I shall now endeavour to separate, in the Diurnal Tide, the effects of the Sun and Moon. In equation (1.), the effect of the Sun, represented by the first member of the right-hand side of the equation, when observed at High Water, may be considered to owe its periodical change almost altogether to the change in $\cos(s-i_s)$, the angle s increasing day by day as the tide becomes later and later; for the angle $\bar{\sigma}$ may be regarded as sensibly constant during the semilunation. On the other hand, the Lunar portion of the Diurnal Tide owes its change to the change of $\bar{\mu}$, the moon's declination, for the angle $(m-i_m)$ is sensibly constant. The Solar Diurnal Tide disappears at the equinox, because then $\bar{\sigma}=0$; hence we may find the Lunar Diurnal Tide, at that period of the year, uncomplicated by the coexistence of the Solar Tide.

Taking the means of the Diurnal Tide Ranges at High and Low Water for March and April*, I find

At High Water

$$D=1\,462 \text{ ft.}=M\sin(2\bar{\mu}).\cos(m-i_m);$$

at Low Water

$$D=0\cdot852 \text{ ft.}=M\sin(2\bar{\mu}).\cos(m-i_m+90^\circ),$$

m denoting the moon's hour-angle at High Water, and $\bar{\mu}$ denoting the moon's maximum declination. Dividing one of these equations by the other, we find

$$\cot(m-i_m)=-\frac{1\cdot462}{0\cdot852}; \quad \ldots\ldots\ldots\ldots \quad (4.)$$

from which we deduce

$$m-i_m=149^\circ\ 46'=10^h\ 19^m.$$

m, the moon's hour-angle at High Water, is shown by the observations made in March and April to have at New and Full Moon a mean value of $12^h\ 0^m$. Substituting this value in the preceding equation, I obtain,

(1) *The Lunitidal Diurnal Interval*$=i_m=1^h\ 41^m$.

The coefficient M of the Lunar Diurnal Tide may be found as follows:—

Let H=Range of Diurnal Tide, at High Water, at the equinox.

Let L=Range of Diurnal Tide, at Low Water, at the equinox.

Then

$$2M\sin 2(\text{max. declination of moon})=\sqrt{H^2+L^2}. \quad \ldots\ldots \quad (5.)$$

Substituting in this equation the values of H, L, and $\bar{\mu}$, we find

$$2M\sin 37^\circ=\sqrt{(1\cdot462)^2+(0\cdot852)^2}=1\cdot692 \text{ feet};$$

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* The Tide Ranges used in obtaining these means and marked (*) in Table I.

and therefore, finally,

(2) *Coefficient of Lunar Diurnal Tide*$=\mathrm{M}=1\cdot409$ feet.

The Age of the Lunar Diurnal Tide is found by examining the interval, at the Equinox, from the vanishing of the Moon's declination to the vanishing of the Diurnal Tide, at High and at Low Water. The interval from the Moon's declination vanishing to the Tide vanishing is given in Tables II. and III. for High and Low Water; and the figs. IV. and V. of Plate XI. represent the results of those Tables.

From these figures, or from the Tables which they represent, it is evident that the difference in the time of vanishing of the Diurnal Tide at the times of High and Low Water is not altogether due to the Solar Tide; for at the Equinox, when the Solar Tide has disappeared, the Age of the Lunar Diurnal Tide at High Water is $1^{d}\ 21^{h}$, while the Age of the same Tide at Low Water is only $0^{d}\ 21^{h}$; showing a permanent difference of one whole day in the times of disappearance of the Tide at High and Low Water quite independent of the Solar Tide, which, as I have already shown, tends to increase this difference as we approach the Solstices.

The Means of the Ages of the Diurnal Tide, taken from the Tides of the 13th and 25th of March and the 6th of April, 1849, at High Water, and from the Tides of the 11th and 24th of March and the 5th of April, 1849, at Low Water, are $1^{d}\,21^{h}$ and $0^{d}\,21^{h}$*.

I am unable to explain why the Age of the Lunar Diurnal Tide at High Water should be greater than its Age at Low Water; but there is good reason to believe that it is an established fact, as I found the same kind of difference of Age in the Tidal Observations made in 1850–51, on the coasts of Ireland, by the Royal Irish Academy. The following Table shows the difference in the Irish stations

Age of Lunar Diurnal Tide, deduced from Observations at High and Low Water, on the Coasts of Ireland, 1850–51 (from Transactions of the Royal Irish Academy, vol. xxiii. p. 137).

Station	Age at High Water	Age at Low Water	Difference
	d h	d h	d h
Caherciveen	5 4	4 17	0 11
Bunown	4 9	4 9	0 0
Rathmullan	5 10	4 20	0 14
Portrush	5 9	4 19	0 14
Cushendall	6 19	5 3	1 16
Donaghadee	6 5	5 2	1 3
Kingstown	6 17	4 11	2 6
Courtown	6 22	3 12	3 10
Dunmore East	5 19	5 14	0 5
Means	$5^{d}\ 20^{h}\ 40^{m}$	$4^{d}\ 17^{h}\ 13^{m}$	$1^{d}\ 3^{h}\ 27^{m}$

Bringing together the Constants of the Lunar Diurnal Tide just determined, they are as follow:—

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* These tides are marked (†) in Tables II. and III.

1. Diurnal Lunitidal Interval	1h 41m
2. Age of Lunar Diurnal Tide at High Water	1d 21h
„ „ Low Water	0d 21h
3. Coefficient of Lunar Diurnal Tide	1·409 ft.

It remains now to determine, if possible, the corresponding Constants of the Solar Diurnal Tide. In order to effect this object, I laid down the Lunar Tide, both at High and Low Water, from the preceding constants, on the observed Diurnal Tide at the time of the Solstices, and thus obtained the constants of the Solar Tide, which at those periods of the year is a maximum.

Having thus constructed the Lunar Tide, I found, by the difference between it and the Observed Diurnal Tide, that the maximum Solar Diurnal Tide was as follows:—

	ft.
Range of Solidiurnal Tide at High Water:—	
Summer Solstice, 1849	0·82
Winter Solstice, 1848	0·91
Range of Solidiurnal Tide at Low Water:—	
Summer Solstice, 1849	0·86
Winter Solstice, 1848	0·88
Mean	0·867 ft.

The following Table shows the time at which the Solar Diurnal Tide vanished at the Solstices.

Vanishing of Solar Diurnal Tide at High Water.

Summer Solstice, 1849. Tide passing from — to +.	h	m	Summer Solstice, 1849. Tide passing from + to —.	h	m
May 31	8	34	June 16	20	10
June 29	8	20	July 14	18	50
Winter Solstice, 1848. Tide passing from + to —.			Winter Solstice, 1848. Tide passing from — to +.		
Dec. 19	8	15	Dec. 6	20	50
Jan. 17	8	35	Jan. 4	20	10
Mean	8h	26m	Mean	20h	0m

Vanishing of Solar Diurnal Tide at Low Water.

Summer Solstice, 1849. Tide passing from — to +.	h	m	Summer Solstice, 1849. Tide passing from + to —.	h	m
June 23	7	30	June 19	19	45
July 23	8	19	July 8	19	48
Winter Solstice, 1848. Tide passing from + to —.			Winter Solstice, 1848. Tide passing from — to +.		
Dec. 28	7	30	Dec. 13	19	40
Jan. 27	8	30	Jan. 12	20	0
Mean	7h	57m	Mean	19h	48m

It will be observed in the preceding Table, that the time of the Diurnal Solar Tide vanishing may be referred to one or other of two hours, which differ by 12^h, and that the times of passing from + to − at the two solstices are reversed. These changes are evident from the consideration of the expression for the Solar Diurnal Tide,

$$S \sin 2\bar{\sigma} . \cos(s-i_s),$$

which changes sign, from Solstice to Solstice, by the change of sign of $(\bar{\sigma})$, the sun's declination, and also changes sign at the two high waters or low waters of the same day by the increment of 180° which the sun's hour-angle s undergoes. Combining all the results together, I find that the Solar Diurnal Tide vanishes at High Water when

$$8^h\ 26^m - i_s = 18^h,$$

and

$$20^h \quad - i_s = 6^h,$$

or

$$i_s = -9^h\ 34^m, \text{ and } +14^h.$$

$$\text{Mean value of } i_s = 14^h\ 13^m;$$

and that the Solar Diurnal Tide vanishes at Low Water when

$$7^h\ 57^m - i_s = 18^h,$$

and

$$19^h\ 48^m - i_s = 6^h,$$

or

$$i_s = -10^h\ 3^m, \text{ and } +13^h\ 48^m.$$

$$\text{Mean value of } i_s = 13^h\ 52^m\ 30^s.$$

The Mean of the values of the Solitidal Interval, at High and Low Water, is

$$i_s = 14^h\ 2^m\ 45^s.$$

From the preceding data we can readily find the coefficient of the Solar Diurnal Tide; for

$$S \times \sin(\text{max. declination of Sun}) = 0{\cdot}867 \text{ feet},$$

or

$$S = \frac{0{\cdot}867}{\sin(47^\circ)} = 1{\cdot}186 \text{ feet}.$$

The Age of the Solar Diurnal Tide cannot be deduced from observations such as those under discussion, because the Sun's declination changes so slowly at the Solstices, that it may be considered constant during a fortnight, and therefore the Coefficient $S \sin(2\bar{\sigma})$ is also constant during that period. The Constants of the Solar Diurnal Tide, as just found, are as follow:—

1. Diurnal Solitidal Interval . . .	$14^h\ 2^m\ 45^s$.
2. Age of Solar Diurnal Tide . . .	Unknown.
3. Coefficient of Solar Diurnal Tide .	1·186 feet.

The ratio of the Solar to the Lunar Coefficient is

$$\frac{S}{M} = \frac{1{\cdot}186}{1{\cdot}409} = 0{\cdot}842.$$

This result differs widely from the ratios of S to M found by me at the Irish Stations, which were as follow:—

Ratio of the Solar to the Lunar Coefficient of the Diurnal Tide, on the Coasts of Ireland, 1850–51 (from Trans. Roy. Irish Acad. vol. xxiii. p. 128).

Station.	$\frac{S}{M}$
Caherciveen	0·698
Bunown	0·529
Rathmullan	0·498
Portrush	0·659
Cushendall	0·427
Donaghadee	0·441
Kingstown	0·504
Courtown	0·570
Dunmore East	0·436
Mean	0·5305

I shall now deduce, according to received theories, the mean depth of the channel of the Atlantic Sea, which conveys the tide from the South Atlantic Ocean to Port Leopold. The theory which I select for this purpose is that given by Mr. AIRY in his 'Tides and Waves,' which is considerably in advance of that given by LAPLACE and the earlier mathematicians, and, as it is directly founded on the motion of water in canals, seems particularly well adapted to the discussion of a tide like that of Port Leopold, which is situated at the extreme northern end of the Atlantic Ocean, which may be regarded as a Canal occupying a meridian circle, and nearly 10,000 miles in length. From the discussion of the Diurnal Tide, in a meridian canal, given by Mr. AIRY (Tides and Waves, p. 356), it may be deduced that the following equation is true, and that it contains the means of finding the mean depth of the Atlantic Canal:

$$\frac{S}{M}=\frac{\text{mass of Sun}}{\text{mass of Moon}}\times\frac{d^3}{D^3}\times\frac{\frac{n^2b}{g}-4\frac{k}{b}}{\frac{N^2b}{g}-4\frac{k}{b}}. \quad \ldots\ldots\ldots \quad (6)$$

In this equation,

S, M are the coefficients of the Solar and Lunar Diurnal Tide, found, as at Port Leopold, by observation.

D, d are the mean distances of the Sun and Moon from the Earth.

N, n are the angular velocities of the Sun and Moon about the Earth.

b is the mean radius of the Earth.

k is the mean depth of the Atlantic Canal.

g is the velocity acquired in a second by a falling body.

The left-hand side of equation (6.) is known by observation, and all the quantities on the right-hand side are known, except k.

Substituting, therefore, for the symbols the following values,

$$\frac{\mathrm{S}}{\mathrm{M}}=0\cdot842,$$

$$\frac{\text{mass of Sun}}{\text{mass of Moon}}=359551\times85,$$

$$\frac{d}{\mathrm{D}}=\frac{59\cdot964}{2\times12032},$$

$$\frac{\text{mass of Sun}}{\text{mass of Moon}}\times\frac{d^3}{\mathrm{D}^3}=0\cdot47288;$$

$$b=\frac{7912}{2}\text{ miles};$$

$$\frac{\mathrm{N}^2b}{g}=0\cdot00345;$$

$$\frac{n^2b}{g}=0\cdot00323;$$

we find

$$0\cdot842=0\cdot473\,\frac{0\cdot00323-4\dfrac{k}{b}}{0\cdot00345-4\dfrac{k}{b}};$$

from which I deduce

$$\frac{k}{b}=\frac{1}{1072},\quad k=3\cdot69\text{ miles}.$$

In discussing the Solar and Lunar Diurnal Tides of nine stations on the Irish Coasts, I found the following results*:—

$$\frac{\mathrm{S}}{\mathrm{M}}\text{ (mean of nine Stations)}=0\cdot5305;$$

$$\frac{k}{b}=\frac{1}{773};\quad k=5\cdot12\text{ miles}.$$

The mean depth of the Atlantic Canal may be also deduced, by means of Mr. AIRY's Theory of Tides with Friction, from a comparison of the Solitidal and Lunitidal Intervals, and from the Lunitidal Interval compared with the Age of the Lunar Tide.

According to the Theory of Tidal Waves without Friction, Low Water should occur at the time of the meridian passage of the luminary; in consequence, however, of friction, the phase of High Water is accelerated by an interval equal to the difference between the Tidal Interval and half the period of a Tidal Oscillation. According to Mr. AIRY's Theory, taking account of friction (supposed proportional to the horizontal velocity of the tidal current), the acceleration of High Water is represented by†

$$\frac{f}{n^2-gkm^2};$$

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* Trans. Royal Irish Academy, vol. xxiii. pp. 128, 131.

† Tides and Waves, p. 332.

where

f = coefficient of friction;
n = angular velocity of Luminary;
g = 32 feet;
k = depth of the sea;
$m=\frac{2\pi}{\lambda}$;
λ = length of the tide-wave.

Therefore

$$\frac{\text{Lunitidal Acceleration}}{\text{Solitidal Acceleration}}=\frac{\mathrm{N}^2-gkm^2}{n^2-gkm^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

and, substituting the following values,

$$n=\frac{2\pi}{89280}, \quad \mathrm{N}=\frac{2\pi}{86400},$$

$$m=\frac{2\pi}{25000\times 5280};$$

we find, k being expressed in miles,

$$\frac{\text{Acceleration of Lunar Diurnal Tide}}{\text{Acceleration of Solar Diurnal Tide}}=\frac{13{\cdot}815-k}{12{\cdot}938-k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

To find the Lunitidal and Solitidal Accelerations, we must subtract the Lunitidal and Solitidal Intervals, i_m and i_s, from $12^h\ 24^m$ and 12^h, respectively; but

$$i_m=\ 1^h\ 41^m,$$
$$i_s=13^h\ 52^m{\cdot}5;$$

therefore

Acceleration of Lunar Diurnal Tide . $=+10^h\ 43^m$,
Acceleration (*Retardation*) of Solar Diurnal Tide $=-\ 1^h\ 52^m{\cdot}5$.

Substituting these values in equation (8.), I find

$$k=13{\cdot}07 \text{ miles.}$$

Again, according to Mr. Airy's Theory of Tidal Waves with friction*, the greatest tide follows the greatest force by an interval (Age of Tide),

$$\frac{f(n^2+gkm^2)}{(n^2-gkm^2)^2};$$

but the acceleration of the Tide is

$$\frac{f}{n^2-gkm^2}.$$

Therefore

$$\frac{\text{Age of Lunidiurnal Tide}}{\text{Acceleration of Lunidiurnal Tide}}=\frac{n^2+gkm^2}{n^2-gkm^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

or

$$\frac{\text{Age of Lunidiurnal Tide}}{\text{Acceleration of Lunidiurnal Tide}}=\frac{12{\cdot}938+k}{12{\cdot}938-k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.)$$

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* Tides and Waves, p. 333.

In applying this equation to determine the depth of the sea, the difficulty already noticed, as to the Age of the Lunar Tide, deduced from High Water and Low Water observations, meets us again. The Age at High Water is $1^d\ 21^h$, and at Low Water $0^d\ 21^h$.

Substituting these values respectively, I find

	miles.
Depth of sea (k) deduced from Age of Lunar Diurnal Tide at High Water	=7·96
Depth of sea deduced from Age of Lunar Diurnal Tide at Low Water	=4·19
Mean	=6·07

Bringing together all the preceding results, we find the following mean depths of the Atlantic Canal, as deduced by the various methods described:—

	miles.
1. Depth deduced from Heights of Solar and Lunar Diurnal Tides .	3·69
2. Depth deduced from Accelerations of Solar and Lunar Diurnal Tides, caused by friction	13·07
3. Depth deduced from Acceleration and Age of Lunar Diurnal Tide, caused by friction	6·07

Of the three methods just given for finding the mean depth of the sea, the first is the most trustworthy, for the following reasons:—

1st. The determination of Heights of the Solar and Lunar Diurnal Tide by observation is more accurate than the determination of Acceleration and Age.

2nd. The theory by which the depth of the sea is deduced from Heights is independent of friction, the introduction of which requires additional hypotheses, which are, at best, of a doubtful character.

At the same time it should be remarked that the depth of the sea deduced from Acceleration and Age, at eight stations on the coasts of Ireland, exceeded the depth deduced from Heights, in a manner similar to that which is found to occur at Port Leopold.

The Irish depths are—

	miles.
1. Depth of sea deduced from Heights of Solar and Lunar Diurnal Tides	5·12
2. Depth deduced from Accelerations of Solar and Lunar Diurnal Tides, caused by friction	11·98
3. Depth deduced from Acceleration and Age of the Lunar Diurnal Tide	11·32

In the present state of our knowledge of the Theory of the Tides, I think it is safer to adopt the results deduced from Heights as the most reliable, and to wait until mathematical researches shall have further perfected the Theory of friction in Tidal Waves, before we draw conclusions from it as to the depth of the sea, especially when we consider that this Theory has not yet explained the anomaly discovered by observation as to the difference in Age of the Diurnal Tides deduced from High and Low Waters.

PART II.—*The Semidiurnal Tides of Port Leopold, North Somerset.*

Received October 8,—Read November 27, 1862.

When the daily height of High and Low Water has been cleared of the Diurnal Tide, as explained in Part I., and as is shown in the MS. diagrams that accompany this paper, it is easy to estimate the successive Heights of Spring and Neap Tides, cleared of the Diurnal Tide.

Bringing together the Spring Tides and the Neap Tides, the following Tables I. and II. are constructed; and from the second column of these Tables the diagram No. 1 Plate XII. is prepared, of which the following explanation may be useful.

The interval in the abscissæ corresponds to five Lunar weeks, or intervals between the greatest Spring Tide and least Neap Tide. The ordinates are divided, as usual, into feet.

The Curve *a*, drawn through alternate Spring Tide Heights, is the curve of New Moon Springs.

The Curve *a′* is the Curve of Full Moon Springs.

The Curve *b* is the Curve of First Quarter Neap Tides.

The Curve *b′* is the Curve of Third Quarter Neaps.

These Curves are constructed from Tables I. and II., which are themselves formed from the Curves of the MS. diagram.

The diagram No. 2, Plate XII. is formed from the first column of Table I. Its abscissæ are the same as those of diagram No. 1, and its ordinates are the Solar Hours at which the Maximum Spring Tide occurred.

Curve *a* represents the Time of New Moon Springs.

Curve *a′* the Time of Full Moon Springs.

Table I.—Semidiurnal Maximum Spring Tide Ranges, 1848–49.

	Time.			Range.	Moon's Hour-Angle.	
	d	h	m	ft	h	m
1848. October	28	0	30	5·42	1	0
„ November	12	1	0	6·67	1	48
„ „	27	0	45	5·01	1	8
„ December	12	1	0	6·42	2	29
„ „	27	1	15	5·03	1	31
1849. January	10	0	50	6·56	2	4
„ „	26	1	15	5·65	1	54
„ February	9	1	0	6·28	2	21
„ „	25	1	45	6·50	2	17
„ March	10	1	0	6·11	1	45
„ „	26	1	30	6·63	1	53
„ April	7	0	30	5·76	2	24
„ „	25	1	20	6·60	2	32
„ May	7	0	30	5·50	2	35
„ „	24	1	15	6·47	2	17
„ June	8	1	30	4·60	2	28
„ „	21	0	30	6·28	1	0
„ July	8	1	30	5·06	2	48
„ „	22	1	15	6·46	2	33
Mean ...	$1^h\ 4^m$			......	$2^h\ 2^m$	

TABLE II.—Semidiurnal Minimum Neap Ranges, 1848–49.

	Sun's Hour-Angle.			Range.	Moon's Hour-Angle.	
	d	h	m	ft.	h	m
1848. November	4	5	30	2·33	6	38
„ „	18	5	45	2·83	7	23
„ December	4	6	0	2·95	7	0
„ „	18	6	20	2·89	7	32
1849. January	2	6	0	3·55	6	36
„ „	17	6	30	2·54	7	40
„ February	2	7	0	3·65	8	8
„ „	17	7	30	1·98	8	40
„ March	3	6	45	3·25	7	58
„ „	18	6	45	2·00	8	8
„ April	1	7	0	3·42	7	46
„ „	16	6	30	2·27	7	37
„ May	1	7	30	3·18	8	17
„ „	16	7	30	2·76	7	54
„ „	30	7	45	3·38	8	33
„ June	15	6	30	3·14	8	16
„ „	29	8	0	3·15	7	59
„ July	14	7	0	3·27	7	53
„ „	29	8	10	2·68	8	11
Mean ...	6^h	50^m		......	7^h	48^m

A.—*Parallactic Inequality of Semidiurnal Tide.*

The general expression for the Semidiurnal Tides is, as is well known,

$$T=S\left(\frac{P}{P_m}\right)^3\cos^2\bar{\sigma}\cos 2(s-i_s)+M\left(\frac{p}{p_m}\right)^3\cos^2\bar{\mu}\cos 2(m-i_m), \quad . \quad . \quad . \quad . \quad (1.)$$

where

S and M are the Solar and Lunar Coefficients;

P and p the Solar and Lunar parallax; and P_m, p_m the mean values of the same.

$\bar{\sigma}$, $\bar{\mu}$ the declinations of the Sun and Moon, at periods preceding that of observation by unknown intervals called the Solar and Lunar Age of the Semidiurnal Tide.

s, m the hour-angles of the Sun and Moon, west of the meridian at the time of observation.

i_s, i_m the Solar and Lunar Tidal intervals, or time after southing of the luminary, at which its high water is found to occur.

From an inspection of diagrams 1 and 2, Plate XII., it is plain that the conditions of the tide are different at New and Full Moon, and in the 1st and 3rd Quarter. But the Solar conditions may be supposed constant for a fortnight, and as the moon's declination only enters equation (1.) by the square of the cosine, it must be *q. p.* the same at the beginning and end of a Lunar fortnight. The difference, therefore, shown in the diagrams must depend on the Moon's Parallax, which takes four weeks to complete its changes.

From diagram No. 1, it appears that the greatest difference in the Parallactic Tide, between the opposite quarters of the Moon, both at Spring and Neap Tides, amounts to 1·47 ft. in the range of the Tide.

I shall presently prove that the Lunar Tide Range at the time of this maximum Parallactic inequality is 4·62 ft.; adding, to this, half the Parallactic inequality, and subtracting it from it, we find

$$\frac{5\cdot35}{3\cdot89}=\left(\frac{1+e}{1-e}\right)^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

that is, the cube of the ratio of the Apogee to the Perigee. Solving this equation for e, the eccentricity of the moon's orbit, we obtain

$$e=\frac{112}{2112}=0\cdot05303. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

This value of the eccentricity of the moon's orbit is very near the true value 0·05484*, as near, indeed, as could be expected from any Tidal Observations.

B.—*Solar Semidiurnal Tide.*

In diagram No. 1 the Curves marked A and B are drawn so as to eliminate the Parallactic inequality, and they represent the Curves of Spring and Neap Tide Ranges; but from the expression (1) it appears that if we could find two sets of tides, for which the moon's declination and hour-angle should be the same, we could eliminate the Lunar Tide and calculate the Solar Tide separately.

These conditions are fulfilled by the Solstitial Springs and Equinoctial Neaps—as may be thus shown.

Solstitial Springs:—
New Moon . . . . June $20^d\ 2^h\ 19^m$ (Greenwich).
Declination . . . . 18° 49′ N. *q. p.* maximum.
Equinoctial Neaps:—
First Quarter . . . . . . March $30^d\ 18^h\ 58^m$ (Greenwich).
Declination . . . . 18° 27′ N. *q. p.* maximum.

And the Curve of Spring Tide Range, cleared of Parallax, attains its Solstitial Minimum, June $21^d\ 12^h\ 40^m$ H.W., and its Equinoctial Maximum, April $1^d\ 12^h\ 20^m$ L.W.

We may therefore safely assume the part of equation (1.) which depends on the Moon to be the same at both these times.

We therefore have

$$\left.\begin{aligned} a' &=2\mathrm{S}\,.\cos^2\overline{\sigma'}\,\cos 2(s'-i_s)+2\mathrm{M}\cos^2\overline{\mu'}\cos 2(m'-i_m), \\ a_{\prime\prime} &=2\mathrm{S}\,.\cos^2\overline{\sigma_{\prime\prime}}\cos 2(s_{\prime\prime}-i_s)+2\mathrm{M}\cos^2\overline{\mu_{\prime\prime}}\cos 2(m_{\prime\prime}-i_m), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4.)$$

where

a', $\overline{\sigma'}$, s', $\overline{\mu'}$, m' refer to Solstitial Springs,

a' being the Solstitial Spring Range; and

$a_{\prime\prime}$, $\overline{\sigma_{\prime\prime}}$, $s_{\prime\prime}$, $\overline{\mu_{\prime\prime}}$, $m_{\prime\prime}$ refer to Equinoctial Neaps,

$a_{\prime\prime}$ being the Equinoctial Neap Range.

But

$$a'=5\cdot56 \text{ ft.}$$
$$a_{\prime\prime}=2\cdot67 \text{ ft.}$$

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* HERSCHEL's Astronomy, London, 1850, p. 649.

Subtracting from each other the two equations (4.), we find

$$(a'-a_{\prime\prime})=2S[\cos^2\bar{\sigma}'\cos 2(s'-i_s)-\cos^2\bar{\sigma}_{\prime\prime}\cos 2(s_{\prime\prime}-i_s)], \quad . \ . \ . \ . \ . \quad (5.)$$

and differentiating equations (4.), so as to express that the tide in question is a max. maximorum or min. minimorum, and subtracting, we find

$$0=\cos^2\bar{\sigma}'\sin 2(s'-i_s)-\cos^2\bar{\sigma}_{\prime\prime}\sin 2(s_{\prime\prime}-i_s). \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (6.)$$

In these equations (5.) and (6.), $\bar{\sigma}'$ and $\bar{\sigma}_{\prime\prime}$, are practically the declinations of the Sun at the Equinox and Solstice, *i.e.* $\bar{\sigma}'=0$ and $\bar{\sigma}_{\prime\prime}=23° 28'$; and s' and $s_{\prime\prime}$ are the mean values of the first column of Tables I. and II.; or $s'=1^h\ 4^m=16°$, and $s_{\prime\prime}=6^h\ 50^m=102°\ 30'$.

Hence equation (5.) becomes

$$2{\cdot}89 \text{ ft.}=2S[0{\cdot}84\cos 2(16°-i_s)-\cos 2(102°\ 30'-i_s)], \quad . \ . \ . \ . \ . \ . \quad (7.)$$

and equation (6.) becomes

$$0=0{\cdot}84\sin 2(16°-i_s)-\sin 2(102°\ 30'-i_s), \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (8.)$$

$$\tan 2i_s=\frac{\sin 25°+0{\cdot}84\sin 32°}{\cos 25°+0{\cdot}84\cos 32°},$$

or finally,

$$i_s=14°\ 5'=56^m\ 20^s. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (9.)$$

Substituting this value in equation (7.), we find

$$2{\cdot}89=2S[0{\cdot}84\cos 17°\ 55'+\cos 10°\ 55'],$$

and, finally,

$$2S=\frac{289}{178}=1{\cdot}624 \text{ ft.} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (10.)$$

Similar reasoning may be used with reference to the Solstitial Neaps and Equinoctial Springs, from which we derive the equations

$$\left.\begin{aligned} a''&=2S\cos^2\bar{\sigma}''\cos 2(s''-i_s)+2M\cos^2\bar{\mu}''\cos 2(m''-i_m),\\ a_\prime&=2S\cos^2\bar{\sigma}_\prime\cos 2(s_\prime-i_s)+2M\cos^2\bar{\mu}_\prime\cos 2(m_\prime-i_m),\end{aligned}\right\} \quad . \ . \ . \ . \quad (11.)$$

but

$$a''=3{\cdot}15 \text{ ft.}$$
$$a_\prime=6{\cdot}33 \text{ ft.}]$$

Substituting these numerical values, and proceeding as above, we find

$$2S=\frac{318}{178}=1{\cdot}786 \text{ ft.}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (12.)$$

and the mean of (10.) and (12) will be

$$2S=1{\cdot}705 \text{ ft.} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (13.)$$

C.—*Lunar Semidiurnal Tide.*

The constants of the Solar Tide being found, (9.) and (13.), nothing was easier than to calculate its amount at each Spring and Neap, and to subtract it from the former and add it to the latter. In this way the curves (α) and (β), diagram No. 1, were constructed, and represent the Lunar Semidiurnal Tide Range at Springs and Neaps respectively.

Its maximum and minimum amounts are

Maximum Lunar Range=4·62 ft.
Minimum Lunar Range=4·23 ft.

If α and α' denote the maximum and minimum Lunar Ranges which correspond with the Spring and Neap Tides of either the Solstice or Equinox, and if m and m' denote the hour-angles of the moon at these tides, we have, if $I=18\frac{1}{2}$ to 19° be the inclination of moon's orbit,

$$\left.\begin{aligned}\alpha&=2M\cos^2 I\,.\,\cos 2(m-i_m),\\ \alpha'&=2M\cos^2(0)\cos 2(m'-i_m).\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots \quad (14.)$$

From which we deduce

$$\frac{\alpha}{\alpha'}=\frac{\cos^2 I\cos 2(m-i_m)}{\cos 2(m'-i_m)},$$

and

$$\tan 2i_m=\frac{\alpha\cos 2m'-\alpha'\cos^2 I\cos 2m}{-\alpha\sin 2m'+\alpha'\cos^2 I\sin 2m}.\quad\ldots\ldots\ldots \quad (15.)$$

But, from Tables I. and II.,

$$m=2^h\ \ 2^m=\ 29°\ 31',$$
$$m'=7^h\ 48^m=113°\ 14'.$$

Substituting these values in (15.), we find

$$-\tan 2i_m=\frac{462\cos(46°\ 28')+423\times\cdot 9\cos(59°\ 2')}{462\sin(46°\ 28')+423\times\cdot 9\sin(59°\ 2')},$$

or

$$-\tan 2i_m=\frac{5141}{6608}$$

$$2i_m=142°\ 7';\ [\text{or }-37°\ 53'];$$
$$i_m=\ 71°\ 3';\ [\text{or }-18°\ 56'];$$

or

$$i_m=4^h\ 54^m;\ [\text{or }-1^h\ 18^m].$$

Substituting the value of i_m in the second of equations (14.), we find

$$2M=\frac{4\cdot 23}{\cos(84°\ 22')}=4\cdot 309\text{ ft.}\quad\ldots\ldots\ldots \quad (16.)$$

I have not been able to deduce from the observations any close approximation to the Age of the Lunar Tide, but think it is probably rather over than under five days. It would require more observations than the heights of high and low water to determine this important constant with accuracy.

D.—*Mass of the Moon and Depth of the Sea deducible from the Semidiurnal Tide.*

According to the Statical Theory of the Tides, excluding the terms depending on Parallax, Declination, Hour of Tide, &c., the ratio of the Solar to the Lunar coefficient is as follows:—

$$\frac{S}{M}=\frac{\text{mass of Sun}}{\text{mass of Moon}}\times\left(\frac{d}{D}\right)^3,\quad\ldots\ldots\ldots \quad (17.)$$

where D and d denote the mean distances of the Sun and Moon from the Earth. Hence we find

$$\frac{\text{mass of Sun}}{\text{mass of Moon}}=\frac{\mathrm{S}}{\mathrm{M}}\times\left(\frac{\mathrm{D}}{d}\right)^3$$

$$=\frac{\mathrm{S}}{\mathrm{M}}\times\left(\frac{2\times 12032}{59{\cdot}964}\right)^3.$$

But since

$$\mathrm{S}=1{\cdot}705,$$

and

$$\mathrm{M}=4{\cdot}309,$$

we have

$$\frac{\mathrm{S}}{\mathrm{M}}=0{\cdot}3956, \quad \ldots\ldots\ldots\ldots \quad (18.)$$

and, finally,

$$\frac{\text{mass of Sun}}{\text{mass of Moon}}=359551\times 71{\cdot}11. \quad \ldots\ldots\ldots \quad (19.)$$

From which we deduce the mass of the Moon to be $\frac{1}{71{\cdot}11}$th of that of the Earth.

From the Dynamical Theory of the Tides given by Mr. AIRY (Tides and Waves, p. 357), the ratio of S to M, for the Semidiurnal Tide, is the following:—

$$\frac{\mathrm{S}}{\mathrm{M}}=\frac{\text{mass of Sun}}{\text{mass of Moon}}\times\left(\frac{d}{\mathrm{D}}\right)^3\times\frac{\frac{n^2b}{g}-\frac{k}{b}}{\frac{\mathrm{N}^2b}{g}-\frac{k}{b}}, \quad \ldots\ldots \quad (20.)$$

where N and n are the angular velocities of the Sun and Moon, k, b the mean depth of sea and radius of earth, and g the force of gravity.

Substituting for these quantities their usual values, we find

$$\frac{\mathrm{S}}{\mathrm{M}}=0{\cdot}47288\times\frac{0{\cdot}00323-\frac{k}{b}}{0{\cdot}00345-\frac{k}{b}};$$

or

$$\frac{\mathrm{S}}{\mathrm{M}}=0{\cdot}47288\times\frac{12{\cdot}778-k}{13{\cdot}648-k}. \quad \ldots\ldots\ldots \quad (21.)$$

Substituting for $\frac{\mathrm{S}}{\mathrm{M}}$ its value 0·3956, we obtain by solving for k,

$$k = \text{depth of sea} = 8{\cdot}32 \text{ miles}. \quad \ldots\ldots\ldots \quad (22.)$$

Whatever doubt may attach to this high value of the mean depth of the Atlantic Canal, and to the depths 13·07 miles and 6·07 miles obtained from the Diurnal Tide in Part I., should properly be considered as belonging to the imperfect condition of the Theory of Canals of equal depth, as applied to such a body of water as the Atlantic Ocean, and not to the observations.

It is, however, well worthy of remark, that we can obtain from the Solitidal and Lunitidal Intervals a value for the mean depth of the sea that is much more probable, and is also very close to the depth derived from the ratio of S to M in the Diurnal Tide.

If we call the Acceleration of the Tide the difference between the Tidal Interval and the period of half a Tide Oscillation, we find, from Mr. AIRY's Theory, the following equation for the Semidiurnal Tide:—

$$\frac{\text{Lunitidal Acceleration}}{\text{Solitidal Acceleration}}=\frac{N^2-gkm^2}{n^2-gkm^2}, \quad \dots\dots\dots\dots \quad (23.)$$

where

$$m=\frac{2\varpi}{\lambda};$$

or

$$\frac{\text{Lunitidal Acceleration}}{\text{Solitidal Acceleration}}=\frac{13{\cdot}815-4k}{12{\cdot}938-4k}. \quad \dots\dots\dots\dots \quad (24.)$$

Substituting in this equation,

$$\text{Lunitidal Acceleration}=6^{h}\;12^{m}-4^{h}\;54^{m}=\;78^{m},$$
$$\text{Solitidal Acceleration}\;=6\quad 0-0\quad 56=304,$$

we find

$$\frac{78}{304}=\frac{13{\cdot}815-4k}{12{\cdot}938-4k},$$

and, finally,

$$k=\frac{3190{\cdot}6}{904}=3{\cdot}529 \text{ miles.}$$

From what we know, independently of the Tides, of the depth of the Atlantic Ocean, this value, and that found from the ratio of S to M in the Diurnal Tide, will seem nearer the truth than the higher values found from other considerations. From this it may be inferred that the theory of Tides, with friction, in these two cases, comes nearer to the truth than it does in the other cases from which the higher values are derived. I do not know why this is so, and would recommend the fact to the notice of those mathematicians who are conversant with the Theory of the Tides.

Her Majesty's Ship 'Investigator' was secured in her Winter position by Noon of the 18th day of October, the Moon's Age being 21 days. The Register of the Tides is commenced with the A.M. High Water on the 25th, four Tides before the change of the Moon.—Lat. 74° N., Long. 90° W.

Register of Tides observed at Leopold Harbour in the Month of October 1848.

Day.	High Water.		Low Water.		Wind.	
	Time.	Height.	Time.	Height.	Direction.	Force.
	h m	ft. in.	h m	ft. in.		
25. A.M.	10 45	22 9½	5 0	17 4½	S.E.	6
P.M.	11 0	23 0	5 0	18 3		
26. A.M.	11 30	23 6	5 30	17 6½	Easterly.	7·8
P.M.	10 0	22 6	5 30	18 6		
27. A.M.	Noon	23 9	5 0	17 10½	S.Easterly.	3
P.M.	……	……	6 45	18 4		
28. A.M.	12 20	23 1	6 10	17 6½	N.Easterly.	2
P.M.	12 30	24 4	6 45	18 10		
29. A.M.	12 30	23 3½	6 30	18 3	S.Easterly.	3·4
P.M.	1 0	24 6	7 30	19 1		
30. A.M.	1 30	22 11	7 15	17 9	Variable.	1
P.M.	1 30	24 0	7 30	18 10	Northerly.	1
31. A.M.	2 0	21 9	7 30	17 8	„	3
P.M.	2 0	23 7	8 0	18 10	N.Westerly.	3

Register of Tides observed at Leopold Harbour in the Month of November 1848.

Day.		High Water.		Low Water.		Wind.	
		Time.	Height.	Time.	Height.	Direction.	Force..
		h m	ft. in.	h m	ft. in.		
1.	A.M.	2 30	21 4	7 45	17 3	N.W.	1
	P.M.	2 40	22 8	9 0	18 4	,,	
2.	A.M.	3 0	20 9	9 30	18 0	,,	
	P.M.	3 30	23 0	9 30	19 10	Northerly.	5·6
3.	A.M.	3 40	21 6	9 40	19 3	,,	,,
	P.M.	3 40	22 8	10 30	19 6	,,	,,
4.	A.M.	4 30	20 6	11 15	19 0	Northerly.	6·7
	P.M.	5 10	22 7	12 30	19 0	N.Westerly.	3
5.	A.M.	6 0	20 0	Noon	19 0	Northerly.	3
	P.M.	6 10	22 0	1 0	18 6	N.Westerly.	2
6.	A.M.	7 30	20 6	1 0	18 6	,,	,,
	P.M.	7 45	20 1	1 10	18 9	Northerly.	5·6
7.	A.M.	8 40	21 2	2 30	18 8	N.Westerly.	5·6
	P.M.	9 30	22 4	3 0	17 9	N.Westerly.	4·5
8.	A.M.	9 10	22 3	3 30	18 7	N.Westerly.	6·7
	P.M.	10 20	22 8	4 0	17 5	N.Westerly.	6
9.	A.M.	10 15	22 9	4 20	18 2	N.Westerly.	2·3
	P.M.	11 0	22 9	5 15	16 10	,,	,,
10.	A.M.	10 45	22 10	5 30	18 1	,,	2
	P.M.	11 30	22 11	5 0	16 6	,,	1·2
11.	A.M.	12 20	22 11	5 45	17 6	S.Easterly.	4·5
	P.M.	12 30	22 9	5 45	16 3	,,	6·7
12.	A.M.	12 30	22 11	6 45	17 4	S.S.E.	4
	P.M.	1 0	23 1	7 0	16 7		
13.	A.M.	12 30	22 3*	7 15	15 7		
	P.M.	12 50	22 11	7 30	17 3		
14.	A.M.	1 40	22 4	7 45	16 5		
	P.M.	1 45	24 2	8 30	17 7		
15.	A.M.	2 30	22 2	8 15	16 9		
	P.M.	2 30	23 10	9 0	17 6		
16.	A.M.	2 45	21 6	9 0	17 6		
	P.M.	3 15	23 7	10 30	18 3		
17.	A.M.	4 0	21 0	10 0	18 3		
	P.M.	5 0	23 3	11 45	18 9		
18.	A.M.	5 0	20 11	11 0	18 9		
	P.M.	5 45	22 7	……	……		
19.	A.M.	7 0	20 10	12 50	18 5		
	P.M.	7 0	22 5	1 0	19 3		
20.	A.M.	8 20	21 5	1 50	18 6		
	P.M.	8 15	22 2	2 0	19 2		
21.	A.M.	9 15	21 9	2 45	18 0		
	P.M.	9 15	21 11	3 30	18 8		
22.	A.M.	10 0	22 3	3 30	17 8		
	P.M.	10 15	22 0	4 0	18 9		
23.	A.M.	11 30	22 11	4 20	17 9		
	P.M.	11 0	22 2	4 45	18 8		
24.	A.M.	11 10	23 5	5 0	17 9		
	P.M.	11 25	22 4	5 30	18 8		
25.	A.M.	11 20	23 10	5 45	17 7		
	P.M.	Midnight	22 2	6 20	18 7		
26.	A.M.	……	……	6 0	17 7		
	P.M.	12 15	24 0	6 30	18 6		
27.	A.M.	12 30	22 9	6 20	17 6		
	P.M.	12 45	24 0	7 00	18 7		
28.	A.M.	1 0	22 0	7 00	17 6		
	P.M.	1 30	24 0	8 15	19 0		
29.	A.M.	1 30	21 11	7 30	17 10		
	P.M.	1 45	24 0	8 20	18 9		
30.	A.M.	2 15	21 9	8 0	17 11		
	P.M.	2 30	23 9	8 45	18 9		

First quarter at $3^{d}\ 2^{h}\ 19^{m}$ Greenwich. Full Moon at $10^{d}\ 12^{h}\ 25^{m}$ Greenwich.

Register of Tides observed at Leopold Harbour in the Month of February 1849.

Day.		High Water.		Low Water.		Wind.	
		Time.	Height.	Time.	Height.	Direction.	Force.
		h m	ft. in.	h m	ft. in.		
1.	A.M.	5 40	22 2	11 35	19 5	N.N.W.	6
	P.M.	5 30	23 1	11 20	19 0	N.Westerly.	8
2.	A.M.	6 55	23 4	......	......	„	8
	P.M.	7 0	21 11	1 30	20 3	„	5
3.	A.M.	8 0	22 5	1 30	17 10	S.S.E.	3
	P.M.	8 20	21 4	2 30	19 1	N.Westerly.	2
4.	A M.	9 15	22 9	2 30	17 6	Northerly.	1
	P.M.	9 30	21 4	4 20	18 8	„	3
5.	A.M.	10 40	23 5	3 30	17 3	N.N.W.	6
	P.M.	10 30	22 0	5 0	18 5	„	6
6.	A.M.	11 15	24 5	4 30	17 5	„	6
	P.M.	11 30	23 5	5 45	18 4	„	6
7.	A.M.	11 45	24 5	5 30	17 3	„	8
	P.M.	......	......	6 30	17 8	„	7
8.	A.M.	12 30	22 3	6 15	16 6	„	7
	P.M.	12 40	24 2	7 0	17 6	N.Westerly.	7
9.	A.M.	1 0	22 6	7 0	16 6	„	8
	P.M.	1 0	23 10	7 40	17 4	N.N.W.	7
10.	A.M.	1 30	22 9	7 45	17 4	„	7
	P.M.	1 50	24 2	8 30	17 8	N.Westerly.	8
11.	A.M.	2 0	22 10	8 15	17 6	„	6
	P.M.	2 30	23 6	8 40	17 9	„	7
12.	A.M.	3 10	22 10	9 15	18 6	„	9
	P.M.	2 45	23 10	9 40	18 3	N.W.	6
13.	A.M.	3 15	22 3	9 30	18 2	N.Westerly.	4
	P.M.	3 30	22 6	9 45	17 8	„	2
14.	A.M.	4 30	21 5	10 30	18 5	S.S.E.	1 to 6
	P.M.	4 30	21 8	10 30	18 4	„	7
15.	A.M.	5 0	21 10	11 20	19 6	Southerly.	7
	P.M.	5 0	21 4	11 30	18 2	S. by E.	5
16.	A.M.	6 30	21 3	12 30	19 6	S.S.E.	5
	P.M.	6 10	21 0	......	......	„	7
17.	A.M.	7 30	32 7	12 30	19 1	„	7
	P.M.	7 30	21 5	2 15	20 6	„	7
18.	A.M.	8 30	22 11	2 0	19 6	„	7
	P.M.	9 0	21 6	3 30	20 3	„	9
19.	A.M.	10 0	23 1	3 0	19 3	„	9
	P.M.	10 0	21 2	4 30	19 6	S.Easterly	6
20.	A.M.	10 40	22 11	4 0	18 1	Southerly.	3·1
	P.M.	11 0	21 1	5 15	18 4	Calm.	0
21.	A.M.	11 30	23 0	5 0	17 7	S.Westerly.	1
	P.M.	......	......	5 30	17 11	Calm.	0
22.	A.M.	12 20	21 4	5 45	17 5	Southerly.	1
	P.M.	12 15	23 2	6 30	17 6	Northerly.	1
23.	A.M.	12 15	21 8	6	16 9	N.N.W.	2
	P.M.	12 30	23 3	7 0	16 8	Calm.	0
24.	A.M.	12 45	21 6	6 30	15 11	Northerly.	1
	P.M.	1 0	22 10	7 15	16 1	Calm.	0
25.	A.M.	12 45	21 9	7 20	15 8	„	0
	P.M.	1 45	22 8	8 0	15 10	N.N.W.	3
26.	A.M.	1 45	21 0	7 40	16 3	N.Easterly.	6
	P.M.	2 0	23 7	8 15	17 3	Easterly.	7
27.	A.M.	2 28	22 11	8 30	17 6	N.Westerly.	5
	P.M.	1 45	23 0	9 0	17 9	„	5
28.	A.M.	3 0	22 9	9 15	17 10	„	5
	P.M.	3 15	22 0	9 30	17 5	N.W. to S.E.	4·2

Full Moon at 6^d 23^h 15^m, Greenwich. Last quarter at 14^d 6^h 2^m, Greenwich.
New Moon at 22^d 13^h 29^m, Greenwich.

Register of Tides observed at Leopold Harbour in the Month of March 1849.

Day.	High Water.		Low Water.		Wind.	
	Time.	Height.	Time.	Height.	Direction.	Force.
	h m	ft. in.	h m	ft. in.		
1. A.M.	4 30	22 4	10 30	18 0	Calm.—N.W.	0·4
P.M.	4 30	21 10	10 30	17 6	N.Westerly.	2
2. A.M.	5 30	22 3	11 20	18 10	S.S.E.	2
P.M.	5 30	21 8	11 30	18 3	S.	2
3. A.M.	6 40	22 8	......	......	Calm and Variable.	2·3
P.M.	6 45	21 6	1 30	19 0		
4. A.M.	7 30	22 9	1 10	18 6	S.Easterly.	4
P.M.	8 20	21 7	2 45	19 5	,,	4·7
5. A.M.	9 0	23 5	2 30	18 8	,,	5
P.M.	9 45	21 11	4 20	19 1	,,	7·3
6. A.M.	10 0	23 4	4 0	17 9	,,	3·2
P.M.	11 15	22 2	4 45	18 4	,,	3·7
7. A.M.	11 10	24 4	4 30	18 4	E.S.E.	6·7
P.M.	11 20	23 0	5 45	18 8	Easterly.	8·9
8. A.M.	Noon	24 7	5 30	18 1	S.E.	7·4
P.M.	......	......	6 25	18 4	S.S.E.	2
9. A.M.	12 15	23 1	6 15	17 4	N.N.W.	1
P.M.	12 30	23 10	6 20	17 7	N.N.E.	2
10. A.M.	12 45	22 11	7 0	17 0	N.Westerly.	2
P.M.	1 0	23 8	7 15	17 3	,,	2
11. A.M.	1 30	22 11	7 45	16 0	,,	3
P.M.	1 40	23 4	8 0	17 5	,,	6
12. A.M.	1 50	23 1	8 15	17 8	,,	5
P.M.	2 30	23 2	8 30	17 6	W.N.W.	6
13. A.M.	2 0	23 0	8 30	18 5	,,	6
P.M.	2 30	23 1	8 20	17 10	,,	7
14. A.M.	3 0	22 10	9 30	18 6	,,	9
P.M.	3 0	22 4	9 20	18 0	,,	6
15. A.M.	3 30	22 8	9 45	19 2	N.Westerly.	5
P.M.	3 30	22 1	10 10	18 7	,,	2
16. A.M.	4 30	22 5	10 30	19 10	N.N.W.—S.S.E.	1·1
P.M.	4 0	21 7	11 0	18 11	S.E.'ly.—S.W.'ly.	1·3
17. A.M.	5 30	22 6	11 45	20 5	Calm.	,,
P.M.	4 45	21 8	12 0	19 8	Easterly.	6
18. A.M.	6 40	22 6	1 30	20 3	,,	4
P.M.	6 45	21 3	......	......	N.Easterly.	4
19. A.M.	7 45	22 5	1 0	19 9	,,	4
P.M.	8 30	21 0	3 15	19 8	,,	5
20. A.M.	9 30	22 8	2 30	19 5	,,	6
P.M.	9 45	21 3	4 20	19 4	,,	5
21. A.M.	10 0	23 2	3 30	19 1	Northerly.	3
P.M.	10 31	21 11	4 30	19 0	,,	1
22. A.M.	10 45	23 9	4 30	19 0	,,	3
P.M.	11 0	22 7	5 30	18 10	,,	4
23. A.M.	11 10	23 11	5 0	18 7	,,	2
P.M.	11 40	22 5	5 45	17 11	Calm.	,,
24. A.M.	......	......	5 30	17 6	,,	,,
P.M.	12 5	23 9	6 20	17 4	Southerly.	2
25. A.M.	12 25	22 9	6 15	16 11	S.S.Easterly.	3
P.M.	12 40	23 1	6 45	16 0	,,	3
26. A.M.	1 15	22 11	7 0	16 4	,,	5
P.M.	1 30	22 9	7 10	16 2	,,	5
27. A.M.	1 30	23 5	7 50	16 10	,,	5
P.M.	2 20	23 0	8 0	16 6	Southerly.	4
28. A.M.	2 30	23 7	8 30	17 2	Easterly.	4
P.M.	2 20	22 8	8 45	17 0	Northerly.	1
29. A.M.	3 15	23 8	9 10	18 1	Easterly.	7
P.M.	3 0	22 7	9 15	17 10	,,	7
30. A.M.	3 45	23 6	10 30	18 9	N.Easterly.	5
P.M.	3 50	21 9	10 15	17 11	N.Westerly.	5
31. A.M.	5 0	22 11	11 30	19 0	Northerly.	2
P.M.	5 15	21 1	11 30	18 5	Southerly.	3

First quarter at 1^{d} 12^{h}, Greenwich. Full Moon at 8^{d} 13^{h}, Greenwich. Last quarter at 16^{d} 12^{h}, Greenwich.
New Moon at 24^{d} 2^{h}, Greenwich. First quarter at 30^{d} 19^{h}, Greenwich.

Register of Tides observed at Leopold Harbour in the Month of April 1849.

Day.	High Water.		Low Water.		Wind.	
	Time.	Height.	Time.	Height.	Direction.	Force.
	h m	ft. in.	h m	ft. in.		
1. A.M.	6 20	23 0	……	……	E.S.E.	6
P.M.	7 20	21 6	1 0	19 5	,,	8
2. A.M.	7 20	23 0	12 20	19 0	,,	8
P.M.	8 30	21 6	2 15	19 3	N.Easterly.	4
3. A.M.	8 40	23 4	2 20	19 1	,,	4
P.M.	9 50	22 4	4 0	19 1	,,	4
4. A.M.	10 0	23 11	3 30	19 4	N.Westerly.	3
P.M.	10 30	22 7	4 30	18 11	,,	2
5. A.M.	10 30	23 3	4 15	18 5	,,	4
P.M.	11 0	22 6	5 20	17 10	Calm.	0
6. A.M.	11 20	23 3	5 15	17 9	,,	0
P.M.	11 50	23 10	5 30	17 8	Northerly.	1
7. A.M.	12 0	23 6	6 0	17 10	Calm.	
P.M.	……	……	6 15	17 11	N.Westerly.	2
8. A.M.	0 20	23 9	6 30	18 2	,,	3
P.M.	0 30	23 7	6 30	17 9	N.N.W.	3
9. A.M.	0 45	23 8	5 45	17 10	,,	2
P.M.	1 0	22 11	7 10	17 7	N.N.E.	4
10. A.M.	1 0	23 8	7 30	18 2	N.Easterly.	4
P.M.	1 30	22 7	7 30	18 0	,,	5
11. A.M.	1 50	23 7	8 30	18 3	Northerly.	2
P.M.	2 30	22 3	8 10	18 1	Southerly.	2
12. A.M.	2 30	24 0	8 45	19 3	Northerly.	3
P.M.	2 30	22 7	8 45	18 3	N.N.W.	3
13. A.M.	3 15	23 0	9 30	18 8	Variable.	2
P.M.	3 0	21 3	9 0	18 0	S.S.E.	1
14. A.M.	3 35	22 6	10 35	18 10	,,	3
P.M.	3 30	20 11	9 45	18 5	,,	5
15. A.M.	5 0	22 2	11 30	19 1	,,	6
P.M.	4 45	20 9	10 45	18 9	,,	6
16. A.M.	5 30	22 2	……	……	,,	4
P.M.	6 30	20 7	{ 12 30 11 30	19 4 19 3 }	,,	3
17. A.M.	7 0	22 3	2 30	19 4	,,	3
P.M.	8 30	21 0	……	……	,,	,,
18. A.M.	8 30	23 0	1 30	19 8	Southerly.	2
P.M.	9 30	21 9	3 30	19 6	N.Westerly.	9
19. A.M.	9 15	23 2	3 15	19 5	,,	6
P.M.	9 45	21 6	4 20	19 0	,,	8
20. A.M.	10 0	23 2	4 0	19 0	,,	7
P.M.	11 0	23 1	4 15	18 6	,,	4
21. A.M.	10 50	23 9	4 30	18 11	S.S.E.	2
P.M.	11 15	23 9	5 15	18 2	,,	4
22. A.M.	11 40	23 9	5 30	18 5	N.N.W.	2
P.M.	12 0	24 2	5 45	17 8	,,	2
23. A.M.	12 0	23 9	5 45	18 0	Northerly.	3
P.M.	……	……	6 20	17 5	,,	2
24. A.M.	12 40	24 7	7 0	17 10	,,	3
P.M.	12 40	23 3	7 0	16 11	,,	5
25. A.M.	1 15	24 7	7 30	17 9	N.N.E.	4
P.M.	1 20	22 11	7 30	17 0	,,	6
26. A.M.	2 0	24 6	8 20	18 0	N.Easterly.	5
P.M.	2 15	22 6	8 20	17 0	N.N.W.	3
27. A.M.	3 0	24 0	9 0	17 11	N. by E.	4
P.M.	3 30	21 7	9 10	17 3	N.N.E.	5
28. A.M.	4 0	23 5	10 30	18 0	,,	3
P.M.	4 15	22 1	10 0	17 11	N.Westerly.	6
29. A.M.	4 45	23 5	11 30	18 9	,,	5
P.M.	5 10	21 5	11 20	18 11	,,	2
30. A.M.	5 45	23 4	……	……	S.S.E.	2
P.M.	6 40	21 6	12 50	19 2	,,	1

Full Moon at $7^{d}\ 3^{h}\ 49^{m}$, Greenwich. Last quarter at $15^{d}\ 7^{h}\ 7^{m}$, Greenwich.
New Moon at $22^{d}\ 11^{h}\ 54^{m}$, Greenwich. First quarter at $29^{d}\ 2^{h}\ 17^{m}$, Greenwich.

Register of Tides observed at Leopold Harbour in the Month of May 1849.

Day.		High Water.		Low Water.		Wind.	
		Time.	Height.	Time.	Height.	Direction.	Force.
		h m	ft. in.	h m	ft. in.		
1.	A.M.	7 15	23 2	12 35	19 5	N.N.W.	1
	P.M.	8 30	21 9	2 30	18 11	N. Westerly.	3
2.	A.M.	8 30	23 1	2 0	19 5	N.N.E.	3
	P.M.	9 15	22 3	3 30	18 8	,,	4
3.	A.M.	......	......	2 45	19 2	Northerly.	3
	P.M.	10 15	22 9	4 20	18 5	,,	3
4.	A.M.	10 15	23 1	4 15	19 0	,,	5
	P.M.	10 20	23 9	4 30	18 5	N.N.E.	4
5.	A.M.	11 0	23 4	5 0	18 9	,,	3
	P.M.	11 45	23 10	5 0	18 5	,,	3
6.	A.M.	11 30	23 0	5 50	18 11	Northerly.	3
	P.M.	12 10	23 9	5 30	17 9	N.N.W.	5
7.	A.M.	12 15	23 7	6 15	18 6	S.Easterly.	2
	P.M.	12 30	23 10	6 0	17 6	Southerly.	2
8.	A.M.	12 30	22 4	6 45	18 4	Variable.	1·3
	P.M.	......	......	6 30	17 9	Easterly.	7
9.	A.M.	0 45	24 2	7 0	18 11	,,	7
	P.M.	0 50	22 10	7 0	18 3	Southerly.	2
10.	A.M.	1 30	24 2	8 0	18 7	Variable.	2·4
	P.M.	1 40	21 9	8 0	17 9	,,	3
11.	A.M.	2 20	23 7	8 40	18 7	S.S.E.	5
	P.M.	2 30	21 8	8 30	18 1	Southerly.	5
12.	A.M.	3 0	23 8	9 15	19 2	Variable.	2
	P.M.	2 40	21 9	9 15	18 6	S.S.E.	2
13.	A.M.	3 20	23 5	10 15	19 2	,,	2
	P.M.	3 20	21 4	9 30	18 10	,,	2
14.	A.M.	4 15	23 1	11 10	19 2	,,	4
	P.M.	4 10	21 0	10 19	19 3	Easterly.	5
15.	A.M.	5 0	22 11	12 0	19 4	S.Easterly.	4
	P.M.	6 0	21 2	11 0	19 7	,,	4
16.	A.M.	6 0	22 11	......	......	,,	5
	P.M.	7 30	21 7	1 20	19 5	,,	4
17.	A.M.	7 15	22 11	0 45	19 11	Easterly.	3
	P.M.	8 30	21 11	2 30	19 1	S.E.	3
18.	A.M.	8 30	22 8	2 0	19 8	N.Easterly.	5
	P.M.	9 30	22 0	3 0	18 4	Northerly.	1
19.	A.M.	9 15	22 5	3 20	18 8	N.Westerly.	4
	P.M.	10 15	22 7	4 0	17 7	,,	4
20.	A.M.	10 15	22 7	4 15	18 6	N.N.W.	5
	P.M.	10 40	23 2	4 20	17 3	N.Westerly.	4
21.	A.M.	11 0	22 5	5 0	18 2	N.N.W.	2
	P.M.	12 0	24 4	5 20	17 0	S.S.E.	2
22.	A.M.	11 30	23 2	6 0	18 4	Northerly.	2
	P.M.	12 15	24 11	6 5	17 1	S.S.E.	2
23.	A.M.	12 20	23 3	6 45	18 5	,,	2
	P.M.	......	......	6 30	17 3	N.N.W.	3
24.	A.M.	1 0	25 3	7 30	18 4	,,	2
	P.M.	1 15	23 4	7 0	17 7	N.Westerly.	3
25.	A.M.	2 0	25 5	8 15	18 1	Calm.	0
	P.M.	1 30	22 10	8 15	17 8	N.Westerly.	6
26.	A M.	2 45	25 0	9 30	18 7	,,	7
	P.M.	2 45	22 6	9 0	18 1	,,	5
27.	A.M.	3 30	24 7	10 30	18 9	,,	5
	P.M.	4 0	22 0	9 45	18 5	,,	6
28.	A.M.	4 40	23 11	11 20	18 8	,,	5
	P.M.	4 0	21 4	10 45	18 11	,,	3
29.	A.M.	5 30	23 5	11 45	18 11	,,	2
	P.M.	6 40	21 10	11 40	19 8	Variable.	1
30.	A.M.	6 30	23 5	......	......	Easterly.	2
	P.M.	8 0	22 3	1 30	19 1	,,	1
31.	A.M.	8 15	23 1	1 30	19 11	Northerly.	2
	P.M.	8 15	22 8	2 20	19 1	,,	2

Full Moon at 6ᵈ 19ʰ, Greenwich.
New Moon at 21ᵈ 19ʰ, Greenwich.

Last quarter at 14ᵈ 22ʰ, Greenwich.
First quarter at 28ᵈ 11ʰ, Greenwich.

Register of Tides observed at Leopold Harbour in the Month of June 1849.

Day.		High Water.		Low Water.		Wind.	
		Time.	Height.	Time.	Height.	Direction.	Force.
		h m	ft. in.	h m	ft. in.		
1.	A.M.	8 45	22 9	2 30	19 9	Northerly.	1
	P.M.	9 30	23 6	3 0	18 8	N.N.W.	2
2.	A.M.	9 20	22 11	3 30	19 9	,,	4
	P.M.	10 30	23 7	3 45	18 9	,,	3
3.	A.M.	10 20	22 9	4 30	19 9	,,	4
	P.M.	11 0	24 2	4 30	18 10	,,	4
4.	A.M.	11 0	22 9	5 15	19 9	,,	5
	P.M.	11 40	24 4	5 0	18 10	,,	7
5.	A.M.	11 30	22 9	5 50	19 7	,,	6
	P.M.	12 10	24 6	5 30	18 8	,,	7
6.	A.M.	......	......	6 45	19 7	N.Westerly.	7
	P.M.	12 15	22 8	6 10	18 8	,,	7
7.	A.M.	12 45	24 6	7 20	19 2	,,	7
	P.M.	1 10	22 4	6 45	18 5	,,	5
8.	A.M.	1 30	24 5	7 45	19 2	Northerly.	2
	P.M.	1 30	22 5	7 30	18 4	,,	1
9.	A.M.	1 45	24 2	8 30	19 4	N.Easterly.	3
	P.M.	2 0	22 3	7 30	18 11	E.N.E.	6
10.	A.M.	2 20	24 4	9 0	19 4	Easterly.	5
	P.M.	2 40	22 2	8 30	19 1	N.N.E.	2
11.	A.M.	3 15	24 3	9 15	18 10	E.N.E.	6
	P.M.	3 0	21 4	9 15	18 7	,,	6
12.	A.M.	3 20	23 5	10 20	18 11	Variable.	2
	P.M.	3 40	21 7	10 0	19 4	N.Westerly.	1
13.	A.M.	4 30	23 7	11 20	19 4	N.N.E.	3
	P.M.	5 0	21 10	11 0	19 8	,,	1
14.	A.M.	5 15	23 5	12 0	19 5	Variable.	1
	P.M.	5 30	22 0	12 0	20 0	Easterly.	2
15.	A.M.	6 30	23 1	......	......	,,	6
	P.M.	7 30	22 4	1 15	18 11	,,	6
16.	A.M.	7 50	22 9	1 30	19 9	S.Easterly.	2
	P.M.	9 0	22 10	2 15	18 6	Variable.	1
17.	A.M.	8 30	22 9	1 45	19 7	S.Easterly.	4
	P.M.	10 45	23 6	3 0	18 1	,,	2
18.	A.M.	9 30	22 8	3 40	19 3	E.S.E.	3
	P.M.	10 30	24 2	3 45	17 9	,,	5
19.	A.M.	10 30	22 11	5 0	19 0	Easterly.	2
	P.M.	11 30	24 9	4 40	17 7	Northerly.	2
20.	A.M.	11 15	22 10	6 0	18 8	Easterly.	2
	P.M.	12 0	25 0	5 30	17 1	,,	4
21.	A.M.	......	......	6 30	18 4	,,	4
	P.M.	12 30	23 0	6 10	17 3	Variable.	2
22.	A.M.	12 40	25 5	7 15	18 3	N.N.E.	2
	P.M.	1 10	22 0	7 15	17 3	N.Easterly.	3
23.	A.M.	1 30	25 4	8 30	18 4	,,	2
	P.M.	1 50	23 6	7 40	17 10	,,	2
24.	A.M.	2 15	25 3	8 50	18 5	Northerly.	1
	P.M.	2 30	22 9	8 30	18 3	Southerly.	2
25.	A.M.	3 15	24 11	9 30	18 5	Variable.	1
	P.M.	3 20	22 5	9 30	18 9	S.S.E.	2
26.	A.M.	3 50	24 4	10 30	18 5	,,	5
	P.M.	4 30	22 3	10 30	19 2	S.Westerly.	5
27.	A.M.	4 45	23 7	10 30	18 9	S.S.E.	5
	P.M.	5 30	22 1	11 30	19 5	,,	5
28.	A.M.	5 40	23 2	......	......	,,	4
	P.M.	7 0	22 5	12 30	18 11	,,	5
29.	A.M.	7 0	22 10	12 50	20 2	,,	3
	P.M.	8 20	22 7	1 30	19 0	,,	1
30.	A.M.	8 0	22 7	2 30	20 0	,,	1
	P.M.	9 0	23 3	2 30	19 0	,,	2

Full Moon at 5^{d} 16^{h}, Greenwich.
New Moon at 20^{d} 2^{h}, Greenwich.

Last quarter at 13^{d} 10^{h}, Greenwich.
First quarter at 26^{d} 22^{h}, Greenwich.

Register of Tides observed at Leopold Harbour in the Month of July 1849.

Day.	High Water.		Low Water.		Wind.	
	Time.	Height.	Time.	Height.	Direction.	Force.
	h m	ft. in.	h m	ft. in.		
1. A.M.	9 30	22 7	3 15	20 4	S.S.E.	4
P.M.	9 45	23 10	3 15	19 0	„	4
2. A.M.	10 0	22 6	4 15	20 3	Variable.	1
P.M.	10 45	24 0	4 30	18 11	Northerly.	2
3. A.M.	10 40	22 5	5 15	19 2	N.Westerly.	4
P.M.	11 15	24 2	5 0	18 7	„	5
4. A.M.	11 30	22 5	5 50	19 8	„	4
P.M.	11 50	24 6	5 15	18 6	„	4
5. A.M.	11 50	22 8	6 0	19 6	„	5
P.M.	Midnight.	24 9	5 45	18 7	„	2
6. A.M.	......	......	6 45	19 4	„	3
P.M.	12 30	22 10	6 20	18 4	„	2
7* A.M.	12 50	24 9	7 10	19 4	„	2
P.M.	12 50	22 9	7 0	18 6	„	3
8. A.M.	1 20	24 10	8 0	18 11	„	5·6
P.M.	1 30	22 6	7 20	18 2	„	6
9. A.M.	2 0	24 6	8 45	18 10	Variable.	2
P.M.	2 0	22 6	8 0	18 7	S.S.E.	3
10. A.M.	3 0	24 6	9 0	19 1	S.Easterly.	4
P.M.	3 0	22 7	8 50	19 0	Northerly.	2
11. A.M.	3 20	24 5	9 50	19 3	„	3
P.M.	3 20	23 0	9 20	19 8	„	2
12. A.M.	3 45	24 8	10 40	19 10	Variable.	2
P.M.	4 0	23 2	10 15	20 2	N.Westerly.	4
13. A.M.	5 0	24 0	11 30	19 5	„	4
P.M.	5 40	22 11	11 0	20 5	Variable.	2
14. A.M.	5 30	23 10	12 20	19 7	N.Westerly.	2
P.M.	7 0	23 2	......	......	„	3
15. A.M.	6 45	23 1	1 0	20 4	„	4
P.M.	8 0	23 6	1 0	19 10	„	3
16. A.M.	8 0	22 10	2 10	20 4	„	2
P.M.	9 15	23 10	2 30	18 7	„	2
17. A.M.	9 0	22 9	3 20	19 10	S.S.E.	2
P.M.	10 10	24 8	3 10	18 8	„	3
18. A.M.	10 0	23 0	4 15	19 7	„	3
P.M.	10 45	25 3	4 15	18 0	N.Westerly.	3
19. A.M.	11 0	23 1	5 30	19 1	„	5
P.M.	11 45	25 5	5 0	17 8	„	5
20. A.M.	11 45	23 3	6 30	18 8	„	3
P.M.	......	......	6 15	17 8	„	4
21. A.M.	12 30	25 8	7 15	18 7	Northerly.	4
P.M.	12 30	23 6	6 40	17 6	„	3
22. A.M.	1 15	25 5	7 50	18 1	„	1
P.M.	1 15	23 2	7 30	17 5	„	2
23. A.M.	2 0	25 1	8 30	17 11	„	1
P.M.	2 0	22 11	8 0	17 7	N.N.E.	2
24. A.M.	2 30	24 5	9 20	17 11	„	3
P.M.	2 45	22 9	9 15	18 2	„	4
25. A.M.	3 15	23 11	9 45	18 1	Northerly.	5
P.M.	3 40	22 4	9 40	18 7	N.Westerly.	4
26. A.M.	4 0	23 3	10 30	18 6	N.W.	6
P.M.	4 0	22 4	10 20	19 4	„	7
27. A.M.	4 30	22 10	11 20	18 10	„	6
P.M.	5 40	22 4	11 30	20 0	„	6
28. A.M.	5 30	22 6	......	......	N.Westerly.	5
P.M.	7 0	22 9	12 30	19 3		
29. A.M.	6 30	22 5	1 0	20 8		
P.M.	8 10	23 0	1 40	19 5		
30. A.M.	8 10	22 2	2 30	20 7	N.Easterly.	3
P.M.	9 0	23 6	2 30	19 5	E.N.E.	4
31. A.M.	9 15	22 3	3 45	20 6	„	6
P.M.	10 10	23 4	3 50	19 4	East.	6

Full Moon at 5^{d} 1^{h}, Greenwich.
New Moon at 19^{d} 9^{h} Greenwich.

Last quarter at 12^{d} 19^{h} Greenwich.
First quarter at 26^{d} 12^{h} Greenwich.

Register of Tides observed at Leopold Harbour in the Month of August 1849.

Day		High Water		Low Water		Wind	
		Time	Height.	Time	Height	Direction.	Force
		h m	ft in	h m	ft. in		
1.	A.M.	10 15	22 5	4 45	20 3	E.S.E.	7
	P.M.	10 45	24 8	4 0	19 4	N.Easterly.	6
2.	A.M.	10 50	22 9	5 20	20 3	E.S.E.	5·6
	P.M.	11 45	23 0	6 0	19 10	"	6
3.	A.M.	11 55	25 0	5 15	18 10	Easterly.	4
	P.M.	12 0	23 2	6 30	19 5	S.Easterly	5
4.	A.M.	Noon.	23 2	6 30	19 5	"	6
	P.M.	"	. .	6 0	18 9	Variable.	3·7
5.	A.M.	12 30	25 3	7 30	19 6	S.Easterly.	7
	P.M.	12 0	23 5	6 30	18 4	"	6
6.	A.M.	1 0	25 1	7 20	18 11	Southerly.	4
	P.M.	1 15	23 3				

Full Moon at 3^d 13^h.

Explanation of the MS. Diagram accompanying this Paper, referred to in page 243.

This diagram was formed by laying down on ruled paper the observations of height recorded in the Tables of Observation.

The alternate high waters were then joined by lines, forming two curves, the distance between which is double the Diurnal Tide at the time of High Water.

The same construction was made for the Low Waters. As the Diurnal Tide vanishes with the declination, the equinoctial Diurnal Tide is altogether Lunar; accordingly the Lunar Diurnal Tide was found from the equinoctial portion of the diagram.

When found, it was superposed on the Solstitial portion of the diagrams,—the points of intersection being found by means of the interval from $\mu=0$ to $D=0$, or age of tide already obtained from the equinoctial observations.

The part of the diurnal curves remaining, after the superposition of the Lunar Tide, was the Solar Diurnal Tide.

The Lunar and Solar Tides at High and Low Water having been thus found, their remaining constants, viz. Coefficients and Tidal Intervals, were worked out, as shown in the paper itself.

A Curve, drawn bisecting the range of the Diurnal Tide at High Water, and another curve in like manner at Low Water, give the High and Low Water Curves of the Semidiurnal Tides that I have used in the second part of my paper on the Arctic Tides. This part of the subject is given in detail in the paper itself.

XII. *Results of the Magnetic Observations at the Kew Observatory, from* 1857 *and* 1858 *to* 1862 *inclusive.*—No. I.

By Major-General Edward Sabine, *R.A.*, *President of the Royal Society.*

Received May 21,—Read June 18, 1863.

§ 1. *A tabular synopsis of ninety-five of the principal disturbances of the magnetic declination recorded by the Kew Photograms between January* 1858 *and December* 1862 *inclusive; and a comparison of the Laws of the Disturbances derived therefrom, with the Laws derived by the more usual method.*

It seems difficult to understand how any one having the opportunity of examining the daily photographic records of a magnetic observatory, and viewing them with an intelligent eye, can fail to discern in the magnetic disturbances the systematic operation of laws depending upon the solar hours; and to perceive that these laws are different from those which govern the regular solar-diurnal variation (upon which the disturbances, whensoever occurring, are superposed).

There are, however, many persons who have not the opportunity of examining for themselves these full and complete records, but who may, nevertheless, be desirous of obtaining a clearer and more distinct understanding of the true character of these remarkable phenomena, in the belief that such knowledge is indispensable as the *first step* of an inductive inquiry which may ultimately reveal to us their causes, and the nature of the causation by which they are produced; and also from the prominency which is given to such an investigation in the Report of the Royal Society in 1840, wherein it is asserted that "the progressive and periodical variations are so mixed up with the casual and transitory changes, that it is *impossible* to separate them so as to obtain a correct knowledge and analysis of the progressive and periodical variations, *without taking express account of and eliminating the casual and transitory changes.*" The elimination of the disturbances was thus early foreseen to be an essential preliminary step in the systematic investigation of the periodical magnetic variations generally. The whole course of subsequent research has manifested the sagacity and importance of this early precept, and the necessity of placing this fundamental point of our investigations on a secure basis. I have thought, therefore, that it might be desirable to place before the Royal Society a synopsis of the deflections from the normal positions of the declinometer, tabulated from the photograms of the Kew Observatory, in a large portion of the most notable disturbances which occurred between January 1858 and December 1862, showing the direction and the amount of disturbance at twenty-four equidistant epochs in each of the disturbed days—in the belief that those who may desire to do so will obtain, by a careful examination of such a tabular view, and of the appended comments, a more distinct and definite perception of the character of the magnetic disturbances than appears to be usually possessed.

In forming the Table which occupies pages 276 and 277, the principle of selection adopted, and invariably adhered to, has been to take all those days in which twelve at the least of the twenty-four equidistant epochs have been disturbed to an amount equalling or exceeding 0·15 inch of the photographic scale, or 3′·3 of arc, on either side of the normal of the month and hour to which the recorded position corresponds, the normal itself having been obtained by recomputation after the omission of all disturbances amounting to 3′·3. The figures in the Table are the differences of the disturbed positions from the normals as above defined. By the process thus described the solar-diurnal and other minor variations are eliminated. There have been ninety-five such days in the five years. The Summary at the close of Table I. shows the resulting aggregate values, both of Easterly and of Westerly deflection, at each of the twenty-four equidistant epochs in each of the five years, as well as in the whole period. The hours of astronomical time at the Kew Observatory have been taken for the twenty-four equidistant epochs.

It is obvious, on the most cursory view of the Summary at the close of the Table (page 277), that the Easterly and Westerly deflections are both subject to systematic laws, and that these laws are distinct and dissimilar in the two cases. Thus the easterly deflections prevail during the hours of the night, and the westerly during the hours of the day. In the day-hours the easterly are small, and vary but slightly; they begin to increase about 5 or 6 P.M., and augment progressively until 11 or 12 P.M., when they attain a value (speaking always of aggregate values) nine or ten times as great as on the average of the day-hours. This great development of easterly disturbance continues until one or two hours after midnight, when it as steadily and progressively subsides until 5 or 6 A.M. The westerly deflections, on the other hand, are distinguished not only by their great prevalence at the hours when the easterly deflections are small, viz. 5 A.M. to 6 P.M., but also by having two distinct epochs of maximum about eight or nine hours apart, viz. one about 6 or 7 A.M., and the other about 3 P.M. This last-named distinction between the two classes of deflection, viz. a single maximum in the one, and a double maximum in the other, is the more worthy of notice, because, as will be shown hereafter, a similar distinction prevails at the greater part of the stations where the laws of the disturbances have been investigated, although, whilst in certain localities of the globe it is, as at Kew, the easterly disturbances which have the single maximum, and the westerly the double maximum, in other localities the converse is found to take place. The increased prevalence of each of the two classes of deflection for about half the twenty-four hours, and diminished prevalence during the other half, appears also to be a usual characteristic,—but with the reservation, that the hours of the prevalence of each class are not the same in different localities, and that they vary *independently of each other*—so much so that at some stations the two classes of disturbance, instead of affecting opposite parts of the twenty-four hours as at Kew, may even have their greatest prevalence at the same hours.

If we now take the pains to compare the summaries of the easterly and westerly deflections in *each* year with those of the means of the *five* years, the accordance is too manifest to admit of a doubt remaining as to the general and systematic character of the laws which have been thus placed in evidence. And if we further proceed to examine *seriatim* the general progression of disturbance in each of the ninety-five days, we shall see reason to conclude that by far the greater part of the disturbances are in conformity with these laws (which are of course more fully and clearly shown by the annual and quinquennial summaries)—thus manifesting the general prevalence of a common type in the disturbing action, even when the days are regarded individually.

In the greater part of the ninety-five days it is easy to trace the presence of *both* the features which may be regarded as the leading characteristics of a disturbance: viz., 1, a deflection (of very considerable amount at certain hours) from the mean or normal position of the magnet; and 2, rapid fluctuations on either side of the deflected position. All days of disturbance are marked by one or the other of these two features, and frequently by both. The *deflections* from the normal are variable in *amount*, but in *direction* they are generally conformable to the systematic laws which have been already adverted to, and which will be more fully discussed in the sequel. The *fluctuations* are extremely irregular both in direction and amount, conveying the impression that the magnet at such times is under the action of two opposing forces, of which sometimes the one and sometimes the other preponderates. A tremulous motion of the magnet is occasionally shown by the photographic traces unaccompanied by changes of direction, as if both the opposing forces were at such times in a state of agitation, but without more than a merely momentary preponderance of either. When large and rapid fluctuations present themselves, we sometimes find considerable and apparently irregular differences in the successive tabulated directions of the magnet (taken, as must be remembered, at the precise instants of the equidistant epochs); but the more regular and systematic prevalence of easterly deflection at particular hours, and of westerly deflection at other hours, usually overrides, even in the individual cases as it does altogether in the means, the partial influence of the fluctuations.

The excess of easterly over westerly, or of westerly over easterly deflection at the several hours in the ninety-five days is a measure of the influence which the disturbances would necessarily exercise on the "diurnal inequality" derived from the hourly means of the ninety-five days, if the elimination of the disturbances were unattended to: the excess thus referred to constitutes, in fact, the *disturbance-diurnal variation* due to that portion of the disturbances occurring in the five years which is included in the ninety-five days contained in the Table. This part of the subject will be resumed in the third section of this paper.

TABLE I.—Synopsis of Ninety-five days of most notable Disturbance of the Magnetic Declination in the years 1858 to 1862 inclusive. From the Kew Photograms.

Dates.	Kew Astronomical Hours.																							
	0h.	1h.	2h.	3h.	4h.	5h.	6h.	7h.	8h.	9h.	10h.	11h.	12h.	13h.	14h.	15h.	16h.	17h.	18h.	19h.	20h.	21h.	22h.	23h.
1858.	′	′	′	′	′	′	′	′	′	′	′	′	′	′	′	′	′	′	′	′	′	′	′	′
Jan. 8.	...	...	...	...	E. 9·8	...	E. 3·6	E. 4·7	...	...	...	E. 24·3	E. 10·0	W. 4·4	E. 4·0	. .	...	W. 9·1	W. 10·8	W. 21·0	...	W. 12·2	W. 5·7	...
Mar. 13	W. 7·6	W. 5·1	W. 5·2	W. 10·3	W. 10·4	W. 10·1	W. 9 8	W. 7·1	E. 10·9	W. 7·4	..	...	W. 14·0	E. 7·1	E. 14·5	E. 12·0	E. 15·8	...	E. 5·5	...	...	...	...	...
14.	...	...	W. 7·2	W. 6·5	W. 7·1	W. 7·6	. .	E. 11·1	E. 6·4	E. 10·1	E. 3·4	...	E. 4 1	E. 4·5	W. 6·6	...	W. 5·9	...	...	W. 9·2	...	...	...	...
April 9.	...	W. 4·4	W. 11·0	W. 14·2	W. 43·9	W. 18·5	W. 36 6	. .	W. 5·6	W. 5·3	E. 4·7	E. 4 3	E. 10 6	E. 7·2	E. 3·5	..	...	...	...	...	W. 10·7	W. 3·7	...	...
10.	W. 5·9	W. 4·2	W. 5·5	W. 8·3	W. 5·7	W. 5·9	W. 5·0	E. 15·5	E. 3·4	...	...	E. 8·4	E. 13·4	...	E. 4·0	E. 7·3	E. 7·2	...	...	W. 3·4	W. 3·8	W. 4·1	...	W. 5·1
24.	E. 13·5	E. 14·8	E. 13·5	E. 9·9	E. 6·7	E. 4·6	...	..	..	...	...	...	...	...	..	...	...	W. 4·6	W. 6·6	W. 9·5	W. 10·5	W. 6·2	...	E. 6·3
May 18.	E. 12·2	E. 12·9	E. 11·5	E. 7·9	E. 5·7	E. 4·2	...	...	...	...	...	...	...	..	...	...	W. 6 0	W. 7·9	W. 10 5	W. 12·6	W. 9·6	W. 4·1	...	E. 5·3
19.	E. 10·7	E. 12 5	E. 10·5	E. 8·0	E. 5·3	E. 4·1	...	...	...	...	...	...	...	...	...	...	W. 3·8	W. 5·3	W. 7·3	W. 9·2	W. 8 3	W. 4·3	...	...
June 3.	W. 4·2	W. 6·7	W. 5·5	W. 4·8	...	...	W. 3·9	...	...	W. 6·1	W. 3·7	. .	...	W. 10·5	...	E. 6·6	..	W. 9·1	W. 6·2	...	...	...	...	W. 4·1
22.	...	...	...	E. 3·6	E. 4·6	E. 4·3	...	...	...	...	E. 13·1	...	..	E. 7·9	E. 4·5	E. 4·8	E. 7·0	E. 3·8	W. 6·1	W. 21·0	W. 8·7	...	..	...
23.	...	...	...	...	W. 7·2	E. 3 5	W. 8·4	E. 5·8	E 9·5	...	E 26·2	...	...	E. 10·3	E. 3 6	W. 9·3	W. 8·1	...	W. 9·9	W. 8·3	W. 10·6	W. 6·2	W. 7·8	W. 3·3
24.	W. 3·2	...	...	...	...	...	E. 3·4	E. 10·1	E. 6·1	E. 9·0	E. 6·1	E. 5 1	E. 7·6	E. 9·5	E. 7·5	W. 3·7	W. 9·2	W. 4·7	W. 11·0	W. 10·5	W. 5·8	W. 6·2	...	...
July 5.	E. 4·8	...	.	...	...	...	...	E. 6·1	E. 4·6	E. 4·1	E. 10·3	E. 8 3	E. 14 9	E. 7·2	...	W. 5·0	E. 4·8	E. 4·8	E. 4·0	E. 3·2	...	...	...	...
Sept. 20	W. 8 7	...	W. 6·2	W 5·3	E. 4·2	W. 3·9	W. 3·8	...	...	E. 4·4	...	E. 11 5	...	E. 14·8	E. 9 1	E. 6·5	...	..	...	W. 8 2	W. 6·6	W. 5·3	...	...
21.	..	...	...	W. 4·6	E. 5 8	...	...	...	E. 6 4	W. 6·1	E. 10·1	E. 11 8	E. 10 3	E. 12·5	W. 4·1	E. 4·3	W. 4 4	W. 6·7	W. 5·2	W. 5·6	...	...	...	...
Oct. 18.	...	..	E. 4·5	E. 3·8	E. 4·0	...	...	...	...	...	W. 6 8	E. 6·4	E. 16·1	E. 3·8	E. 5·1	E. 5·3	E. 6·2	...	..	W. 3·3	W. 8·3	W. 8·3	...	...
Dec. 4.	...	...	...	...	...	...	...	...	W. 6·2	E. 13·2	E. 6·6	E. 10 7	E. 16·5	E. 10·1	E. 15·0	E. 23·5	W. 3·5	W. 12·7	W. 13·0	W. 20·6	W. 6·8	...	W. 3·3	...
22.	...	...	W. 4·1	W. 5·7	W. 4·0	...	...	...	...	E. 4·3	..	..	E. 4·3	E. 5·4	. .	...	...	W. 6·0	W. 6·3	W. 3·5	W. 3·4	W. 4·6	W. 6·3	W. 3·7
23.	W. 9 8	W. 4·8	...	W. 4·9	W 4·9	W. 3·7	W. 4·0	E. 5·4	...	...	...	..	E. 3 8	E. 18 4	E. 9·4	...	...	...	W. 4·4	...	W. 3·3	...	...	W. 3·9
1859																								
Jan. 15.	W. 6·9	W. 7·1	W. 4·7	W. 8·3	W. 8·7	W. 6·0	W. 7·6	E. 5·4	E. 5·6	E. 6·1	E. 7·2	E. 12·4	E. 6·7	E. 10 5	E. 11·1	E. 8·0	..	W. 3·4	W. 3·5	W. 5·5	W. 6·5	W. 6·1	W. 4·2	W. 4·3
Feb. 9.	...	.	W. 4·9	W. 6·0	W. 6·5	W. 5·5	W. 13·0	...	...	E. 6·5	E. 6·3	..	E. 3 2	..	E. 12·3	E. 4·5	W. 6·7	W. 13·8	W. 3·4	W. 5·2	...	...	W. 4·5	...
23.	..	...	...	...	...	. . .	E. 7·0	E. 10 9	E. 5·9	..	E. 3·9	E. 15 3	E. 8·6	E. 5·2	E. 3·9	E. 3 9	...	W. 4·0	W. 7·8	W. 12·4	W. 8·6	W. 11·7	W. 11·4	W. 5·9
April 21.	...	...	W. 6·8	W. 10·4	W. 21·7	W. 17·4	W. 16·8	W. 15·0	...	E. 5·7	E. 16·9	E. 9·0	E. 4 7	E. 8·9	E. 14·8	E. 14·9	...	E. 4·8	...	...	...	...	...	...
29.	W. 3·5	W. 6·8	..	W. 7·1	W. 12·2	W. 5·7	..	...	E. 4·9	E. 3·7	E. 12·4	E. 8·3	E. 5·8	E. 9·9	W. 4·4	..	W. 4·2	...	...	W. 8 0	W. 8·5	W. 6·6	...	...
June 8.	W. 9·9	...	E. 6·1	W. 8·6	W. 14·9	W. 14·0	W. 13·5	E. 9·2	...	W. 3·7	...	. .	...	...	...	...	W. 24·7	E. 3·4	W. 15·2	W. 11·1	W. 11·3	W. 5·9	W. 6 0	W. 4·6
16.	. . .	W. 5·7	W. 6·5	W. 5 4	W. 5·8	W. 3 6	..	...	.	...	E. 3 3	...	E. 4·0	E. 3·5	..	W. 3·3	W. 3·6	...	W. 3·4	...	W. 4·4	W. 3·7	...	...
July 11.	...	..	...	W. 3·6	..	W. 4·5	E. 5·4	...	...	W 4·5	...	E. 3 5	E. 7 3	E. 6·1	E. 14·0	E. 10·8	E. 11·5	E. 8·4	E. 7·9	...	...	W. 4·2	W. 5·0	...
18.	W. 4·3	W. 6·1	W. 12·2	W. 10 8	...	W. 6·1	W. 3·5	.	...	...	...	W. 4·0	...	...	W. 4·2	...	W. 4·8	W. 7·4	W. 5·6	W. 4·8	...	W. 4·4	...	...
Aug. 26.	W. 5·3	W. 6·2	W. 4·8	. .	. .	...	..	..	...	E. 4·9	E. 3·4	E. 5 3	E. 4·3	E. 4·9	E. 5·9	...	...	...	E. 5·3	E. 5·1	E. 5·1	E. 3·3	...	...
28.	.	...	...	..	...	W. 3·5	W. 4·2	E. 8·9	E. 5·9	...	E. 3·4	E. 6·4	...	E. 24·7	E. 32·6	...	W. 17·8	W. 47·8	W. 34·4	...	...	W. 26·1	W. 25·2	...
Sept. 1.	W. 8·3	W. 4·7	W. 4·5	W 4·5	...	..	...	...	.	..	E. 3 9	..	..	...	..	.	E. 8·5	W. 52·3	W 55 1	E. 20·9	E. 27·8	E. 4·4	E. 13·7	...
2.	E. 17·3	..	..	W. 6·2	W. 4·3	W. 6·9	E. 6·8	..	W. 4·8	E 25·9	W. 8 4	E. 7·6	W. 15·5	..	E. 11·4	E. 5·3	W. 7·4	..	...	...	...	...	...	...
3.	W. 6·8	W. 9·0	W. 11·7	W. 25·3	W. 16·4	W. 13·4	E. 36·3	...	. . .	...	...	W. 10·7	W. 12·0	...	..	.	W. 5·9	...	W. 5 1	W. 7·4	...	...	...	...
4.	W. 4·0	W. 7·9	W. 6·0	...	...	..	.	..	W. 3·4	..	...	E 7 6	W. 7·9	E. 4·8	W. 5 4	...	.	...	E. 4·4	E. 4·9	...	W. 5·3	...	W. 9·8
5.	..	..	.	W. 8·3	W. 8·5	..	.	W. 5·9	W. 4·1	W. 12·3	. .	.	...	W. 6·3	W. 5·5	...	W. 4·1	W. 3·5	W. 3·8	W. 4 9	W. 9·4	W. 4·2	W. 3·5	W. 11·5
24.	..	W. 4·8	W. 5·4	W. 3·3	W. 4 1	W. 5·2	W. 13·9	W. 5·0	W. 7·4	..	E. 15 9	E. 4·7	E. 5 0	E. 8·7	.	..	. . .	...	...	...	...	...	...	...
Oct. 17.	W. 3·7	...	. .	...	...	...	...	..	..	E. 32·3	E. 12·9	E. 16·9	..	E. 3·9	E. 3·8	E. 7·0	E. 12·6	W. 6·7	W. 4·3	W. 16·0	W. 10·3	W. 10·2	W. 14·1	W. 10·0
18	W. 6·4	W. 4·9	...	.	W. 3·5	...	..	...	...	...	E. 6·0	E. 10·2	E. 8·8	W. 5·2	E. 8·6	E. 6·6	...	..	W. 5·5	...	W. 5·1	W. 4·1	...	W. 4 0
20.	W. 5 5	W. 7·0	W. 3·6	W. 7·9	W. 5·0	.	...	..	...	E. 5·8	.	E. 5·9	...	..	E. 4·7	...	...	...	W. 5·1	W. 3·4	W. 5·5	W. 4·1	W. 4·0	...
21.	......	W. 4·5	W. 3·9	W. 7·3	W. 10·0	W. 4·8	..	W. 4·1	W. 3·5	..	..	...	...	...	..	...	E. 3·5	...	W. 5·2	...	W. 5·0	W. 5·5	W. 4·1	...
Nov. 12.	...	W. 3·9	W. 3·6	W. 5·3	W. 5·3	...	E. 4·1	..	...	E. 7·9	E. 10·6	E. 9·7	E. 9·1	E. 18·5	E. 5·4	...	...	W. 7·0	W. 7·6	W. 14·8	W. 10·9	...	W. 10·1	W. 4·3
Dec. 6.	W. 7·1	W. 5·1	...	W. 4·6	.	E. 6·5	W. 5·0	. .	E. 5·2	.	.	...	E. 4·2	E. 4·1	.	W. 11·7	W. 4·1	...	W. 3·4	W. 3·3	...	...	...	...
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Feb. 21.	W. 4·2	W. 11·5	...	W. 8·5	W. 7·8	...	...	...	...	E. 5·0	E. 9·9	W. 20·3	E. 14·2	E. 11·0	E. 11·9	E. 10·4	E. 10·5	E. 8·2	...	...	...	W. 5·5	W. 8·3	W. 3·7
22.	W. 7·7	W. 7·7	W. 6·1	W. 6·4	W. 7·9	E. 6·5	...	..	E. 6·8	E. 4·9	E. 6·6	E. 3·2	. .	W. 3·3	W. 4·2	..	...	...	...	...	...	...	...	...
Mar. 17.	...	W. 3·8	...	...	W. 5·2	W. 3·4	E. 7·3	...	...	E. 5 9	E. 7·0	E. 7·9	E. 8·1	E. 11·8	E. 6·0	E. 4·6	E. 6 1	...	W. 4·4	...	W. 4·0	W. 5·8	W. 7·2	W. 7·2
28.	W. 17·6	W. 7·4	W. 8·4	...	...	...	..	...	...	..	..	E. 3·7	W. 3 9	E. 4·2	E. 12·8	W. 4·4	W. 5·3	W. 10·0	W. 20·2	W. 14·2	W. 9·5	W. 14·1	W. 7·3	W. 8·3
29.	W. 10·7	W. 4·8	W. 12·7	W. 30 0	W. 37 3	...	W. 3·2	. .	E. 12·8	..	.	..	E. 8·3	E. 3·6	..	W. 3·4	...	...	...	...	...	W. 5·9	W. 6·2	W. 5·6
Apr. 9.	W. 6 2	W. 9·5	W. 8·1	W. 20·4	W. 12·5	W. 15·7	W. 16·2	.	W. 4·5	E. 7·9	W. 3·5	E. 14·9	E. 16·1	E. 6·3	E. 8·3	.	...	...	...	W. 4·0	W. 6·3	W. 9·3	W. 7·3	W. 8·4
13.	...	...	...	..	...	..	..	E. 4 5	...	E. 4·8	..	E. 7·0	E. 9·6	E. 3·4	E. 6·6	E. 9·5	...	...	W. 3·3	W. 3·3	W. 4·7	W. 6·2	W. 6·9	...
May 5.	...	W. 3·2	W. 6·7	W. 7·8	W. 5·9	...	...		...	.	. .	...	E. 4·0	E. 3·3	E. 5·5	E. 8·1	E. 4·4	...	...	...	...	E. 6·0	E. 6·0	E. 3·5
6.	...	W. 5·0	W. 9·1	W. 9·2	W. 3·3	...	...	E. 7·8	E. 5·3	..	E. 3·3	E. 3·3	E. 9·4	E. 4·1	E. 8·2	E. 10·4	E. 7·6	E. 7·2	..	...	...	...	...	...
June 30.	W. 5·3	W. 3·2	W. 8·7	..	E. 5·9	E. 4·2	E. 4·9	E. 7·9	E. 6·5	E. 9·7	E. 10 7	E. 17·2	E. 7·6	E. 21·2	E. 27·1	E. 6·7	E. 6·4	E. 14·3	...	W. 6·2	E. 4·1	E. 8·8	E. 14·1	E. 21·1
July 1.	...	...	...	E. 7·1	...	E. 5·4	E. 5·4	E. 4·4	E. 20·6	E. 20·7	E. 14·3	E. 5·4	E. 9·6	E. 7·5	E. 7·0	E. 8·3	E. 9·4	E. 7·5	E. 7·7	E. 5·1	E. 10·4	E. 4·5	E. 7·7	E. 5·7
4.	W. 3·7	W. 5·2	W. 4·6	W. 4·4	W. 6·2	W. 4·3	..	...	W. 3·4	W. 3·4	..		E. 9·2	E. 7·8	E. 18·2	E. 12·9	...	E. 11·3	...	E. 6·0	...	...	...	W. 6·2

Aug. 6.	...	E. 3·7	E. 3·5	E. 4·6	...	...	E. 4·5	...	..	..	...	..	E. 5·1	E. 7·8	E. 5·4	...	E. 12·4	E. 5·6	E. 8·4	E. 26·2	...	E. 13·9	W. 6·2	E. 6·1
10.	...	...	...	E. 4·4	W. 4·9	..	..	W. 4·9	W. 5·5	W. 5·5	E. 3·4	W. 7·2	W. 14·1	W. 17·3	E. 5·4	W. 3·7	...	...	...	...	...	...	...	E. 4·2
12.	W. 5·5	...	W. 9·0	W. 21·7	E. 3 5	E. 13·0	W. 3·7	W. 7·8	...	..	...	W. 17·4	W. 19·2		W. 13·3	E. 15·5	E. 8·0	E. 4·8	...	...	...	...	...	E. 6·9
Sept. 15.	..	...	...	...	...	...	..	...	...	E. 11·3	E. 4·6	E. 7·9	E. 4·0	E. 11·0	E. 10·1	E. 8·4	E. 13·2	E. 7·4	W. 4·8	W. 12·7	W. 9·0	W. 8·7	W. 11·3	...
Oct. 4.	W. 6·9	...	W. 3·4	...	...	W. 4·8	...	E. 12·6	...	..	E. 3·8	E. 5·0	E. 5·7	E. 5·4	E. 5·6	E. 5·4	...	...	W. 5·5	...	...	W. 3·3	W. 3·3	...
Nov. 2.	W. 4·6	W. 4·9	W. 7·7	W. 10 0	W. 3·9	...	W. 7 8	W. 8 7	W. 21 5	W. 23·1	W. 24 0	..	W. 3·7	W. 4·6	W. 6 3	...	...	..	...	...	W. 3·3	W. 3·6	...	W. 3·7
4.	W. 3·8	...	...	..	..	...	W. 3 8	W. 3·6	W. 4·9	W. 4·2	W. 4 1	..	E. 4·6	E. 14·1	E. 9·8	E. 6·6	E. 3·4	E. 3·7	...	...	W. 3·4	...	...	W. 7·8
Dec. 10.	...	W. 4·5	W. 12·2	W. 10·7	W. 10·8	W. 10·6	...	E. 3·8	E. 16·5	E. 14·1	E. 5·1	E. 10·4	E. 12·0	E. 15·5	E. 15·4	..	E. 7·6	...	W. 8·3	W. 8·4	W. 7·2	W. 9·0	W. 3·8	W. 7·3
1861.																								
Jan. 24.	W. 6·4	W. 4·8	W. 14·9	W. 5·0	W. 6·1	...	...	W. 4·5	E. 13·6	...	E. 10·6	E. 15·9	..	W. 3 6	E. 3 3	E. 4·0	...	...	W. 3·3	...	...	...	W. 3·2	W. 7·9
25.	W. 3·9	W. 7·2	W. 7·7	W. 3·5	W. 9·4	E. 10·9		E. 6·7	...	E. 4 8	E. 3·5	E. 4·4	E. 3·7	E. 4·0	E. 9·8	E. 3·3	W. 10·8	...	W. 8·0	W. 6·1	W. 7·3	W. 3·3	...	...
26.	W. 6·4	...	W. 6·8	W. 4 5	W. 5·9	...	W. 4·7	...	..	E. 8·5	E. 3·9	E. 32 8	E. 16·4	.	W. 3·6	...	..	W. 5·0	W. 21·8	W. 15·0	W. 8·5	W. 4·4	W. 8·1	W. 6·7
Feb. 27.	W. 3·3	W. 7·2	W. 6 9	W. 7·9	W. 6 8	W. 4·9	...	W. 9·5	E. 6·5	..	E. 7·9	E. 14·1	E. 20·3	E. 10·1	E. 9·4	E. 12·5	E. 12·7	..	..	...	...	W. 3·5	W. 4·1	W. 5·9
28.	W. 6·1	W. 7·9	...	W. 5·5	W. 5 0	W. 5·2	...	E. 5·9	...	E. 12·9	E. 8·7	E. 6·4	E. 12·3	E. 15·1	...	E. 9 3	E. 4·9	W. 9·5	W. 9·1	W. 6·1	W. 5·5	...	...	...
Mar. 1.	...	. .	W. 4·9	W. 6 2	W. 5 7	E. 12 4	...	W. 5·3	E. 9·7	E. 4·4	...	E. 3·7	E. 7·8	...	..	E. 4·5	..	...	...	W. 4·5	W. 5 1	W. 4·0	...	...
9.	...	W. 6·4	W. 3·9	...	...	W. 7·9	...	..	E. 7·2	E. 6·5	E. 29·6	E. 13·8	E. 10·3	E. 14·2	E. 13·7	E. 9·9	E. 19·1	E. 20·9	E. 21·9	E. 19·9	E. 19·1	E. 17·6	E. 17·4	...
25.	W. 5·2	W. 6·5	W. 4·7	W. 4·0	W. 4 5	W. 5·1	...	E. 4·1	..	E. 8·2	...	E. 3·9	E. 5 5	E. 5·0	E. 10·6	E. 7·2	E. 6·9	...	...	...	W. 5·2	W. 4·7	W. 6·4	W. 6·2
Aug. 2.	W. 4·8	W. 9 9	W. 9·4	W. 5·7	W. 5·3	W. 3·9	W. 4·4	W. 3 7	..	...	E. 3·7	E. 6·1	E. 11·8	...	E. 3·9	...	E. 3·6	. .	...	...	...	...	...	...
19.	. .	...	...	. .	E. 6·7	. .	E. 4·8	E. 13·9	. .	. .	..	E. 3·3	...	...	W. 3·3	..	...	W. 5·2	W. 4·3	W. 3·6	W. 5·9	W. 4·2	W. 5·2	W. 4·5
Sept. 19.	W. 4·4	W. 7·0	W. 3·5	W. 5·5	E. 4·2	..	...	...	E. 3·4	...	...	...	.	E. 10 1	...	...	...	W. 11·0	W. 4·3	W. 5·0	W. 7·8	W. 4·9	...	...
Oct. 19.	W. 8·5	W. 13·1	W. 14·9	W. 6 8	W. 4 5	. .	...	E. 16·4	..	E. 20·8	E. 10·4	E. 24 7	E. 16·7	..	E. 16·1	...	..	W. 5·5	W. 4·0	...	...	...	...	...
24.	W. 5.5	W. 7·3	W. 9·3	.	W. 15·6	W. 9·8	...	...	E. 6·3	E 15·3	E. 9·6	E. 7·9	..	.	E. 7·0	E. 12·3	E. 8·0	W. 8·3	...	W. 11·0	...	...	...	...
Nov. 7.	W. 4·1	W. 3·3	W. 3·7	W. 6·2	W. 5 6	...	.	E. 16·4	E. 5·2	E 7·7	..	. .	E. 8·7	...	..	...	...	W. 4·2	W. 4·3	W. 13·0	W. 9·1	W. 6·4	W. 4·4	...
18.	...	...	..	...	...	..		...		E. 6·5	E. 3·7	E. 4·5	E. 4·1	E. 3·9	E. 7·8	E. 3·6	...	..	...	W. 7·5	W. 6·5	W. 6·4	W. 5·2	W. 3·9
Dec. 4.	...	W. 3·6	W. 7·3	W. 6·8	W. 11·5	W. 5·4	...	E. 3·5	E. 6·6	W. 11·8	E. 5·8	..	..	..	W. 5·7	..	...	..	...	..	W. 3 3	W. 6·9	W. 5·6	...
9.	..	...	...	..	...	E. 8·5	...	. .	E. 11·1	...	E. 7·4	E. 10 0	E. 4 5	E. 4·8	E. 4·3	. .	...	W. 8·1	W. 5·3	W. 4·4	W. 6·4	W. 6·7	W. 6·6	W. 5·8
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Jan. 15.	.	..	.	...	..	W. 5·0	E. 4·2	E. 3·9	E. 4·2	.	E. 12·2	E. 5·6	E. 12·7	E. 18·4	E. 15·4	E. 11·5	E. 7·2	...	W. 4·5	W. 7·1	W. 8·5	E. 19·6	E. 19·1	...
Feb. 21.	W. 4·4	W. 6·7	W. 11·7	W. 7·2	W. 12·8	W. 7·4	W. 8·7	E. 7·4	E. 26·7	...	E. 5·9	E 3·5	E. 8·3	E. 10·7	. .	...	..	...	...	...	...	...	...	...
Mar. 16.	...	..	...	...	...	...	..	..	...	. .	E. 4·4	E. 5·0	E 5·2	E. 5·5	E. 6·7	E. 3·8	E. 7·3	E. 4·9	...	W. 4·9	W. 8·0	W. 9·9	W. 9·5	W. 8·1
April 2.	W. 7·9	W 6·4	W. 14·5	W. 6 0	W. 6·3	..	E. 8·9	E. 4 2	E. 6·5	E. 13·2	E. 18·6	E. 5·4	. .	.	...	...	.	W. 5·1	...	...	...	...	...	...
10.	W. 5·1	W. 5·9	W. 8·8	W. 8 9	W. 7 4	W. 5·2	W. 3·5	...	...	. .	E 5·1	E. 4·7	E. 4·8	E. 7·3	E. 11·7	E. 16·4	E. 20 5	E. 7·6	...	W. 11·7	W. 9·8	W. 8·3	W. 8·3	W. 7·2
27.	...	...	...	...	..	..	..	E. 5·1	...	E. 7·7	E. 9 0	E. 10·7	E. 18·7	E. 9·9	E. 14·1	E. 9 5	E. 5·2	...	..	. .	W. 9·0	W. 6·8	W. 8·5	W. 4·2
July 5.	..	E. 4·2	...	..	..	...	W. 6·1	..	E. 12·4	...	E. 4·6	E. 3·8	E 3 3	...	E. 4·4	. .	...	...	W. 5·4	W. 9·5	W. 8·7	W. 3·3	W. 4·9	W. 4·3
30.	..	...	W. 4·3	W. 4·4	W. 3·3	.	...	E. 4·4	...	...	...	E. 5·4	E. 7·0	E. 5·6	..	..	...	...	...	E. 3·3	E. 4·7	E. 6·2	E. 5·5	E. 4·1
August 4.	...	W. 10·0	W. 7·5	W. 8·5	..	W. 5·6	...	...	W. 3·6	..	..	..	..	W. 11·1	W. 8·7	E. 9·1	W. 8·7	W. 5·3	W. 16·7	W. 3·3	W. 4·9	W. 10·9	...	...
28.	..	. .	...	W. 3·6	W. 4·9	..	W. 4·8	. .	...	E. 19·3	E. 7·5	E. 6·4	E. 6·6	..	W. 7·4	W. 6·8	...	...	...	W. 5 3	W. 9·0	W. 10·6	W. 5·9	W. 3·6
30.	.	W. 3·3	...	..	..	...	E. 18·0	E. 4·9	...	. .	...	..	E. 3 8	W. 9·8	E. 6·6	E. 8·2	W. 4·6	W. 7·3	W. 10·8	W. 14·5	W. 9·5	W. 6·6	...	...
Sept. 29.	..	...	...	E. 3·7	E. 5·4	E. 4·3	...	. .	.	.	W. 4·4	W. 5·7	W. 5·1	W. 4·6	W. 6·8	W. 5·2	W. 7·0	W. 7·2	W. 4·5	...	...	...	...	...
Oct. 3.	...	W. 3·8	W. 4·4	...	...	... .	...	E. 4·4	E. 24·0	E 15·4	E. 49·9	E. 24·4	E. 19·2	E. 14·6	E. 14·8	E. 8·6	W. 3·8	W. 25·1	W. 37·0	W. 33·4	W. 20·4	W. 28·2	W. 9·9	W. 4·2
22.	W 7·9	W 11·3	W. 5·6	W. 10·3	E. 21·0	W. 3·3	..	E. 6·3	.	W. 16 5	...	E. 16·5	.	W 4·8	...	...	...	W. 10·7	W. 4·9	W. 4·4	W. 6·1	W. 6·2	...	W. 5·7
Dec. 14.	..	..	..	W. 11·2	W. 6 3	E. 12 2	E. 5·3	W. 23·1	E. 14·6	E. 32·9	E. 28·1	E. 13·7	E. 29·5	E. 20·5	E. 14·9	E. 8·5	...	...	W. 3·6	...	...	...	...	...
24.	W. 4·2	..	W. 3·9	...	E. 3·4		E. 9·9	E. 7·0	E. 13·2	E. 7·5	E. 18·1	E. 19 4	E. 20·3	E. 22·4	E. 29·9	E. 16·1	..	E. 6·9	...	W. 10·9	W. 11·7	W. 5·2	W. 5·9	W. 4·8

Summary.

Easterly Deflections. 1858.	41′·2	40′·2	40′·0	33′·2	46′·1	20′·7	7′·0	58′·7	47′·3	45′ 1	80′·5	90′·8	111′·6	118′·7	80′ 2	70′ 3	41′·0	8′·6	9′·5	3′·2	′ ...	′ ..	′ . .	11′·6
1859	17·3		6·1	.	.	6·5	59·6	34·4	27·5	98·8	106·1	122 8	71·7	113·7	128·5	61·0	36·1	16·6	17·6	30·9	32·9	7·7	13·7	. .
1860	.	3 7	3·5	16·1	9·4	29·1	22·1	41·0	68·5	84·3	68·7	85·9	127·5	138·0	163·3	106·8	89·0	70·0	16·1	37·3	14·5	33·2	27·8	47·5
1861.	...	.	..	.	10·9	31·8	4 8	66·9	69·6	95·6	104·8	151·5	122·1	67 2	85·9	66·6	55·2	20·9	21·9	19·9	19·1	17·6	17·4	...
1862	..	4·2	..	3·7	29·8	16·5	46·3	47·6	101·6	96 0	163·4	124·5	139·4	114·9	118·5	91·7	40·2	19·4	...	3·3	4·7	25·8	24·6	4·1
Sums ..	58·5	48·1	49·6	53 0	96·2	104·6	139·8	248·6	314·5	419·8	523·5	575·5	572·3	552 5	576·4	396·4	261·5	135·5	65·1	94·6	71·2	84·3	83·5	63·2
Westerly Deflections. 1858.	39·4	25·2	44·7	64·6	83 2	49 7	71·5	7·1	11·8	24·9	10·5	. .	14 0	14·9	10·7	18·0	40·9	66·1	97·3	145·9	96·4	65·2	23·1	20·1
1859.	71·7	83·7	78·6	132 9	126 9	96·6	77·5	30·0	23·2	20·5	8·4	14·7	35·4	11·5	19·5	15·0	83·3	145·9	168·4	96 8	85·5	102·1	92·1	54·4
1860.	76·7	70·7	96·7	129·1	105·7	38·8	34·7	25·0	39·8	36 2	31·6	44·9	40·9	25·2	23·8	11·5	5·3	10·0	46·5	48·8	47·4	71·4	67 8	58·2
1861.	58·6	84·2	97·9	67·6	85·9	42 2	9·1	23·0	...	11·8	...	.	.	3·6	12·6	..	10·8	56·8	64·4	76·2	70·6	55 4	48·8	40·9
1862.	29·5	47·4	60·7	60 1	41·0	26·5	23·1	23·1	3·6	16 5	4·4	5·7	5 1	30·3	22·9	12·0	24·1	60·7	87·4	105·0	105·6	96 0	52·9	42·1
Sums ..	275·9	311·2	378·6	454·3	442·7	253·8	215·9	108·2	78·4	109·9	54·9	65·3	95·4	85·5	89·5	56·5	164·4	339·5	464 0	472 7	405·5	390·1	284·7	215·7

§ 2. *Comparison of the laws of the disturbance-diurnal variation derived from the ninety-five days of disturbance tabulated in the first section of this paper, with the conclusions derived, at the same place and for the same period, from the wider basis of investigation supplied by the process first introduced and published by myself eighteen years ago.*

The process here referred to consists, as is well known, in separating from the whole body of observations employed, *all*, without exception and whensoever occurring, which differ from their respective normals of the same month and hour by a certain value, constant for the same element at the same station—the amount of this arbitrary standard, or minimum value of a disturbance, being regulated by one condition only, viz. that it shall not be so small as to endanger the inclusion amongst the separated observations of any in which the cause of the irregularity may with probability be ascribed to any other source than that of the class of phenomena whose laws we desire to study. In the case of the hourly positions tabulated from the Kew Photograms from January 1858 to December 1862, 0·15 inch of the photographic scale, or 3·3 minutes of arc measured from the normal of the same month and hour after the omission of the disturbed observations, has been taken as the standard or minimum value of a disturbance. There are altogether in the photograms of the five years at Kew the effective records of 43,456 hourly positions; the number of failures in the photographic registration from *all* causes being only 368. Of these 43,456 recorded positions, 5941, being about 1 in 7 of the whole body, differed by an amount equalling or exceeding 3′·3 from their respective normals. The aggregate value of the differences of the disturbed positions, measured from the normals, was 36,580·8 minutes of arc, of which 19,748′·7 were easterly, and 16,832′·1 were westerly deflections.

Table II. exhibits the *aggregate values* of the disturbances distributed into easterly and westerly deflections, and into the several *hours* of their occurrence. The easterly deflections derived from the ninety-five days are in column 2, and those derived from the 5941 disturbed positions (*i. e.* from all disturbances equalling or exceeding 3′·3) in column 3; the westerly deflections derived from the ninety-five days occupy column 4, and those obtained from the 5941 disturbed positions column 5. The *Ratios* which the aggregate values of easterly and westerly deflection at the different hours bear to their respective mean hourly values are shown in the same Table (II.), the easterly in columns 6 and 7; the westerly in columns 8 and 9. By comparing the values in columns 6 and 7 with each other, it will be seen that the Ratios of the easterly deflections exhibit approximately the same law, whether obtained from the ninety-five days, or from all disturbances equalling or exceeding 3′·3; and by comparing the ratios in columns 8 and 9, it will be seen, in like manner, that there is a similar general accordance in the ratios of the westerly deflections, whether obtained from the ninety-five days, or from the more extensive induction: the *laws*, when examined by the *ratios*, are seen to be approximately the same when derived by either process, although the aggregate values are very dissimilar—being more than three times as great when the method of investigation is such as to comprehend all disturbances equalling or exceeding 3′·3, as when it is limited to the disturbances in ninety-five days of principal note.

TABLE II.

Kew Astronomical Hours.	Aggregate Values.				Ratios.				Kew Astronomical Hours.
	Easterly Deflections.		Westerly Deflections.		Easterly Deflections.		Westerly Deflections.		
	From 95 Days.	From all Disturbances.	From 95 Days.	From all Disturbances.	From 95 Days.	From all Disturbances.	From 95 Days.	From all Disturbances.	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0	59	247	276	1074	0·25	0·30	1·15	1·53	0
1	48	269	311	1157	0·21	0·33	1·57	1·65	1
2	50	308	379	1232	0·21	0·37	1·56	1·76	2
3	53	245	454	1175	0·23	0·30	1·88	1·67	3
4	96	371	443	1039	0·41	0·45	1·86	1·48	4
5	105	478	254	718	0·46	0·58	1·07	1·02	5
6	140	647	216	506	0·60	0·79	0·89	0·72	6
7	249	999	108	314	1·07	1·21	0·44	0·45	7
8	315	1322	78	233	1·35	1·61	0·33	0·33	8
9	420	1562	110	257	1·81	1·90	0·45	0·37	9
10	524	1983	55	160	2·26	2·41	0·23	0·23	10
11	576	1978	65	156	2·47	2·40	0·27	0·22	11
12	572	1946	95	301	2·46	2·36	0·40	0·43	12
13	553	1706	86	371	2·38	2·07	0·35	0·53	13
14	576	1558	90	441	2·48	1·89	0·37	0·63	14
15	396	1245	57	376	1·71	1·51	0·24	0·54	15
16	262	862	164	488	1·12	1·05	0·68	0·70	16
17	136	419	340	839	0·58	0·51	1·40	1·20	17
18	65	243	464	1058	0·28	0·30	1·91	1·51	18
19	95	262	473	1175	0·41	0·32	1·95	1·67	19
20	71	289	406	1039	0·31	0·35	1·67	1·48	20
21	84	274	390	981	0·36	0·33	1·61	1·40	21
22	84	287	285	838	0·36	0·35	1·18	1·19	22
23	63	250	216	910	0·27	0·30	0·89	1·30	23
Sums...	5592	19750	5815	16838	233=1·00	823=1·00	243=1·00	701=1·00	Mean hourly values.
Means..	233	823	243	701					

For the convenience of those who prefer graphical to tabular representation, the diurnal course of the easterly deflections, corresponding to the Ratios in columns 6 and 7 of Table II., is exhibited in Plate XIII. figure 1, where the broken line shows the diurnal march indicated by the ratios obtained from the ninety-five days, and the unbroken line the diurnal march obtained from all the disturbances equalling or exceeding 3′·3. Figure 2 is a similar representation of the diurnal march of westerly disturbance-deflection, obtained, as shown by the broken line, from the ninety-five days, and by the unbroken line from the more comprehensive investigation. The general aspect of the two figures seems to establish in the most conclusive manner—

1. That the disturbances have systematic laws:

2. That the easterly and westerly deflections have each their own systematic laws, distinct and different from each other:

3. That these laws are approximately the same, whether derived from the more limited or from the more comprehensive basis, although in the latter case the *aggregate values* of disturbance are more than three times as great as when the disturbances of the ninety-five days only are taken into account.

Hence it follows that by taking only the most notable days of disturbance in five years (averaging nineteen in each year), we may gain an approximately correct view of the *character* of the disturbance-diurnal variation; but if we desire not only to learn its character, but also to *eliminate its influence*, in compliance with the prescribed condition of "eliminating the casual and transitory changes as a first and essential step towards a correct knowledge of the more regular periodical variations," then we see that *the mere omission of those ninety-five days is altogether inadequate for the desired object*, as it would scarcely eliminate a third part of the systematically disturbing element, shown to admit of elimination by a more suitable process.

§ 3. *Disturbance-diurnal Variation.*

Table III. exhibits the excess of easterly over westerly, or of westerly over easterly deflection at twenty-four equidistant epochs of the solar day, derived, in column 2, from the disturbances in the ninety-five days, and in column 3, from all disturbances equalling or exceeding 3′·3 from their respective normals. These columns consequently show the disturbance-diurnal variation corresponding to the more complete, and to the less complete, process of elimination. The *character* of the progression is seen to be substantially the same in both cases, but the *amount* of disturbance is between three and four times as great in column 3 as in column 2.

Table III.—Disturbance-diurnal Variation; or Excess of Easterly over Westerly, or of Westerly over Easterly Deflection, at twenty-four equidistant epochs in the twenty-four hours.

Kew Astronomical Hours. (1)	Derived from the ninety-five days of most notable disturbance. (2)	Derived from all disturbances equalling or exceeding 3′·3. (3)	Kew Astronomical Hours. (4)
0	217′ w.	826′ w.	0
1	263 w.	889 w.	1
2	329 w.	924 w.	2
3	401 w.	930 w.	3
4	347 w.	607 w.	4
5	149 w.	240 w.	5
6	76 w.	141 E.	6
7	141 E.	686 E.	7
8	237 E.	1089 E.	8
9	310 E.	1305 E.	9
10	469 E.	1823 E.	10
11	511 E.	1822 E.	11
12	477 E.	1644 E.	12
13	467 E.	1336 E.	13
14	486 E.	1117 E.	14
15	340 E.	869 E.	15
16	97 E.	374 E.	16
17	204 w.	420 w.	17
18	399 w.	815 w.	18
19	378 w.	913 w.	19
20	334 w.	750 w.	20
21	206 w.	706 w.	21
22	201 w.	551 w.	22
23	153 w.	660 w.	23

For those who prefer graphical representation, the curved line in Plate XIII. fig. 3 exhibits the excess of easterly over westerly, or of westerly over easterly deflection, *i. e.* the disturbance-diurnal variation, obtained from the 5941 disturbances equalling or exceeding 3′·3 from the respective normals, as shown in column 3 of Table III. The straight horizontal line in figure 3 represents the mean or normal position of the magnet at the several hours, after the omission of the disturbances. It is figured for convenience as a straight line, though in reality it is itself a curve following the progression of the solar-diurnal variation. The lengths of the ordinates which are *above* the normal line indicate the excess of the easterly over the westerly deflections at the hours when the easterly preponderate, and those which are *below* the normal line the excess of the westerly over the easterly at the hours when the westerly deflections predominate.

The easterly portion of the disturbance-diurnal variation is seen to be continuous for about ten hours, or from about 6 P.M. to 4 A.M. The westerly portion is also continuous, extending over the remaining fourteen hours, or from about 4 A.M. to 6 P.M. The easterly has a single maximum occurring about midway between its commencement and its termination. The westerly is more complex, having two maxima separated by an interval of about 8 or 9 hours. But whilst the westerly excess extends over more hours than the easterly, the areas of the two portions have nearly the same dimensions; or, in other words, the sums of the hourly deflections in opposite directions are at Kew nearly equal, and any small difference between them is not a persistent one, the easterly exceeding in some years and in others the westerly. The equality or otherwise of the sum of the deflections in opposite directions is apparently a point of some theoretical significance, as will be further noticed when the analogous phenomena in other localities come to be discussed.

As we find the same general forms of the two portions of the disturbance-diurnal variation, which have been thus derived from the Kew photograms, reproduced in other localities in the separated portions of the easterly and westerly deflections (with only such slight variations as may well be supposed to be due to accidental or subordinate causes), it may be desirable to examine somewhat more closely what may be viewed as the characteristic differences of the deflections in the two directions. The easterly deflection is represented, as we have already seen, in Plate XIII. fig. 1: it is distinguished by its approximately conical form and single maximum, and by the small and nearly equable amount of variation during the ten or eleven hours when the ratios are least. Its general form thus bears a striking resemblance to the diurnal curve of the solar-diurnal variation (as obtained after the careful separation and omission of the casual and transitory changes); but the two phenomena differ from each other in the important circumstance, that in the solar-diurnal variation the solar hours corresponding to its different features are *the same* in all meridians in the extra-tropical parts of the same hemisphere, whilst in the portion of the disturbance-diurnal variation which is now under notice, the solar hours corresponding to its different features *vary*, apparently without limit, in different meridians. This is a distinction which may well be supposed to indicate a difference in the *mode* of causation, although it would not justify an inference that the sun may not be the originating cause in both cases.

The westerly deflections at Kew, represented in Plate XIII. fig. 2, have a decided double maximum, with an intervening interval of about eight or nine hours. The analogous form in other localities has the double maximum sometimes more and sometimes less decidedly marked. The interval intervening between the maxima is usually of about the same duration at stations in the northern hemisphere; at some stations in the southern hemisphere it is apparently somewhat longer.

The conical form and single maximum which characterize the *easterly* deflections at Kew belong also to the easterly deflections in all localities in North America where the laws of the disturbances have been investigated. But when we view the phenomena at Nertschinsk and Pekin, which are the only two localities in Northern Asia for which the investigation has yet been made, we find, on the contrary, that the conical form and single maximum characterize the *westerly* deflections, whilst the easterly have the double maximum. Further, we find that at the two Asiatic stations the aggregate values of the *westerly* deflections decidedly predominate, whilst in America the *easterly* deflections are no less decidedly predominant; and at Kew, which we may regard as an intermediate locality, the amount of deflection in the two directions may be said to be balanced, there being in some years a slight preponderance of westerly, and in other years of easterly deflection.

There is another circumstance which seems to connect, in what may prove even a more instructive relation, the westerly deflections in Northern Asia with the easterly in other parts of the northern hemisphere. I refer here to an approximate accordance in *absolute time* which appears in the most marked features of the diurnal curve at the widely separated localities of Pekin, Nertschinsk, Kew, and Toronto, at each and all of which the curves as they are presented in Plate XIII. figs. 1, 4, 5, and 6 are the mean result of several years of hourly observation*. These localities appear to be particularly well suited for a comparison of this nature, being not very dissimilar in geographical latitude, whilst they include a difference in longitude of no less than 195°. If we select the epoch of the maximum deflection (or the apex of the curve) as the most marked feature, the comparison would stand nearly as follows; commencing with the most easterly, and proceeding in succession from east to west:—

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* The figures 1, 4, 5, and 6 in Plate XIII., representing respectively the Easterly deflections at Kew and Toronto and the Westerly at Nertschinsk and Pekin, are delineated from the following formulæ, in which a, expressed in degrees 15 to the hour, is reckoned from the mean noon at the station:—

Kew . . . $1+0{\cdot}98\,(\sin a+280^{\circ}\;22')-0{\cdot}417\,(\sin 2a+286^{\circ}\;29')$:
Toronto . . $1+1{\cdot}05\,(\sin a+285\;\;58)-0{\cdot}332\,(\sin 2a+334\;\;07)$:
Nertschinsk . $1-0{\cdot}94\,(\sin a+309\;\;02)-0{\cdot}238\,(\sin 2a+\;\;13\;\;11)$:
Pekin . . . $1-0{\cdot}76\,(\sin a+289\;\;12)-0{\cdot}200\,(\sin 2a+\;\;\;1\;\;42)$.

Assuming that the formulæ represent correctly the ratios at the several hours, the observed values are in very tolerable accord with them; at Kew and Nertschinsk they are the most so; at Kew the probable error of a single hourly ratio is $\pm 0{\cdot}056$, at Nertschinsk $\pm 0{\cdot}062$.

Deflections.	Localities.	Latitudes.	Longitudes.	Approximate	
				Local solar Hour.	Absolute Hour at Kew.
		° ′	° ′ h		
Westerly......	Pekin	39 54 N.	116 6 E. = 7·8	22	14
	Nertschinsk	51 19 N.	114 9 E. = 7·7	21	13
Easterly	Kew...............	51 29 N.	0 = 0·0	11	11
	Toronto	43 40 N.	79 0 W. = 5·3	10	15

It must be remembered that the time of the occurrence of the apex (or maximum of deflection) scarcely admits of very *precise* determination; and further, that assuming for the disturbing impulse a common origin at any other point of the terrestrial surface than at the geographical pole, and an equable but appreciable velocity of propagation, the difference of the geographical meridians would not be the sole consideration in deducing the absolute epoch from the local hours at different stations.

Could we thus identify the westerly deflections in Asia with the easterly in Europe and America, we should have a confirmation on a very extended scale of M. GAUSS's conclusion derived from the comparison of synchronous disturbances at stations remote from each other, viz. that "the synchronous disturbances of the same element not only differ widely in amount, but occasionally appear to be even *reversed* in direction."

It may be that this may prove the first step in the inductive inquiry which may lead eventually to a complete understanding of the systematic distinction which we find in comparing the solar-diurnal with the disturbance-diurnal variations,—by referring the first to causes which, within the sphere of their operation, produce the same phenomena at the same solar hours; and the second to effects originating (as far as the terrestrial surface is concerned) in special localities from whence they are propagated, and admitting of classification by means of the absolute hours to which they approximately correspond. For a conclusion of such moment, however, much preliminary investigation is still required, for which materials either do not yet exist, or have not yet been submitted to the necessary processes of examination. It seems especially important that the laws of the disturbances, and of their respective easterly and westerly deflections, should be known at a station or stations intermediate between Nertschinsk and Kew.

The propriety of making the easterly and the westerly deflections the subjects of distinct investigation will be still more apparent by reverting to Plate XIII. fig. 3, and remembering that the areas containing respectively the ordinates above and below the normal line are subject at different stations to horizontal displacements, each independent of the other; and thus that at some stations the opposite deflections may have a tendency to mask each other's influence in the resultant mean deflection (*i. e.* in the *excess* of easterly over westerly, or of westerly over easterly deflection). It happens at Kew that the large disturbances in opposite directions take place at opposite hours of the twenty-four, and that they thus record themselves in great measure independently of each other; but experience has already shown that there are stations where large disturbances show themselves in both directions, on different days, at the same hours; and such deflections would of course tend to neutralize each other in the resultant mean, thus masking the operation of the general causes whose laws we desire to learn. This inconvenience is in great measure remedied by the method of analysis which has

§ 4. TABLE IV.—Kew, Diurnal Inequality of the Declination from January 1, 1858 to December 31, 1862.

Months.		Astronomical Hours.																						
		0^{h}.	1^{h}.	2^{h}.	3^{h}.	4^{h}.	5^{h}.	6^{h}.	7^{h}.	8^{h}.	9^{h}.	10^{h}.	11^{h}.	12^{h}.	13^{h}.	14^{h}.	15^{h}.	16^{h}.	17^{h}.	18^{h}.	19^{h}.	20^{h}.	21^{h}.	22^{h}.	23^{h}.
January		w. 3′·67	w. 4′·19	w. 3′·95	w. 2′·74	w. 1′·75	w. 1′·05	w. 0′·84	E. 0′·10	E. 1′·21	E. 1′·80	E. 2′·63	E. 2′·86	E. 2′·26	E. 1′·87	E. 1′·53	E. 1′·31	E. 1′·08	E. 0′·86	E. 0′·70	E. 0′·56	E. 0′·93	E. 1′·10	w. 0′·33	w. 2′·23
February		w. 4·47	w. 5·83	w. 5·72	w. 4·51	w. 2·88	w. 1·99	w. 0·77	E. 0·17	E. 1·46	E. 2·33	E. 2·65	E. 2·92	E. 2·75	E. 2·12	E. 1·86	E. 1·77	E. 1·42	E. 1·34	E. 1·38	E. 1·52	E. 2·06	E. 2·10	E. 0·52	w. 2·14
March		w. 6·57	w. 8·07	w. 7·49	w. 6·04	w. 3·74	w. 1·35	E. 0·04	E. 0·91	E. 1·77	E. 1·77	E. 2·52	E. 2·41	E. 2·65	E. 2·42	E. 2·26	E. 2·14	E. 2·37	E. 2·25	E. 2·21	E. 2·73	E. 3·98	E. 3·42	E. 0·79	w. 3·46
April		w. 6·91	w. 8·77	w. 8·29	w. 6·16	w. 4·27	w. 1·84	w. 0·45	E. 0·91	E. 1·17	E. 1·25	E. 2·06	E. 2·57	E. 2·54	E. 2·20	E. 2·25	E. 2·21	E. 2·29	E. 2·35	E. 3·25	E. 4·43	E. 5·43	E. 4·12	E. 1·01	w. 3·28
May		w. 5·90	w. 6·81	w. 6·34	w. 4·95	w. 3·31	w. 1·67	w. 0·26	E. 0·42	E. 0·57	E. 0·60	E. 0·82	E. 1·25	E. 1·30	E. 1·11	E. 1·27	E. 2·05	E. 2·39	E. 3·31	E. 4·17	E. 4·88	E. 4·78	E. 3·23	E. 0·32	w. 3·20
June		w. 5·89	w. 7·15	w. 7·23	w. 6·06	w. 4·33	w. 2·38	w. 1·03	E. 0·02	E. 0·25	E. 0·44	E. 0·86	E. 0·87	E. 0·98	E. 1·27	E. 1·99	E. 2·25	E. 2·90	E. 4·42	E. 4·96	E. 5·64	E. 5·15	E. 3·85	E. 0·90	w. 2·87
July		w. 5·65	w. 7·32	w. 7·31	w. 6·07	w. 4·33	w. 2·58	w. 1·11	w. 0·35	E. 0·20	E. 0·24	E. 0·74	E. 1·19	E. 2·02	E. 1·99	E. 1·96	E. 2·08	E. 2·78	E. 4·37	E. 5·14	E. 5·27	E. 4·78	E. 3·23	E. 1·05	w. 2·45
August		w. 6·88	w. 8·32	w. 7·76	w. 5·59	w. 3·14	w. 0·63	E. 0·50	E. 1·03	E. 0·78	E. 1·04	E. 1·47	E. 1·72	E. 1·93	E. 1·89	E. 2·36	E. 2·53	E. 2·91	E. 3·42	E. 4·09	E. 5·00	E. 4·43	E. 2·30	w. 0·96	w. 4·17
September		w. 7·25	w. 8·26	w. 7·08	w. 5·04	w. 2·62	w. 0·80	E. 0·69	E. 1·16	E. 1·49	E. 2·05	E. 3·05	E. 2·42	E. 2·04	E. 2·22	E. 2·20	E. 2·59	E. 2·75	E. 2·26	E. 2·52	E. 3·50	E. 3·86	E. 2·15	w. 1·06	w. 4·88
October		w. 6·50	w. 7·40	w. 6·75	w. 4·73	w. 2·53	w. 1·21	w. 0·38	E. 0·90	E. 2·01	E. 3·14	E. 3·78	E. 3·71	E. 3·29	E. 2·36	E. 2·32	E. 2·05	E. 1·62	E. 0·69	E. 0·98	E. 1·67	E. 2·70	E. 2·30	w. 0·12	w. 3·86
November		w. 4·62	w. 5·11	w. 4·73	w. 3·62	w. 1·89	w. 1·14	w. 0·17	E. 0·75	E. 1·76	E. 2·65	E. 2·35	E. 2·78	E. 2·67	E. 1·81	E. 1·34	E. 1·24	E. 1·14	E. 1·00	E. 0·98	E. 0·89	E. 1·26	E. 1·45	w. 0·10	w. 2·73
December		w. 3·52	w. 4·06	w. 3·95	w. 2·94	w. 1·67	w. 0·94	w. 0·76	E. 0·29	E. 1·86	E. 2·33	E. 3·21	E. 2·91	E. 2·79	E. 2·13	E. 1·92	E. 0·92	E. 0·54	E. 0·25	E. 0·21	E. 0·20	E. 0·40	E. 0·63	w. 0·51	w. 2·15
Semiannual Means ...	Apr. to Sept.	w. 6·41	w. 7·77	w. 7·34	w. 5·65	w. 3·67	w. 1·65	w. 0·28	E. 0·53	E. 0·74	E. 0·94	E. 1·50	E. 1·67	E. 1·80	E. 1·78	E. 2·01	E. 2·29	E. 2·67	E. 3·36	E. 4·02	E. 4·79	E. 4·74	E. 3·15	E. 0·21	w. 3·48
	Oct. to Mar.	w. 4·89	w. 5·78	w. 5·43	w. 4·10	w. 2·41	w. 1·28	w. 0·48	E. 0·52	E. 1·68	E. 2·34	E. 2·86	E. 2·94	E. 2·74	E. 2·12	E. 1·87	E. 1·57	E. 1·36	E. 1·06	E. 1·08	E. 1·26	E. 1·89	E. 1·83	E. 0·04	w. 2·76
Annual Means		w. 5·65	w. 6·77	w. 6·38	w. 4·87	w. 3·04	w. 1·46	w. 0·38	E. 0·53	E. 1·21	E. 1·64	E. 2·18	E. 2·30	E. 2·27	E. 1·95	E. 1·94	E. 1·93	E. 2·02	E. 2·21	E. 2·55	E. 3·02	E. 3·31	E. 2·49	E. 0·13	w. 3·12

§ 4. TABLE V.—Kew, Solar-diurnal Variation of the Declination from January 1, 1858 to December 31, 1862.

January		w. 3·10	w. 3·78	w. 3·50	w. 2·12	w. 1·32	w. 0·89	w. 0·48	E. 0·09	E. 0·92	E. 1·40	E. 1·68	E. 1·66	E. 1·48	E. 1·13	E. 1·01	E. 0·66	E. 0·89	E. 1·11	E. 1·09	E. 1·21	E. 1·47	E. 1·25	w. 0·10	w. 1·83
February		w. 3·65	w. 4·72	w. 5·07	w. 3·78	w. 2·30	w. 1·35	w. 0·76	w. 0·14	E. 0·68	E. 1·22	E. 1·90	E. 1·76	E. 1·69	E. 1·58	E. 1·23	E. 1·17	E. 1·21	E. 1·56	E. 1·72	E. 1·79	E. 2·32	E. 2·48	E. 1·09	w. 1·70
March		w. 5·87	w. 7·24	w. 6·81	w. 5·13	w. 2·94	w. 1·11	w. 0·22	E. 0·47	E. 0·80	E. 1·36	E. 1·66	E. 1·76	E. 1·74	E. 1·31	E. 1·49	E. 1·46	E. 1·74	E. 2·41	E. 2·48	E. 3·39	E. 4·43	E. 4·01	E. 1·41	w. 2·66
April		w. 6·62	w. 8·41	w. 7·87	w. 5·73	w. 3·34	w. 1·43	w. 0·12	E. 0·45	E. 0·81	E. 0·95	E. 1·00	E. 1·33	E. 1·21	E. 1·22	E. 1·53	E. 1·79	E. 2·07	E. 2·47	E. 3·39	E. 5·26	E. 6·39	E. 4·86	E. 1·60	w. 2·79
May		w. 5·79	w. 6·63	w. 6·04	w. 4·61	w. 2·91	w. 1·74	w. 0·44	E. 0·31	E. 0·26	E. 0·20	E. 0·32	E. 0·62	E. 0·72	E. 0·85	E. 1·09	E. 1·68	E. 2·30	E. 3·52	E. 5·00	E. 5·36	E. 5·12	E. 5·37	E. 0·48	w. 2·94
June		w. 5·66	w. 6·86	w. 6·83	w. 5·86	w. 4·17	w. 2·30	w. 0·85	w. 0·12	w. 0·03	E. 0·15	E. 0·17	E. 0·49	E. 0·61	E. 0·96	E. 1·41	E. 1·98	E. 2·94	E. 4·66	E. 5·75	E. 6·16	E. 5·37	E. 3·90	E. 0·80	w. 2·64
July		w. 5·50	w. 6·96	w. 6·77	w. 5·45	w. 3·96	w. 2·14	w. 1·10	w. 0·58	w. 0·26	E. 0·02	E. 0·19	E. 0·47	E. 1·36	E. 1·12	E. 1·48	E. 1·68	E. 2·81	E. 4·23	E. 5·38	E. 5·73	E. 5·23	E. 3·69	E. 0·95	w. 1·94
August		w. 6·94	w. 8·21	w. 7·66	w. 5·40	w. 2·97	w. 0·68	E. 0·46	E. 0·67	E. 0·87	E. 0·76	E. 1·18	E. 1·09	E. 1·57	E. 1·63	E. 1·84	E. 2·29	E. 2·83	E. 3·92	E. 4·81	E. 5·37	E. 4·91	E. 2·72	w. 0·65	w. 4·24
September		w. 6·38	w. 7·43	w. 6·45	w. 4·22	w. 2·13	w. 0·53	E. 0·16	E. 0·60	E. 1·00	E. 1·05	E. 1·34	E. 1·43	E. 1·66	E. 1·60	E. 2·01	E. 2·18	E. 2·55	E. 2·77	E. 3·22	E. 3·96	E. 4·18	E. 2·88	w. 0·89	w. 4·54
October		w. 5·51	w. 6·24	w. 5·61	w. 3·79	w. 2·11	w. 0·93	w. 0·48	E. 0·20	E. 0·91	E. 1·48	E. 1·85	E. 2·03	E. 1·92	E. 1·54	E. 1·41	E. 1·64	E. 1·47	E. 1·58	E. 1·89	E. 2·46	E. 3·56	E. 3·33	E. 0·60	w. 3·14
November		w. 3·92	w. 4·41	w. 3·90	w. 2·96	w. 1·82	w. 1·20	w. 0·26	E. 0·37	E. 1·24	E. 1·81	E. 1·92	E. 1·95	E. 1·70	E. 1·28	E. 1·14	E. 0·96	E. 1·02	E. 1·12	E. 1·22	E. 1·42	E. 1·65	E. 1·69	E. 0·27	w. 2·18
December		w. 2·66	w. 3·39	w. 3·15	w. 2·35	w. 1·23	w. 0·80	w. 0·55	E. 0·27	E. 0·98	E. 1·42	E. 1·61	E. 1·89	E. 1·49	E. 1·17	E. 1·07	E. 0·68	E. 0·69	E. 0·78	E. 0·84	E. 0·84	E. 0·93	E. 1·18	w. 0·03	w. 1·59
Semiannual Means ...	Apr. to Sept.	w. 6·15	w. 7·42	w. 6·94	w. 5·21	w. 3·25	w. 1·47	w. 0·32	E. 0·22	E. 0·44	E. 0·52	E. 0·70	E. 0·90	E. 1·19	E. 1·23	E. 1·56	E. 1·93	E. 2·58	E. 3·60	E. 4·59	E. 5·31	E. 5·20	E. 3·57	E. 0·38	w. 3·18
	Oct. to Mar.	w. 4·12	w. 4·96	w. 4·67	w. 3·35	w. 1·95	w. 1·05	w. 0·46	E. 0·21	E. 0·92	E. 1·45	E. 1·77	E. 1·84	E. 1·67	E. 1·34	E. 1·22	E. 1·09	E. 1·17	E. 1·43	E. 1·54	E. 1·85	E. 2·40	E. 2·32	E. 0·54	w. 2·18
Annual Means		w. 5·13	w. 6·19	w. 5·81	w. 4·28	w. 2·60	w. 1·26	w. 0·39	E. 0·22	E. 0·68	E. 0·99	E. 1·24	E. 1·37	E. 1·43	E. 1·29	E. 1·39	E. 1·51	E. 1·88	E. 2·51	E. 3·07	E. 3·58	E. 3·80	E. 2·95	E. 0·46	w. 2·68

§ 4. *Diurnal Inequality and Solar-diurnal Variation.* Tables IV. and V. p. 284.

The diurnal inequality (which is by no means identical with the solar-diurnal variation as has been sometimes assumed) has in fact two principal constituents, viz. the solar-diurnal variation itself, and the disturbance-diurnal variation. It is obtained for each month by taking the differences between the mean positions of the magnet at each of the twenty-four hours and the mean position in the month (the latter being the mean of all the days and all the hours in the month). It is the first step in the process of obtaining in a separate form the several periodical variations from the combination in which they appear in the photographic records, and includes all the positions tabulated from the records, without the exception of any. Table IV. (page 284) exhibits the diurnal inequality in each month, on the average of the five years, from January 1858 to December 1862 inclusive.

The *solar-diurnal variation*, shown in Table V. (page 284), is obtained by a similar process from the hourly positions in the same period, exclusive of those which differed 3'·3 or more from their respective normals of the same month and hour—the normals being the hourly means in each month after the exclusion of all the disturbed positions. By this process the effects of the "casual and transitory changes" become in a very great degree "eliminated;" and we obtain a measure of the solar-diurnal variation which is only very slightly affected by the small portion of the disturbance-diurnal variation which remains after the separation and omission of the disturbances equalling or exceeding 3'·3 from their respective normals.

Plate XIV. exhibits the solar-diurnal variation at Kew (fig. 2), in comparison with the same at Toronto (fig. 1), Nertschinsk (fig. 3), Pekin (fig. 4), St. Helena (fig. 5), Cape of Good Hope (fig. 6), and Hobarton (fig. 7). Figs. 1, 2, 3, & 4 show the march of the solar-diurnal variation at stations in the middle latitudes of the northern hemisphere, fig. 5 in the equatorial region, and figs. 6 & 7 in the middle latitudes of the southern hemisphere.

In figs. 3 & 4, compared with 1 & 2, we see the gradual flattening of the curve as the magnetic equatorial region is approached. In fig. 5 (geographical latitude of St. Helena, 15° 55' S.) we perceive the incipient reversal of the diurnal march; whilst in figs. 6 & 7, and particularly in fig. 7 (Hobarton, where the dip is more than 70°, and the total force 13·6 in British units), we see the reversal completed, and the full development of the characteristic features appertaining to the southern hemisphere.

It is seen in the Plate that at the stations in the northern hemisphere *generally*, the north end of the magnet passes rapidly from its extreme eastern limit (about 8 A.M., or nearer 9 A.M. at Pekin) to its extreme western limit (about 1 P.M., or rather later at Pekin), the motion being more rapid between 10 and 11 A.M. than at any other hour of the twenty-four; and that during the remaining nineteen hours the north end returns to its eastern limit by a progression tolerably rapid from about 2 to 7 P.M., scarcely sensible from 7 P.M. to 3 or 4 A.M., and again more rapid until 8 A.M. The turning hours are approximately the same at the four stations, having apparently no relation whatsoever to the varied circumstances of sea or land in the vicinity. Taking Hobarton as the best representative which we possess of the southern hemisphere, we see in the Plate the analogy of its phenomena to those of the northern stations. We have the same rapid movement from one extreme to the other, occupying the same portion of time, viz. five hours, and the return occupying the remaining nineteen hours; but the *directions* of the two movements are inverted, the 5-hour movement being in the southern hemisphere from West to East, and the slower, or 19-hour return, being from East to West. The epochs are nearly but not quite the same, being apparently about an hour later in the southern hemisphere.

In the curve of the Cape of Good Hope (fig. 6) we have the same general features as at Hobarton, but with a more flattened curve, indicating a nearer proximity to the equatorial region.

If, now, we permit ourselves to depart from the general custom of expressing the variations of the Declination in the northern as well as in the southern hemisphere by the directions of the *north* end of the magnet, and to speak of the solar-diurnal variation in the southern portion of the magnetic sphere as a movement of the *south* end of the magnet (using the same phraseology as before for the northern hemisphere), we appear to gain a greater simplicity in describing the general characteristics of the phenomena in the two hemispheres. In such case the Hobarton curve is reversed,—the westerly deflections of the north end becoming easterly deflections of the south end, and *vice versâ*,—the inflections of the curve of the annual solar-diurnal variation at Hobarton then appear altogether as the counterparts of those at Kew and Toronto (excepting in the one feature peculiar to the southern hemisphere, of the turning hours being about an hour later than in the northern hemisphere). This correspondence is shown in plate 1 of the first volume of the Hobarton Observations, published in 1850, and in fig. 7 of Plate XIII. accompanying this paper. In like manner the annual curve at the Cape of Good Hope, when reversed, becomes the counterpart of those at Pekin and Nertschinsk; whilst at St. Helena, so near the dividing line between the hemispheres, the *annual* solar-diurnal variation has almost entirely disappeared, the small remaining inflections, seen in Plate XIV. fig. 5, being due, for the most part at least, to the *semiannual inequality*, which is the subject of the next section (§ 5).

§ 5. *Semiannual Inequality of the Solar-diurnal Variation.*

The solar-diurnal variation exhibited in Table V. is seen by the semiannual means, April to September, and October to March, to be subject to a systematic difference in the two halves of the year, coinciding, or nearly so, with the sun's position on opposite sides of the equator. In Plate XV. fig. 1, the curve corresponding to the mean solar-diurnal variation at Toronto, from April to September, is represented by the black line, and the curve corresponding to October to March by the red line. The systematic character of this half-yearly variation is shown by the corresponding curves similarly represented by black and by red lines for Kew (fig. 2), Nertschinsk (fig. 3), Pekin (fig. 4), St. Helena (fig. 5), Cape of Good Hope (fig. 6), and Hobarton (fig. 7). The scale is the same in all the figures. It will be seen that at all these stations the curve from April to September is on the upper or East side of the October to March curve, from about midnight, or a little later, to about 9 or 10 A.M. in the northern hemisphere, and about 10 or 11 A.M. in the southern hemisphere; and on the lower or West side of the October to March curve during the remainder of the twenty-four hours. On successively considering the figures in Plates XIV. and XV., we perceive that the *annual* curves progressively lessen as the equatorial region is approached, reappearing in a reversed direction in the southern hemisphere, and gradually increasing in magnitude so as to have at Hobarton, in the middle latitudes of the southern hemisphere, nearly the same magnitude as at Toronto and Kew in the northern hemisphere; but that through all these changes both of magnitude and direction in the annual curves, the *semiannual variation* (or the difference between the two semiannual curves in each case) remains persistent throughout; the same in direction at the same hours, and the amount approximately the same in all parts of the globe. Thus in the equatorial region, where the *annual* inflection almost or entirely disappears, the semiannual portion still subsists, and presents in each of the half years, separately viewed, the phenomenon of a solar-diurnal variation. This is approximately exemplified at St. Helena, which, however, is a little on the southern side of the magnetically dividing line between the hemispheres. As the southern magnetic latitude increases, the annual solar-diurnal variation, as shown in Plate XIV., progresively increases in magnitude, but in a reversed direction from those of the analogous phenomena in the north, as has already been noticed. Thus it will be seen that the two portions, viz. the annual and the semiannual, both of which we recognize to be due to the sun's action, inasmuch as they follow the order of the solar hours, evince apparently a dissimilarity in the mode of operation of the producing cause; in the one class of effects, viz. in the annual, the north end of the magnet is deflected in opposite directions in the two hemispheres, the deflection disappearing altogether at the magnetic equator; whilst in the other class, viz. the semiannual difference, no such inversion takes place, and the deflections are approximately the same in amount and direction in the equatorial as in all other parts of the terrestrial surface*.

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* The general custom of speaking always of the north end of the Declination magnet is here followed: if this were modified as suggested in page 286, the reasoning upon the characteristic distinction between the two phenomena would, it is obvious, remain essentially the same.

§ 6. *Lunar-diurnal Variation.*

Table VI. contains the lunar-diurnal variation on the mean of each year, from 1858 to 1862 inclusive, and a general average taken for the five years.

TABLE VI.—Lunar-diurnal Variation in Seconds of Arc.

Lunar Hours.	Years ending December 31,					Means.	Lunar Hours.
	1858.	1859.	1860.	1861.	1862.		
0	W. 6″·0	E. 0″·6	W. 12″·6	W. 5″·4	W. 7″·8	W. 6″·2	0
1	W. 14·4	W. 7·2	W. 12·6	W. 6·0	W. 6·6	W. 9·6	1
2	W. 10·8	W. 9·6	W. 5·4	W. 7·2	W. 9·0	W. 8·4	2
3	W. 7·8	W. 4·2	W. 3·0	E. 2·4	E. 2·4	W. 2·0	3
4	W. 3 0	W. 4·2	W. 2·4	E. 4·2	E. 2·4	W. 0·6	4
5	E. 5·4	W. 6·6	E. 5·4	E. 6·6	E. 9·0	E. 4·0	5
6	E. 12·0	E. 1·2	E. 3·0	E. 14·4	E. 14·4	E. 9·0	6
7	E. 9·0	E. 4·2	E. 9·6	E. 16·2	E. 17·4	E. 11·3	7
8	E. 9·6	E. 8·4	E. 7·8	E. 6·6	E. 14·4	E. 9·6	8
9	E. 7·2	E. 6·6	W. 0·9	W. 1·2	E. 12·0	E. 4·7	9
10	E. 3·0	E. 7·2	W. 1·8	W. 6·6	W. 2·4	W. 0·1	10
11	W. 3·6	W. 1·2	W. 4·2	W. 10·8	W. 7·8	W. 5·5	11
12	W. 4·8	W. 9·0	W. 18·0	W. 8·4	W. 7·8	W. 9·6	12
13	W. 3·0	W. 13·2	W. 15·0	W. 13·2	W. 12·0	W. 11·3	13
14	W. 3·0	W. 8·4	W. 9·6	W. 10·8	W. 15·6	W. 9·5	14
15	W. 7·2	W. 3·6	E. 4·2	W. 8·4	W. 12·0	W. 5·4	15
16	E. 3·0	E. 3·6	E. 7·8	W. 6·6	W. 10·8	W. 0·6	16
17	E. 7·8	E. 9·6	E. 13·8	W. 2·4	W. 4·2	E. 5·1	17
18	E. 7·8	E. 14·4	E. 17·4	E. 3·0	0·0	E. 8·5	18
19	E. 4·8	E. 18·0	E. 15·0	E. 9·0	E. 2·4	E. 9·8	19
20	E. 3·0	E. 12·6	E. 6·0	E. 10·2	E. 12·0	E. 8·8	20
21	W. 2·4	E. 18·6	E. 2·4	E. 10·8	E. 7·8	E. 7·4	21
22	W. 7·8	E. 9·6	W. 3·0	E. 7·8	E. 5·4	E. 2·4	22
23	W. 6·0	E. 5·4	W. 3·6	E. 0·6	W. 4·2	W. 1·6	23

We see in this Table a form of diurnal variation systematically and essentially different from that of the solar-diurnal variation. This characteristic form, which is shown alike by each of the three magnetic elements in all parts of the globe for which the investigation has been made, consists in a double fluctuation taking place in every twenty-four hours, with two extreme deflections in each direction,—the zero-line, or line in which the moon's action produces no deflection, being passed through four times at nearly equal intervals of six lunar hours. At Kew the extreme westerly deflections occur at 1 and 13 hours, and the extreme easterly at 7 and 19 hours. The extremes at 7 and 13 hours appear to be somewhat larger than those at 1 and 19 hours (the two greater elongations being each 11″·3, and the lesser 9″·8 and 9″·6, on the average of the five years). This difference may have an important theoretical bearing if confirmed by the results in future years, and in other parts of the globe.

In considering the lunar-diurnal variation of the three elements in different parts of the globe, the division of the lunar day into four alternate and nearly equal deflections in opposite directions appears, as already stated, to be a general feature; but the *amount* of deflection (speaking of the declination) appears to diminish as the equator is approached, reincreasing in the southern hemisphere, and attaining at Hobarton nearly the same value as at Kew. The hours of extreme deflection are not the same at all stations: the north end of the magnet has its extreme westerly deflections at Kew (in the northern hemisphere), and its extreme easterly deflections at Hobarton (in the southern hemisphere) at the same hours, and *vice versâ*; there is a similar correspondence of hours in the opposite deflections of the same end of the magnet at Pekin in the northern, and at the Cape of Good Hope in the southern hemisphere; but the hours at Kew and Hobarton are different from those at Pekin and the Cape of Good Hope: however, results at more stations must be obtained before we can draw any certain inferences as to the systematic character and theoretical bearing of such differences. There are six stations where the lunar-diurnal variation of the declination has been computed by myself, viz. Toronto, Kew, Pekin, St. Helena, the Cape of Good Hope, and Hobarton: the results at these stations are published in the second volume of the Magnetical and Meteorological Observations at St. Helena, pp. cxlvi–cxlviii.

Results of the Magnetic Observations at the Kew Observatory, from 1857 *to* 1862 *inclusive.*

No. II.

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§ 7. *Secular Change and Annual Variation of the Declination.*

It is desirable to advert briefly to the process by which these results are elaborated from the photographic records. The twenty-four equidistant hourly positions having been tabulated from the photograms, are written in monthly tables, having the days of the month arranged vertically, and the twenty-four hourly positions in each day horizontally. The hourly positions in each *vertical* line are then examined, and those in which the difference from the *normal* of the same hour equals or exceeds 0·15 in. in the photographic scale, or 3'·3 in arc, are marked as disturbed positions, and are put aside for separate consideration. This process is repeated until the *final normals* are the means of the positions in each vertical line after the omission of all those which differ from them by an amount equal to 3'·3 or upwards. A mean is then taken of the positions which remain in each *horizontal* line after the exclusion of the disturbed positions, omitting only days on which the disturbed hours equalled or exceeded six in number, or one-fourth of the whole number of the tabulated positions. The means thus obtained are considered to show the mean declination at the observatory for each day. The daily values are then collected in weekly groups, of which there are consequently fifty-two in each year, and mean *weekly* values are taken, such as are exhibited in columns 2 to 6 of Table VII. (page 292), for the five years commencing in January 1858 and ending in December 1862. The mean of the weekly values in each year corresponds to the mean declination on the 1st of July of that year; and these mean values are placed at the foot of each annual column in Table VII., whilst the means of the values in the several horizontal lines, seen in column 7, show the weekly values *in a mean or typical year*, derived from the hourly positions in the five years, and corresponding chronologically, in the case of Table VII., to the successive weeks in the year 1860. The mean declination of the whole Table, corresponding to July 1, 1860, is seen at the foot of column 7; it is 21° 39' 18''·1 W., and is based upon 260 weekly values, or upon 6240 hourly positions (diminished by the positions omitted, as above stated, on account of disturbance). The differences from this mean value seen in the several weekly means in the typical year (column 7) are ascribable (partly, of course, to casual errors, but) chiefly, as will be seen, to the effects of systematic variations. The presence of one of these, known commonly by the name of *secular change* (inasmuch as its period is of long and yet undetermined duration), is conspicuous, and its mean amount during the five years embraced by Table VII. becomes known by comparing with each other the mean declination in each successive year, placed at the foot of the respective columns. Here we find that

	′	″
From July 1858 to July 1859, the West Declination decreased	6	45·9
„ 1859 „ 1860, „ „ „	7	31·0
„ 1860 „ 1861, „ „ „	8	15·7
„ 1861 „ 1862, „ „ „	8	03·6

whence we have 7′ 39″ as the mean annual amount of decrease in the West Declination at Kew in the five years, corresponding, (as a precise deduction,) to July 1, 1860, the middle epoch of the mean or typical year.

It is obvious that if we apply a proportional part of this secular change to the several weekly values in the mean or typical year, we obtain fifty-two corrected values of the declination, each of which, if there were no other systematic variation than that of the secular change, should agree with 21° 39′ 18″·1; or should show only such small and unsystematic differences as might reasonably be ascribed to casual errors. The character of the differences actually presented sufficed to show that something more was involved, not explicable by the small variation in the rate of secular change itself which appeared to be pointed out by the Table. Small, however, as was this last-named variation, it seemed proper that it should be taken into account before we should be prepared to take a final view of the results.

It is well known that a few years ago the secular change in London was a small annual *increase* of west declination, and that from causes yet but imperfectly understood, this increase first diminished and then ceased, giving place to a change in the opposite direction, at first slow, but becoming progressively more rapid; so that at present the rate of decrease is very nearly if not quite equal to the rate of increase which existed at the time first spoken of. Thus the secular change at Kew (which we may regard as the same as at London) appears to have been somewhat less in 1858 and 1859 than in 1861 and 1862, and therefore, inferentially, less in the earlier than in the later portions of each year; so that we may possibly obtain more exact values of the corrections to be applied for secular change in the different parts of the mean or typical year by substituting for a *mean* value of $\frac{7' \, 39''}{52} = 8''\cdot83$, weekly corrections commencing with 8″·5 and progressively increasing to 9″·1. These corrections are shown in column 8, and produce the corrected values in column 9. The differences of the values in column 9, from 21° 39′ 18″·1, have been placed in column 10, to which I desire to direct attention. The mere aspect of the + and − signs in this column appears to point to a semiannual inequality coinciding very nearly with the sun's position in respect to the equator. If we arrange the differences in two categories, one including the twenty-six weeks from March 26 to September 23, and the other the twenty-six weeks from September 24 to March 25 (which is the division of weeks most nearly according with the equinoxes), the almost constant prevalence of the − sign in the first, and of the + sign in the second category, indicates with a very high degree of probability an *annual variation*, whereby the north end of the magnet points more towards the east when the sun is north, and towards the west when the sun is south of the equator; and we obtain in the first category (corresponding to the interval between March 26 and September 23) an average weekly diminution of 28″·95 of West Declination, and in the second category (corresponding to the interval between September 24 and March 25) an average weekly augmentation of 29″·90 of West Declination,—making together an annual variation amounting to 58″·85.

TABLE VII.—Weekly Means of West Declination at Kew, from January 1, 1858 to December 31, 1862.

Period.	1858. 21°+	1859. 21°+	1860. 21°+	1861. 21°+	1862. 21°+	Means. 21°+	Corrections for secular change.	Mean weekly values corrected for secular change. 21°+	Differences of the several weekly corrected values from 21° 39′ 18″·1.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Jan. 4.	56′ 32″·6	52′ 30″·6	43′ 15″·2	35′ 00″·6	28′ 47″·7	43′ 13″·3	−3′ 41″·1	39′ 32″·2	+14″·1
11.	54 49·5	52 46·5	42 26·3	34 24·9	28 50·3	42 39·5	−3 32·6	39 06·9	−11·2
18.	55 23·8	52 42·5	42 44·8	33 43·9	28 37·1	42 38·4	−3 24·1	39 14·3	− 3·8
25.	55 56·9	52 35·9	42 17·0	34 22·3	28 33·1	42 45·0	−3 15·5	39 29·5	+11·4
Feb. 1.	57 11·0	52 30·6	42 14·4	34 51·4	27 24·4	42 50·4	−3 06·9	39 43·5	+25·4
8.	57 53·3	52 39·8	41 57·2	34 48·7	27 04·6	42 52·7	−2 58·4	39 54·3	+36·2
15.	58 34·3	51 54·9	41 46·6	35 00·6	26 56·6	42 50·6	−2 49·8	40 00·8	+42·7
22.	57 00·4	51 41·7	41 22·8	35 20·5	27 15·1	42 32·1	−2 41·2	39 50·9	+32·8
March 1.	56 32·6	51 08·6	41 04·3	35 32·3	26 44·7	42 12·5	−2 32·6	39 39·9	+21·8
8.	56 23·3	51 16·6	40 49·7	35 04·6	26 50·0	42 04·8	−2 23·9	39 40·9	+22·8
15.	56 53·8	51 41·7	41 54·5	34 55·3	27 07·2	42 30·5	−2 15·3	40 15·2	+57·1
22.	56 20·7	51 16·6	40 36·5	34 27·5	26 20·9	41 48·4	−2 06·7	39 41·7	+23·6
29.	56 19·4	51 25·8	40 51·1	33 36·0	26 07·7	41 40·0	−1 58·0	39 42·0	+23·9
April 5.	56 06·2	50 38·2	41 00·3	33 08·2	24 58·9	41 10·4	−1 49·3	39 21·1	+ 3·0
12.	54 40·2	49 58·6	41 17·5	33 45·2	24 47·0	40 53·7	−1 40·6	39 13·1	− 5·0
19.	55 11·9	49 54·5	40 44·5	34 03·8	25 10·9	41 01·1	−1 33·0	39 29·1	+11 0
26.	54 17·7	50 02·5	40 28·6	33 10·9	25 12·2	40 38·4	−1 23·3	39 15·1	− 3·0
May 3.	53 52·6	48 51·2	40 02·1	32 23·3	24 12·7	39 52·4	−1 14·6	38 37·8	−40·3
10.	54 03·2	47 11·9	40 35·2	32 23·3	24 06·1	39 39·9	−1 05·8	38 34·1	−44·0
17.	53 24·8	47 09·2	40 19·3	32 07·4	24 32·5	39 30·6	−0 57·1	38 33·5	−44·6
24.	53 57·9	47 17·2	40 51·1	31 59·5	23 23·7	39 29·9	−0 48·3	38 41·6	−36·5
31.	54 11·1	46 33·5	39 54·2	32 25·9	23 05·2	39 14·0	−0 39·6	38 34·4	−43·7
June 7.	53 55·3	46 07·1	39 48·9	31 37·0	22 57·3	38 53·1	−0 30·8	38 22·3	−55·8
14.	54 03·2	45 47·3	39 58·2	30 37·4	23 01·3	38 41·5	−0 22·0	38 19·5	−58·6
21.	54 52·1	45 22·1	40 03·5	31 10·5	22 08·3	38 43·3	−0 13·2	38 30·1	−48·0
28.	53 27·5	46 01·8	40 51·1	31 15·8	21 55·1	38 42·3	−0 04·4	38 37·9	−40·2
July 5.	53 08·9	44 46·4	38 11·1	30 29·5	21 52·5	37 41·7	+0 04·4	37 46·1	−92·0
12.	53 34·1	45 06·3	39 07·9	30 46·7	21 48·5	38 04·7	+0 13·2	38 17·9	−60·2
19.	53 15·6	44 57·0	39 05·3	30 30·8	21 20·8	37 49·9	+0 22·1	38 12·0	−66·1
26.	53 12·9	44 49·1	38 01·8	30 52·0	21 45·8	37 44·3	+0 31·0	38 15·3	−62·8
August 2.	52 39·8	45 01·0	37 27·4	31 06·5	22 00·4	37 39·0	+0 39·8	38 18·8	−59·3
9.	53 23·5	45 06·3	38 23·0	31 26·4	21 51·2	38 02·1	+0 48·7	38 50·8	−27·3
16.	54 09·8	46 08·5	38 20·3	31 51·5	21 34·0	38 24·8	+0 57·6	39 22·4	+47·3
23.	53 16·9	45 34·1	38 49·4	31 14·5	21 03·5	37 59·7	+1 06·6	39 06·3	−11·8
30.	52 18·7	45 30·1	39 22·5	30 34·8	21 39·2	37 53·1	+1 15·5	39 08·6	− 9·5
Sept. 6.	52 01·5	44 21·3	40 33·9	30 28·2	21 14·1	37 43·8	+1 24·4	39 08·2	− 9·9
13.	51 39·1	45 27·4	40 33·9	30 01·7	21 35·3	37 51·5	+1 33·4	39 24·9	+ 6·8
20.	52 43·9	44 28·0	40 20·7	30 28·2	21 22·1	37 52·6	+1 42·3	39 34·9	+16·8
27.	52 10·8	44 55·7	41 13·5	29 08·9	21 41·9	37 50·2	+1 51·3	39 41·5	+23·4
Oct. 4.	52 45·2	45 10·3	40 27·3	28 25·2	22 30·8	37 51·8	+2 00·3	39 52·1	+34·0
11.	53 14·3	44 53·0	40 15·4	28 51·7	22 14·9	37 53·9	+2 09·3	40 03·2	+45·1
18.	52 51·8	44 54·4	40 22·0	29 02·3	22 30·8	37 56·3	+2 18·3	40 14·6	+56·5
25.	52 30·6	44 21·3	40 35·2	29 07·6	22 44·0	37 51·7	+2 27·3	40 19·0	+62·9
Nov. 1.	52 41·2	44 47·8	39 15·9	28 35·8	21 47·2	37 25·6	+2 36·3	40 01·9	+43·8
8.	52 08·2	44 18·7	38 19·0	28 51·7	20 59·6	36 55·4	+2 45·4	39 40·8	+22·7
15.	52 46·5	44 18·7	36 45·1	28 23·9	21 34·0	36 45·6	+2 54·4	39 40·0	+21·9
22.	52 16·1	44 45·1	36 47·8	28 43·7	21 48·5	36 52·2	+3 03·5	39 55·7	+37·6
29.	52 26·6	45 53·0	36 59·6	28 45·1	21 23·4	36 53·5	+3 12·5	40 06·0	+47·9
Dec. 6.	52 57·1	45 03·7	36 26·6	28 56·9	20 39·8	36 48·8	+3 21·6	40 10·4	+52·3
13.	53 22·2	44 54·4	36 13·4	28 31·9	20 04·0	36 37·2	+3 30·7	40 07·9	+49·8
20.	52 49·1	44 06·8	35 50·9	28 37·1	19 37·6	36 12·3	+3 39·8	39 52·1	+34·0
27.	52 46·5	43 21·8	35 12·5	28 25·2	19 31·0	35 51·4	+3 49·0	39 40·4	+22·3
Annual Means	54 08·0	47 22·1	39 51·1	31 36·4	23 32·8	39 18·1			

We may compare with Table VII., and the conclusions derived from it, a corresponding Table (VIII. page 294) of the weekly means of the hourly observations of the Declination at the Hobarton Observatory, between October 1843 and September 1848, made by Captain Kaye, R.N., and his assistants in that establishment. The observations themselves are published in the second and third volumes of the Hobarton Observations, and have been treated for the present purpose precisely in the same way as those of the Kew Observatory, 2′·13 having been taken as the standard of a disturbance, instead of 3′·3 as at Kew, a somewhat lower standard being required at Hobarton to separate the same proportion of disturbed observations for the investigation of their laws, and being otherwise unobjectionable. The mean declination in the successive years is placed at the foot of columns 2, 3, 4, 5, and 6 of Table VIII., and from these we obtain the secular change in those years as follows:—

								′	″
From April	1844	to March	1845,	the East	Declination	increased		1	27·6
„	1845	„	1846,	„	„	„		2	02·2
„	1846	„	1847,	„	„	„		1	07·3
„	1847	„	1848,	„	„	„		0	55·8

Whence we have 1′ 23″·2 as the mean annual increase of East Declination at Hobarton in the five years, corresponding precisely to the middle epoch of the mean or typical year, *i. e.* the beginning of April 1846, and which has for its mean declination 9° 56′ 13″·9 E. Column 7 of Table VIII. contains the weekly means in the typical year, each on the average of the five years. Column 8 shows the corrections for secular change, being proportional parts of an annual change of 1′ 23″·2. Column 9 contains the weekly means in column 7 corrected for secular change to the beginning of April 1846; and column 10 the differences in the values in column 9 from the mean declination 9° 56′ 13″·9, derived directly from all the weekly means in the five years.

The aspect of the + and − signs in column 10 appears conclusive in respect to the existence at Hobarton of a semiannual inequality analogous to that which has been shown to exist at Kew. The direction of the inequality in the two semiannual periods is also the same in the two hemispheres, the north end of the magnet pointing more towards the east both at Kew and at Hobarton when the sun is north of the equator and to the west when the sun is south of the equator. If we regard the equinoxes as the approximate epochs of the semiannual change, we find in the weeks from April to September an average increase of east declination of 19″·1, and in the weeks from October to March an average decrease of east declination of 19″·0, making together an annual variation of 38″·1.

TABLE VIII.—Weekly Means of East Declination at Hobarton, from October 1, 1843 to September 30, 1848.

Period.	Years ending September 30,					Means.	Corrections for secular change.	Mean weekly values corrected for secular change.	Difference of the several weekly corrected values from 9° 56′ 13″·9.
	1844. 9°+	1845. 9°+	1846. 9°+	1847. 9°+	1848. 9°+	9°+		9°+	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Oct. 4.	51′ 57″·0	54′ 39″·6	55′ 25″·8	57′ 06″·6	58′ 09″·6	55′ 27″·7	+40″·8	56′ 08″·5	−05″·4
11.	51 57·6	53 56·4	55 48·0	57 16·8	58 09·0	55 25·6	+39·2	56 04·8	−09·1
18.	51 45·6	53 18·4	55 39·6	57 31·4	58 40·8	55 23·2	+37·6	56 00·8	−13·1
25.	51 46·2	53 37·6	55 42·0	57 15·2	57 34·2	55 11·0	+36·0	55 47·0	−26·9
Nov. 1.	52 05·4	53 23·4	55 39·0	57 22·8	58 22·2	55 22·6	+34·4	55 57·0	−16·9
8.	51 51·6	53 34·8	55 57·6	57 26·4	57 40·8	55 18·2	+32·8	55 51·0	−22·9
15.	51 46·8	53 00·0	55 51·6	57 17·4	57 40·2	55 07·2	+31·2	55 38·4	−35·5
22.	51 40·8	53 22·2	55 34·2	57 29·4	57 16·8	55 04·7	+29·6	55 33·8	−40·1
29.	51 52·2	53 08·4	55 01·2	57 06·3	58 24·0	55 06·4	+28·0	55 34·4	−39·5
Dec. 6.	52 06·8	53 11·4	56 02·4	57 01·2	57 46·8	55 13·7	+26·4	55 40·1	−33·8
13.	52 02·6	53 18·6	55 27·9	56 55·2	57 57·6	55 08·4	+24·8	55 33·2	−40·7
20.	52 04·8	53 46·8	55 22·2	56 52·8	57 35·7	55 08·5	+23·2	55 31·7	−42·2
27.	52 05·6	53 49·2	55 30·9	57 10·8	57 39·0	55 15·1	+21·6	55 36·7	−37·2
	1844.	1845.	1846.	1847.	1848.				
Jan. 3.	52 37·8	53 22·8	55 27·6	57 06·3	57 40·2	55 14·9	+20·0	55 34·9	−39·0
10.	52 55·2	54 01·2	55 32·4	56 48·0	57 48·0	55 25 0	+18 4	55 43·4	−30·5
17.	52 27·0	53 55·8	55 40·2	56 57·6	57 47·4	55 21·6	+16·8	55 38·4	−35·5
24.	52 34·2	53 42·6	55 45·6	57 25·5	57 51·6	55 27·9	+15·2	55 43·1	−30·8
31.	52 58·8	53 58·2	55 25·8	57 03·6	58 10·8	55 31·4	+13·6	55 45 0	−28·9
Feb. 7.	52 59·4	54 27·0	55 58·8	57 26·4	58 40·2	55 54·4	+12·0	56 06·4	−07·5
14.	52 55·8	54 19·8	56 12·0	57 13·2	58 05·4	55 45·2	+10·4	56 55·6	−18·3
21.	53 08·4	54 43·8	56 26·4	57 28·4	58 24·3	56 02·3	+ 8·8	56 11·1	−02·8
28.	52 59·4	54 39·6	56 25·2	57 45·6	58 27·6	56 03·5	+ 7·2	56 10·7	−03·2
Mar. 7.	53 55·8	54 31·8	56 21·0	58 03·0	59 03·0	56 22·9	+ 5·6	56 28·5	+14·6
14.	53 42·0	54 43·8	56 53·4	58 07·2	58 47·7	56 26·8	+ 4·0	56 30·8	+16·9
21.	53 42·0	55 07·2	56 42·0	57 57·8	59 30·3	56 35·9	+ 2·4	56 38·3	+24·4
28.	53 31·2	54 46·8	56 58·2	57 52·2	58 52·2	56 24·1	+ 0·8	56 24·9	+11·0
April 4.	53 57·6	55 07·8	57 15·6	58 16·8	58 33·0	56 38·2	− 0·8	56 37·4	+23·5
11.	53 49·8	55 10·2	57 12·6	58 18·3	59 06·9	56 43·6	− 2·4	56 41·2	+27·3
18.	53 40·8	55 07·2	57 20·4	58 06·6	59 13·2	56 41·6	− 4 0	56 37·6	+23·7
25.	53 39·6	55 08·4	57 09·0	58 08·1	58 44·7	56 34·0	− 5·6	56 28·4	+14·5
May 2.	54 03·6	55 06·6	57 23·4	58 33·6	58 58·2	56 49·1	− 7·2	56 41·9	+28·0
9.	53 50·4	55 13·2	57 24·6	58 09·6	59 27·0	56 49·0	− 8·8	56 40·2	+26·3
16.	54 00·0	55 28·8	57 12·3	58 27·6	59 15·0	56 52·7	−10·4	56 42·3	+28·4
23.	54 01·2	55 24·6	57 17·4	58 31·2	59 15·6	56 54·0	−12·0	56 42·0	+28·1
30.	54 10·2	55 35·4	57 30·6	58 27·0	59 18·6	57 00·4	−13·6	56 46·8	+32·9
June 6.	53 57·0	55 01·2	57 24·0	58 27·6	59 17·4	56 49·4	−15·2	56 34·2	+20·3
13.	54 02·4	55 30·0	57 28·2	58 39·6	59 31·8	57 02·4	−16·8	56 45·6	+31·7
20.	53 39·0	55 39·0	57 30·6	58 24·0	59 30·6	56 56·6	−18·4	56 38·2	+24·3
27.	53 45·0	55 33·6	57 31·8	58 22·8	59 21·0	56 54·8	−20·0	56 34·0	+20·1
July 4.	53 37·2	55 24·0	57 59·4	58 16·8	59 27·6	56 57·0	−21·6	56 35·4	+21·5
11.	53 47·4	55 26·4	57 53·4	58 36·6	59 31·8	57 03·1	−23·2	56 39·9	+26·0
18.	53 51·6	55 24·6	57 36·6	58 28·2	59 30·3	56 58·3	−24·8	56 33·5	+19·6
25.	53 52·2	55 24·6	57 40·8	58 25·8	59 14·4	56 55·6	−26·4	56 29·2	+15·3
Aug. 1.	53 59·4	55 26·4	57 49·2	58 10·8	59 36·0	57 00·4	−28·0	56 32·0	+18·1
8.	53 55·4	55 25·2	57 44·4	58 24·6	59 24·0	56 58·7	−29·6	56 29·1	+15·2
15.	53 54·2	55 29·4	57 27·6	58 07·2	59 20·1	56 51·7	−31·2	56 20·5	+06·6
22.	54 06·1	55 30·0	57 40·2	57 54·9	59 40 8	56 58·4	−32·8	56 25·6	+11·7
29.	54 01·9	55 35·4	57 29·4	58 11·4	59 31·2	56 57·9	−34·4	56 23·5	+09·6
Sept. 5.	54 04·2	55 30·6	57 40·2	58 12·6	59 14·4	56 56·4	−36·0	56 20·4	+06·5
12.	54 07·2	55 29·4	57 27·0	58 07·2	59 24·0	56 55·0	−37·6	56 17·4	+03·5
19.	53 40·8	55 33·6	58 00·6	57 42·0	59 39·0	56 55·2	−39·2	56 16·0	+02·1
26.	53 41·4	55 27·6	57 34·2	58 18·3	60 02·4	57 00·8	−40·8	56 20·0	+06·1
Annual means	9 53 12·4	9 54 40·0	9 56 42·2	9 57 49·5	9 58 45·3	9 56 13·9			
Secular change	1′ 27″·6	2′ 02″·2	1′ 07″·2	0′ 55″·8					

In volume II. of the Magnetic Observations at St. Helena, p. v, an examination is made of the monthly values of the declination obtained from eight years of observation, corrected for secular change, and collected in a Table. These also indicate the existence of a semiannual inequality having epochs coincident, or nearly so, with the equinoxes—the north end of the magnet pointing, as at Kew and Hobarton, more to the east in the months from April to September, and to the west from October to March. The amount of the inequality is less than at Kew or Hobarton, "the semiannual difference being about 14 seconds of arc."

The first volume of the Magnetical Observations at the Cape of Good Hope, published in 1851, contains the fortnightly means of the hourly observations of the declination from July 1842 to July 1846; these are corrected for secular change in Table III. of that volume, and the differences of the declination in each fortnight (so corrected) from the mean declination of the whole period, are shown in the final column. The mean of the thirteen fortnights (in the four years) between March 26 and September 23 is $0'\cdot40$ more *easterly*, and of the thirteen fortnights between September 24 and March 25 $0'\cdot40$ more *westerly* than the mean of the year,—thus showing an annual variation of $0'\cdot80$ or $(48''\cdot0)$, or a semiannual inequality averaging $24''$ to the East in the thirteen fortnights from March 26 to September 23, and $24''$ to the West in the thirteen fortnights from September 24 to March 25. This is in accordance with the other stations previously discussed.

The fact of the existence of an annual variation with analogous phenomena at the four widely separated stations of Hobarton, St. Helena, the Cape of Good Hope, and Kew appears to be thus substantiated; its amount is least at St. Helena, intermediate at the Cape and Hobarton, and greatest at Kew; the difference in amount is doubtless to be ascribed, in part at least, to the difference in the amount of the antagonistic force of the earth's magnetism, tending to retain the magnet in its mean place in opposition to all disturbing causes. This force (the horizontal component of the earth's magnetic force) is, in British units, approximately 5·6 at St. Helena, 4·5 at the Cape and Hobarton, and 3·8 at Kew.

§ 8. *Annual Variation, or semiannual inequality, of the Dip, and of the Horizontal and Total Force.*

In the year 1850 I communicated to the Royal Society a paper entitled "On the means adopted in the British Colonial Magnetic Observatories for determining the absolute values, secular changes, and annual variations of the Magnetic Force." This paper is published in the Philosophical Transactions for the same year, No. IX.

In this communication I endeavoured to show the importance of introducing into such determinations greater accuracy than had previously been customary; and by making known the success which had attended the improvements adopted in the instruments and methods employed in the Colonial Magnetic Observatories, I hoped to be the means of promoting the adoption of similar instruments and processes (or the devisal and employment of others which might serve the purpose as well, or still more effectually) in other observatories which had been instituted for the purpose of cooperating with or aiding in the plan of magnetic research proposed by the Royal Society.

Amongst the results referred to in that paper, obtained by means of the instruments and processes therein described, there was one which appeared to myself to be highly deserving the confirmation (or otherwise) which it might receive from similar researches. By a comparison of the monthly determinations of the Dip and of the Horizontal Force at Toronto and Hobarton, between the years 1843 and 1848, there was shown a high probability of the existence of an "annual variation" in the direction and intensity of the magnetic force, common to both hemispheres, the mean values being passed through about the equinoxes, and the intensity of the force being greater, and the inclination more nearly vertical, in the months when the sun is south of the equator than in the months in which the sun is north of the equator. The facts thus made known appeared to indicate the existence of a general affection of the globe having an annual period, and conducting us naturally to the position of the earth in its orbit as the first consideration towards an explanation of the periodic change. The importance of following up without delay, and in the most effective manner, a branch of research which gave so fair a promise of establishing a conclusion of so much theoretical moment upon the basis of competent experiment was earnestly pointed out, and specially so with reference to those national observatories in which magnetical researches were professed objects, and from which exact determinations might most reasonably be expected.

In 1856 the Committee of the Kew Observatory, impressed with the importance of prosecuting an investigation which appeared to lead to the establishment of a previously unsuspected cosmical relation in the minor variations of terrestrial magnetism, and perceiving that no adequate provision had been made for this purpose in any establishment in the British Islands, took the matter in hand, and having obtained permission from the tenant under the Crown, caused a suitable wooden building, copper fastened, to be erected in Richmond Old Deer Park, at a distance of 300 feet from the observatory itself, and having no other buildings in its vicinity. A series of monthly determinations of the dip and of the horizontal force was commenced in this building in April 1857, with inclinometers made by Mr. HENRY BARROW, and with a unifilar magnetometer made by the late Mr. WILLIAM JONES. These instruments were the property of Her Majesty's Government, having been originally made (under my own direction) for the Arctic Expedition under Sir JAMES CLARK ROSS in 1846–1847, and replaced in my charge, on the return of the expedition, for repair and subsequent use. Several minor modifications, which experience had suggested since the publication of the memoir in 1850 already adverted to, were introduced in the instruments previous to April 1857, and in this improved state they have been described and practical directions given for their use in the "Instructions for Magnetic Surveys by Land and Sea," published in 1859 in the third edition of the Admiralty Manual of Scientific Inquiry. The series of determinations with these instruments has been steadily maintained from April 1857 to the present time, and still continues. The unifilar magnetometer employed has been the same throughout, no change whatsoever having been made either in the instrument itself, or in its collimator magnet. In respect to the dip observations, from April 1857 to September 1860 inclusive, twelve dip circles and twenty-four needles, all by BARROW and all of the same size and pattern, were employed, the mean of *all* the observations made in a month with *any* of Barrow's 6-inch circles furnished with microscopes and verniers having been taken as the mean dip of that month. A detailed statement of the results of these observations, specifying in each case the name of the observer and the distinguishing marks of the circle and needle, has been published in the 'Proceedings of the Royal Society,' vol. xi. p. 144–162. In the discussion accompanying that communication it was shown that the probable error of a single determination of the dip with instruments of this pattern does not exceed $\pm 1'\cdot 5$, this being the conclusion derived from 282 determinations on 121 different days, chiefly by four observers, employing twelve different circles and twenty-four needles all of the same size and pattern. Between October 1860 and March 1863, the mean monthly dip has been obtained with *one* of the twelve circles alone, viz. Barrow's circle No. 33 (one of the twelve previously adverted to), and was generally the mean of a single determination in each month with each of the two needles of that circle. This department of the Kew observations has been placed by the Director, Mr. STEWART, in the charge of Mr. CHARLES CHAMBERS, one of the assistants in the establishment, and to that gentleman I am indebted for the results which are embodied in Tables IX. and XI., and which afford most satisfactory evidence of Mr. CHAMBERS's skill and devotion to the duties with which he is charged.

With reference to the values of the Horizontal Force in Table IX. Mr. CHAMBERS remarks, "The constants for the reduction of observations with collimator magnet 'K C 1' are as follows:—

"K the moment of inertia, being the mean of independent determinations with six different inertia-cylinders by the late Mr. WELSH, F.R.S., $=4\cdot4696$ ($\log K=0\cdot65027$ at 60° Fahr.).

"Hence

$$\begin{array}{ll}\log \pi^2 K \text{ at } 30^\circ = 1\cdot64439, & \text{at } 70^\circ = 1\cdot64463 \\ \quad\;\; ,, \qquad 40^\circ = 1\cdot64445, & \text{at } 80^\circ = 1\cdot64469 \\ \quad\;\; ,, \qquad 50^\circ = 1\cdot64451, & \text{at } 90^\circ = 1\cdot64475 \\ \quad\;\; ,, \qquad 60^\circ = 1\cdot64457. & \end{array}$$

"The correction for the decrease of the magnetic moment of the collimator magnet produced by an increase of 1° Fahr. $=(q)=0\cdot000119(t_0-t)+\cdot000000213(t_0-t)^2$, t_0 being the observed temperature, and $t=35^\circ$. The induction coefficient $(\mu)=\cdot000194$. These were both determined by Mr. WELSH. The angular value of one division of the collimator scale$=2'\cdot50$. Comparisons of the deflection-bar with the verified standard measure of the Kew Observatory gave the errors of graduation as follows:—

At 1·0 foot distance $=-\cdot000075$ of a foot at 62° Fahr.
At 1·3 foot distance $=-\cdot000097$.

"The arc of vibration was always too small to require any correction; and none has been applied on account of the rate of the chronometer when the rate was less than five seconds, as was generally the case. The constant P was determined from twenty-four repetitions of experiments of deflection made nearly simultaneously at each of the two distances 1·0 and 1·3 feet, giving P = −·00192.

"Generally there have been three or four observations of deflection and two of vibration made in each month."

Table IX.—Monthly Values of the Horizontal Component of the Magnetic Force at Kew, computed from the Experiments of Deflection and Vibration with the Collimator Magnet "K C 1".

April to September.	1857.	1858.	1859.	1860.	1861.	1862.	Means of the six years.
April	3·7887	3·7932	3·7897	3·8038	3·8078	3·8162	3·7999
May	3·7920	3·7984	3·8008	3·8022	3·8157	3·8209	3·8050
June	3·7901	3·7889	3·8053	3·8142	3·8189	3·8150	3 8054
July	3·7950	3·7980	[3·8052]	3·8065	3·8115	3·8179	3·8057
August	3·7871	3·7942	3·8052	3·7979	3·8113	3·8162	3·8020
September	3·7883	3·7920	3·7995	3·8056	3·8115	3·8158	3·8021
Means, April to September ...	3·7902	3·7941	3·8010	3·8050	3·8128	3·8170	3·8033
October to March.	**1857 and 1858.**	**1858 and 1859.**	**1859 and 1860.**	**1860 and 1861.**	**1861 and 1862.**	**1862 and 1863.**	**Means of the six years.**
October	3·7925	3·7962	3·7914	3·8066	3·8081	3·8144	3·8015
November	[3·7906]	3·7964	3·7963	3·8074	3·8085	3·8161	3·8025
December	[3·7887]	3·7919	3·8056	3·8075	3·8113	3·8124	3·8029
January	3·7868	3·7951	3·8038	3·8101	3·8144	3·8127	3·8038
February	3·7917	[3 7967]	3·8016	3·8071	3·8136	3·8188	3·8052
March	3·7873	3·7983	3·8036	3·8075	3·8125	3·8212	3·8051
Means, October to March ...	3·7896	3·7958	3·8004	3·8077	3·8114	3·8159	3·8035
Yearly means	3·7899	3·7950	3·8007	3·8063	3·8121	3·8165	3·8034

The values within brackets [ ] are interpolated.

The absolute values of the horizontal force, corresponding to the beginning of October in each of the years comprehended in Table IX., and the secular change in each year, were therefore as follows:—

From April 1857 to March 1858 . . . 3·7899
sec. ch. +·0051.
From April 1858 to March 1859 . . . 3·7950
sec. ch. +·0057.
From April 1859 to March 1860 . . . 3·8007
sec. ch. +·0056.
From April 1860 to March 1861 . . . 3·8063
sec. ch. +·0058.
From April 1861 to March 1862 . . . 3·8121
sec. ch. +·0044.
From April 1862 to March 1863 . . . 3·8165

Mean of the six years, corresponding to the middle epoch, April 1860 . . . 3·8034 with a mean annual secular increase of ·0053.

The "Annual Variation" or "Semiannual Inequality" (April to September, and October to March) may be shown from the monthly values in Table IX. to have been as follows:—

TABLE X.

Date.	Corrections for Secular Change.	3·8034 ±Secular Change.	Observed Values.	Observed—Calculated.	
				April to September.	October to March.
July 1, 1857	—·0146	3·7888	3·7902	+·0014	....
Jan. 1, 1858	—·0119	3·7915	3·7896	....	—·0019
July 1, 1858	—·0093	3·7941	3·7941	·0000	....
Jan. 1, 1859	—·0066	3·7968	3·7958	....	—·0010
July 1, 1859	—·0040	3·7994	3·8010	+·0016	....
Jan. 1, 1860	—·0013	3·8021	3·8004	....	—·0017
July 1, 1860	+·0013	3·8047	3·8050	+·0003	....
Jan. 1, 1861	+·0040	3·8074	3·8077	....	+·0003
July 1, 1861	+·0066	3·8100	3·8128	+·0028	....
Jan. 1, 1862	+·0093	3·8127	3·8114	....	—·0013
July 1, 1862	+·0119	3·8153	3·8170	+·0017	....
Jan. 1, 1863	+·0146	3·8180	3·8159	....	—·0021
Mean differences between the observed and calculated values in the respective semiannual periods				+·0013	—·0013

It is seen then by Table X. that there exists a variation in the amount of the horizontal force having an annual period; that the value of this variation is on the average of the six years approximately ·0026; and that it consists of a semiannual inequality, the horizontal force being on the average ·0013 higher in the six months from April to September, and ·0013 lower in the six months from October to March than would be due to its mean value.

I pass to the contemporaneous determinations of the Dip.

TABLE XI.—Monthly Values of the Magnetic Dip at Kew.

April to September.	1857.	1858.	1859.	1860.	1861.	1862.	Means of the six years.
	68°+	68°+	68°+	68°+	68°+	68°+	68°+
April	27′·2	22′·5	21′·1	20′·5	17′·6	18′·1	21′·17
May	24·9	23·0	19·4	19·3	15·7	14·1	19·40
June	24·0	22·7	[20·5]	19·1	17·7	14·0	19·67
July	26·1	23·7	21·6	18·4	16·8	14·0	20·10
August	24·1	21·5	20·6	16·6	18·7	15·1	19·43
September	24·9	21·4	22·0	19·4	17·1	13·8	19·77
Means, April to September ...	25·20	22·47	20·87	18·88	17·27	14·85	19·92

October to March.	1857 and 1858.	1858 and 1859.	1859 and 1860.	1860 and 1861.	1861 and 1862.	1862 and 1863.	Means of the six years.
	68°+	68°+	68°+	68°+	68°+	68°+	68°+
October	24′·3	23′·8	24′·0	19′·6	18′·4	16′·0	21′·02
November	25·6	23·7	22·4	20·8	17·9	15·8	21·03
December	[24·8]	21·2	20·8	18·5	17·9	15·6	19·80
January	24·0	22 3	22 4	19·5	19·0	14·5	20·28
February	24·0	[22·4]	21·1	19·4	15·1	14 2	19·37
March	24·6	22·5	21·0	20·4	17 1	13·5	19·85
Means, October to March ...	24·55	22·65	21·95	19·70	17·57	14·93	20·22
Yearly means...	24·87	22·56	21·41	19·29	17·42	14 89	20·07

The values within brackets [ ] are interpolated.

The absolute values of the dip corresponding to the beginning of October in each of the years comprehended in Table XI., and the secular change in each year, are as follows:—

From April 1857 to March 1858 . . 68° 24′·87
} sec. ch. −2′·31
From April 1858 to March 1859 . . 68 22·56
} sec. ch. −1·15
From April 1859 to March 1860 . . 68 21·41
} sec. ch. −2·12
From April 1860 to March 1861 . . 68 19·29
} sec. ch. −1·87
From April 1861 to March 1862 . . 68 17·42
} sec. ch. −2·53
From April 1862 to March 1863 . . 68 14·89

Mean of the six years, corresponding to middle epoch, April 1, 1860 . } 68 20·07 { with a mean annual secular decrease of 2′·00.

The "Annual Variation" or "Semiannual Inequality" (April to September, and October to March) may be shown from the monthly values in Table XI. to have been as follows:—

TABLE XII.

Date.	Corrections for Secular Change.	68° 20′·07 ±Secular Change.	Observed Values.	Observed—Calculated.	
				April to September.	October to March.
	′	° ′	° ′	′	′
July 1, 1857	+5·50	68 25·57	68 25·20	−0·37	......
Jan. 1, 1858	+4·50	68 24·57	68 24·55	......	−0·02
July 1, 1858	+3·50	68 23·57	68 22·47	−1·10	......
Jan. 1, 1859	+2·50	68 22·57	68 22·65	......	+0·08
July 1, 1859	+1·50	68 21·57	68 20·87	−0·70	......
Jan. 1, 1860	+0·50	68 20·57	68 21·95	......	+1·38
July 1, 1860	−0·50	68 19·57	68 18·88	−0·69	......
Jan. 1, 1861	−1·50	68 18·57	68 19·70	......	+1·13
July 1, 1861	−2·50	68 17·57	68 17·27	−0·30	......
Jan. 1, 1862	−3·50	68 16·57	68 17·57	......	+1·00
July 1, 1862	−4·50	68 15·57	68 14·85	−0·72	......
Jan. 1, 1863	−5·50	68 14·57	68 14·93	......	+0·36
Mean differences between the observed and calculated values in the respective semiannual periods...				−0·65	+0·66

It is seen therefore by Table XII. that there exists a variation in the amount of the Dip having an annual period; that the value of this variation is on the average of the six years approximately 1′·31; and that it consists of a semiannual inequality, the dip being on the average 0′·65 lower in the six months from April to September, and 0′·66 higher in the six months from October to March than would be due to its mean value.

Total Force.—We find in Table IX. that the mean of the April to September values of the *horizontal* component of the force in the six years is 3·8033, corresponding in epoch to January 1, 1860; and in Table XI. that the mean of the April to September values of the dip in the same six years is 68° 19′·92, corresponding to the same epoch.

We find also in Table IX. that the mean in the six years of all the October to March values of the horizontal component is 3·8035, and of the dip (Table XI.) 68° 20′·22, corresponding to the epoch (six months later) of July 1, 1860.

We may reduce these values to a common epoch by applying to either (with the proper signs) a proportional part of the mean secular change derived from the observations of the six years. The mean secular change of the horizontal force is an annual increase of ·0053 (page 298), and of the dip an annual decrease of 2′·00 (page 300). Hence we have the corrections for the secular change (in six months), of the horizontal force =+·00265, and of the dip =−1′·00, to be applied to the mean values of April to September (corresponding in epoch to January 1, 1860) in order to bring them into strict comparison with the mean values, October to March, corresponding to the later epoch of July 1, 1860. The values then become as follows:—

From the April to September observations,		
Values of the Horizontal Force, January 1, 1860	3·8033; and of the Dip	68° 19′·92
Corrections for Secular Change	+0·00265;	— 01·00
Corresponding values July 1, 1860	3·80595	68 18·92

And from the October to March observations (Table IX.), also corresponding to July 1, 1860 } 3·80350 and from Table X. 68 20·22

whence 3·80595 sec. 68° 18′·92=10·30032 from the April to September observations, and 3·80350 sec. 68° 20′·22=10·30349 from the October to March observations, are the values of the total force derived respectively for the same epoch (July 1, 1860) from the determinations of the dip and horizontal force in the two semiannual periods; these show a difference of 0·00317 in British units, as the measure of the greater intensity of the terrestrial magnetic force in the October to March period, than in the April to September period.

For the satisfaction of those who are accustomed to be guided by the theory of probabilities in their estimate of the dependence to be placed on the results of physical investigations, it may be desirable to state the "probable errors" of the mean results of the seventy-two monthly determinations of the Horizontal Force and of the Dip in Tables IX. and XI., as well as the probable error of a single monthly determination of each of these values.

The mean result of the seventy-two monthly determinations of the Horizontal Force, shown in Table IX., is 3·8034 in British units: this has a "probable error" of ±·00027. The mean result of the seventy-two monthly determinations of the Dip (Table XI.) is 68° 20′·07: this has a probable error of ±0′·083.

The probable error of a *single* monthly determination of the Horizontal Force, derived from the seventy-two monthly determinations, and after the application of the corrections for secular change and annual variation have been made, is ±·00233; and of a single monthly determination of the Dip, after the application of the corrections for secular change and annual variation have been made, is ±0′·71.

It has been already stated that for rather more than half the whole period, viz. from April 1857 to September 1860 inclusive, twelve dip circles and twenty-four needles were employed in the monthly determinations of the Dip, the circles and needles being all made by the same artist (Mr. HENRY BARROW), and of the same size and pattern; there were also several observers, but chiefly four, viz. the late Mr. JOHN WELSH, Mr. STEWART, Dr. BERGSMA, Director of the Netherlands Magnetic Observatory at Batavia, and Mr. CHAMBERS. The means of all the observations thus made at the Kew Observatory in the same month, and recorded in the books of the Kew Observatory, have been taken as the mean Dip in that month. From October 1860 to April 1863 there has been only a single observer, Mr. CHAMBERS, with one circle, viz. No. 33, one of the twelve in previous use, with its two needles. Some relative advantages or disadvantages may be supposed to attend observations made by one or by more observers, and with one or with several instruments; and it may therefore be useful to see how far these circumstances have modified the probable error in the two periods. The forty-two monthly determinations from April 1857 to September 1860, give a probable error of ±0′·70 for a single determination; and the thirty from October 1860 to March 1863, give a probable error of ±0′·73; whence we may infer that the greater number of partial results which contributed to produce the monthly mean in the earlier period rather more than counterbalanced the diversities which may be supposed to have been occasioned by the peculiarities of the different observers, and of the different instruments employed. But the small amount of probable error in either case is well worthy of the notice of those who have been engaged, or who are likely to be engaged, in similar investigations.

In Tables XIII. and XIV. are placed the residual errors of the observed monthly determinations of the Horizontal Force and of the Dip, after the application of the corrections for secular change and annual variation.

TABLE XIII.—Residual Errors in the Monthly Determinations of the Horizontal Force.

	1857.	1858.	1859.	1860.	1861.	1862.	1863.	Means.	
April	−·0003	−·0011	−·0099	−·0011	−·0024	+·0007	......	−·0023	·0000 Sun north of the equator.
May	+·0026	+·0037	+·0007	−·0031	+·0051	+·0049	......	+·0024	
June	+·0003	−·0062	+·0048	+·0085	+·0079	−·0013	......	+·0023	
July	+·0047	+·0024	+·0043	+·0004	0000	+·0011	......	+·0021	
August	−·0036	−·0018	+·0038	−·0088	−·0006	−·0010	......	−·0020	
September ...	−·0028	−·0044	−·0023	−·0015	−·0008	−·0018	......	−·0023	
October	+·0035	+·0020	−·0082	+·0017	−·0021	−·0011	......	−·0007	+·0001 Sun south of the equator.
November ...	+·0012	+·0017	−·0038	+·0021	−·0021	+·0001	......	−·0001	
December ...	−·0011	−·0032	+·0051	+·0018	+·0002	−·0040	......	−·0002	
January	......	−·0034	−·0004	+·0031	+·0039	+·0029	−·0041	+·0003	
February ...	......	+·0010	+·0006	+·0005	+·0004	+·0017	+·0015	+·0009	
March	......	−·0039	+·0019	+·0021	+·0004	+·0001	+·0035	+·0006	

TABLE XIV.—Residual Errors in the Monthly Determinations of the Dip.

	1857.	1858.	1859.	1860.	1861.	1862.	1863.	Means.	
April	+1′·8	−0′·8	−0′·2	+1′·2	+0′·3	+2′·8	′	+0·85	+0′·01 Sun north of the equator.
May	−0·3	−0·2	−1·8	+0·1	−1·5	−1·1	......	−0·80	
June	−1·0	−0·3	−0·5	+0·1	+0·7	−1·0	......	−0·33	
July	+1·3	+0·9	+0·8	−0·4	0·0	−0·8	......	+0·30	
August	−0·6	−1·2	−0·1	−2·1	+2·0	+0·4	......	−0·25	
September ...	+0·4	−1·1	+1·5	+0·9	+0·6	−0·7	......	+0·27	
October	−1·4	+0·1	+2·3	−0·1	+0·7	+0·3	......	+0·32	−0′·02 Sun south of the equator.
November ...	+0·1	+0·2	+0·9	+1·3	+0·4	+0·3	......	+0·53	
December ...	−0·6	−2·2	−0·5	−1·1	+0·6	+0·2	......	−0·47	
January	......	−1·2	−0·9	+1·2	+0·3	+1·8	−0·7	+0·09	
February ...	......	−1·0	−0·6	+0·1	+0·4	−1·9	−0·8	−0·63	
March	......	−0·2	−0·3	+0·2	+1·5	+0·2	−1·3	+0·03	

The errors have no systematic appearance; and thus the Tables are thoroughly confirmatory of a semiannual inequality having its epochs coincident, or nearly so, with the sun's passage of the equator.

The second volume of the Hobarton Magnetic Observations, published in 1852, contains the particulars of the monthly determinations of the absolute values of the horizontal force from January 1846 to December 1850 inclusive, all made with the same unifilar magnetometer, and preserving throughout the same experimental process. The mean value, corresponding to July 1, 1848, is 4·50427. The secular change obtained by least squares from the sixty equations of condition is correctly stated in the publication referred to, as an annual diminution of 0·0006. Treating these results in the same manner that the Kew results have been treated in this paper, we obtain 4·5036 in the months from April to September, and 4·5048 in the months from October to March; or a diminution in the horizontal component of the force of 0·0007 in the months when the sun is north of the equator, and an increase of 0·0005 in the months when the sun is south of the equator; constituting a semiannual inequality of 0·0012. When the corrections for secular change and annual variation are applied, the probable error of a single monthly determination is found to be ±0·00125; and the probable error of the mean result of the sixty months is less than 0·0002.

The first volume of the Hobarton Observations, published in 1850, contained the details of a series of monthly determinations of the Inclination, commencing in January 1841 and ending in December 1847. The second volume, published in 1852, contained a similarly detailed account of the continuation of the series to December 1850; comprising, with the observations stated in the preceding volume, an uninterrupted series of monthly determinations during ten years. The mean secular change derived from sixty-eight monthly results obtained with the same circle and needle throughout, was found to be a decrease of 0′·067 in each year—an amount so small as to be practically insignificant in the consideration of the questions at present under notice. The mean value of the Inclination in the ten years, taking all the months into account, was −70° 36′·01; the mean of the months from April to September inclusive was −70° 35′·42, and from October to March inclusive −70° 36′·6. The difference between these half-yearly values is 1′·18, the (south) dip being 0′·59 less in the months from April to September, and 0′·59 greater in the months from October to March, than on the mean of the whole year.

We have therefore for the values of the total force at Hobarton in the two semiannual periods, 4·5048 sec. 70° 36′·6=13·5688 (in British units) from October to March, and 4·5036 sec. 70° 35′·42=13·5520 from April to September. The difference, viz. 0·0168, expresses the greater intensity of the terrestrial magnetic force in the semiannual period from October to March than in the semiannual period from April to September. This value may undergo a slight alteration, when the results of the continuation of the series of monthly determinations of the horizontal force and of the inclination until the final close of the Hobarton Observatory are added to those already stated; but it will be substantially the same. The later results will be published in the fourth Hobarton volume, now preparing for the press.

In the second and third volumes of the Toronto Observations are published the details of the monthly determinations of the Horizontal Force and of the Dip during eight years, viz. 1845 to 1852 inclusive. From these we may form the following Tables, similar to Tables IX. and XI. of the Kew Observations.

Table XV.—Monthly determinations of the Horizontal Component of the Magnetic Force at Toronto, 1845 to 1852 inclusive.

April to September.	1845.	1846.	1847.	1848.	1849.	1850.	1851.	1852.	Means of the 8 years.
April …………	3·5446	3·5414	3·5348	3·5361	3·5378	3·5373	3·5311	3·5054	3·5336
May …………	3·5481	3·5414	3·5386	3·5386	3·5413	3·5366	3·5328	3·5142	3·5365
June …………	3·5514	3·5458	3·5399	3·5366	3·5389	3·5380	3·5311	3·5083	3·5363
July …………	3·5508	3·5446	3·5366	3·5376	3·5428	3·5284	3·5317	3·5139	3·5358
August ………	3·5473	3·5397	3·5424	3·5360	3·5394	3·5199	3·5318	3·5138	3·5338
September ……	3·5466	3·5390	3·5338	3·5332	3·5382	3·5217	3·5286	3·5119	3·5319
Means, April to Sept.	3·5481	3·5420	3·5385	3·5363	3·5397	3·5303	3·5312	3·5113	3·53465

October to March.	1845	1846.	1847.	1848.	1849.	1850.	1851.	1852.	Means of the 8 years.
January ………	3·5472	3·5475	3·5435	3·5329	3·5319	3·5344	3·5249	3·5305	3·5366
February ……	3·5471	3·5413	3·5426	3·5352	3·5312	3·5354	3·5243	3·5231	3·5350
March…………	3·5471	3·5441	3·5386	3·5372	3·5339	3·5387	3·5321	3·5237	3·5369
October ………	3·5466	3·5386	3·5345	3·5263	3·5343	3·5320	3·5311	3·5110	3·5318
November ……	3·5471	3·5360	3·5366	3·5249	3·5366	3·5361	3·5304	3·5140	3·5327
December ……	3·5479	3·5433	3·5347	3·5318	3·5351	3·5283	3·5286	3·5149	3·5331
Means, Oct. to Mar.	3·5472	3·5418	3·5384	3·5314	3·5338	3·5341	3·5286	3·5195	3·53435
Yearly means …	3·5476	3·5419	3·5384	3·5339	3·5367	3·5322	3·5299	3·5154	3·53451

The two half-yearly results are intercomparable, requiring no correction for secular change, as they have both the same mean epoch, viz. January 1, 1849.

TABLE XVI.—Monthly Values of the Magnetic Inclination at Toronto, 1845 to 1852 inclusive.

April to September.	1845.	1846.	1847.	1848.	1849.	1850.	1851.	1852.	Means of the 8 years.
	75°+	75°+	75°+	75°+	75°+	75°+	75°+	75°+	
April	11′·5	14′·3	15′·9	18′·0	18′·4	19′·7	21′·9	20′·0	75° 17′·46
May	15·4	14·4	16·1	17·2	18·4	19·5	20·0	20·8	75 17·73
June	15·2	14·8	13·1	16·8	18·5	19·1	20·7	20·8	75 17·37
July	14·2	14·0	11·6	16·4	18·0	19·9	19·0	19·9	75 16·63
August	14·4	14·4	12·6	19·0	19·3	18·4	19·8	20·0	75 17·24
September......	16·6	15·7	15·4	17·3	21·6	21·0	20·8	21·6	75 18·75
Means, April to Sept.	75 14·55	75 14·60	75 14·12	75 17·45	75 19·03	75 19·60	75 20·37	75 20·52	75 17·53

October to March.	1845.	1846.	1847.	1848.	1849.	1850.	1851.	1852.	Means.
	75°+	75°+	75°+	75°+	75°+	75°+	75°+	75°+	
January	11·4	13·9	15·0	20·3	19·5	19·9	21·6	19·3	75° 18′·49
February	19·5	14·2	15·2	18·7	18·1	18·7	20·0	19·6	75 18·00
March	14·5	13·8	16·3	17·2	16·7	18·0	21·5	19·6	75 17·20
October	14·3	15·4	17·6	19·0	20·6	21·8	20·0	22·2	75 18·86
November	16·8	15·0	17·7	19·4	20·1	21·3	20·4	21·3	75 19·00
December	15·2	15·1	17·0	20·6	18·1	22·5	19·4	21·2	75 18·64
Means, Oct. to March	75 16·45	75 14·57	75 16·47	75 19·20	75 18·85	75 20·37	75 20·48	75 20·53	75 18·36

These two half-yearly results are also intercomparable, requiring no correction for secular change, as they have both the same mean epoch, viz. January 1, 1849.

We have then for the Total Force corresponding to the semiannual period April to September, 3·53465 sec. 75° 17′·53=13·9220 (in British units), and for the Total Force corresponding to the semiannual period October to March, 3·53435 sec. 75° 18′·36 =13·9336; the difference, 0·0116, is the measure of the greater intensity of the terrestrial magnetic force in the October to March period than in the April to September period: or, applying to the values of the horizontal force the induction-correction of —·0040 (Toronto Observations, vol. iii. pp. cxv, cxvi), we have the total force in the April to September period 3·53065 sec. 75° 17′·53=13·9062, and in the October to March period 3·53035 sec. 75° 18′·36=13·9178; and the corresponding difference, ·0116, as the excess of the total force in the October to March period over the April to September period.

The observations of the Inclination at Toronto were carried on previous to 1845 and continued subsequent to 1852, completing a series of fifteen years, for which period, therefore, a corresponding inference, in regard to the annual variation of the Inclination, may be drawn, resting on a still wider basis. The second volume of the Toronto Observations, published in 1853, and the third volume, published in 1857, contain the details of 1920 determinations of the dip nearly equally distributed in the different months of the fifteen years, 1841 to 1855 inclusive, of which the following is a summary, arranged in the two categories, April to September, and October to March:—

April to September.		October to March.	
	° ′		° ′
April	75 18·33	January	75 18·78
May	75 18·08	February	75 18·43
June	75 17·38	March	75 18·17
July	75 17·13	October	75 19·09
August	75 17·33	November . . .	75 19·53
September . . .	75 19·09	December . . .	75 19·15
	75 17·90		75 18·86

The semiannual results require no correction for secular change, as they have both the same mean epoch. They show a semiannual inequality in the Dip at Toronto, causing its value to be, on the average, 0′·96 higher in the months from October to March than in those from April to September. Table XVI., resting on a smaller number of years, gave a semiannual inequality of 0′·83.

We have therefore the concurrent evidence of the three observatories of Toronto, Hobarton, and Kew for the existence of an annual variation in the dip, and in the intensity of the total magnetic force, referable apparently to the earth's position in its orbit, with epochs of maxima and minima coincident, or nearly so, with the solstices. The conclusion terminating the previous section of this paper (§ 7) has shown the probability, resting also on the concurrent evidence obtained at four observatories, Hobarton, the Cape of Good Hope, St. Helena, and Kew, of the existence of a corresponding semiannual inequality in the Declination.

The phenomena thus submitted to the consideration of the Royal Society may be briefly stated to be an increase of the Dip and of the Total Force, and a deflection of the north end of the Declination magnet towards the West, in both hemispheres, in the months from October to March, as compared with those from April to September. It seems difficult to assign to such effects any other than a cosmical cause. The greater proximity of the earth to the sun in the December compared with the June solstice most naturally presents itself as a not improbable cause; but we are as yet too little acquainted with the mode of the sun's action on the magnetism of the earth to enter more deeply into the question at present. The inequalities may in themselves seem to be small, but judged of *scientifically*, *i. e.* by the proportions they bear to their respective probable errors, they are not so.

The tabulation from the Photograms, and the calculation of the values contained in the Tables, have been performed by the Non-commissioned Officers of the Royal Artillery, under the superintendence of the principal clerk, Mr. JOHN MAGRATH, in the Government Establishment at Woolwich for the reduction and publication of magnetic observations.

XIII. *On the Diurnal Inequalities of Terrestrial Magnetism, as deduced from observations made at the Royal Observatory, Greenwich, from* 1841 *to* 1857.

By GEORGE BIDDELL AIRY, *F.R.S., Astronomer Royal.*

Received April 8,—Read April 23, 1863.

IT has been usual for the Royal Society to receive among their communications and to publish in their 'Transactions' the epitomized results of long series of voluminous observations and laborious calculations, of which the fundamental details have been printed in works specially devoted to those subjects. The paper which I have the honour now to submit to the Society consists principally of results of this class. It exhibits in curves the Diurnal Inequalities of Terrestrial Magnetism, as obtained by the use of instruments essentially the same through the whole period of the seventeen years; during the last ten years of which the magnetic indications have been automatically recorded by photographic self-registration, on a system which has been continued to the present time (1863) and is still to be continued. I offer these results to the Royal Society in the hope that they will prove no unimportant contribution to a record of the state of Terrestrial Magnetism at Greenwich, through a period which is likely to be esteemed a very important one in the general history of the science.

The magnets of the three magnetometers (Declination, Horizontal Force, Vertical Force), from which these indications are obtained, are 2-foot magnets, such as were introduced by GAUSS about the time of commencing this series of observations; two of them were prepared at Göttingen. If I had now to establish a magnetical apparatus, I should probably adopt magnets of smaller dimensions. Yet there are advantages in the use of large magnets, as the power of carrying large mirrors, &c., which I would not lightly forego. And, judging from the completeness and delicacy of the registers of magnetic storms made by all three instruments, I have reason to believe that the general accuracy of the records is almost as great as it will be possible to obtain with any instruments. I have therefore not thought it necessary to make any change in the instrumental system.

From the beginning of 1841 for the Declination and Horizontal Force, and from the beginning of 1842 for the Vertical Force, to the end of 1847, the observations were made by eye, every two hours. From the beginning nearly of 1848 (with the exception of the Vertical Force Magnet, of which the auxiliary apparatus was completed so late in the year that it has been thought best to suppress the few observations of 1848 entirely) the positions of the magnetometers are registered by the photographic apparatus planned and established at the Royal Observatory by CHARLES BROOKE, Esq.

The details of the observations, as far as 1847, are printed in the 'Greenwich Magnetical and Meteorological Observations' for each year. The means, however, printed in those volumes are not, in every instance, adopted here. This arises from the circumstance that, in order to give unity to the plan of reduction for this memoir, the days in which there prevailed a certain amount of magnetic disturbance (not defined numerically, but estimated by the judgment of the Superintendent of the Reductions) have now been separated from the rest, in the same manner as had previously been done for the reductions 1848–1857; and the means have been taken without these separated days. The days thus excluded are the following:—

1841: September 24, 25, 26, 27; October 25; November 18, 19; December 3, 14.
1842: January 1; February 24; April 14, 15; July 1, 2, 3; November 10, 21; December 9.
1843: January 2; February 6, 16, 24; May 6; July 24, 25.
1844: March 29, 30; October 1; November 16, 22.
1845: January 9; February 24; March 26; August 29; October 3.
1846: May 12; August 6, 7, 24, 25, 28; September 4, 5, 10, 11, 21, 22; October 2, 7, 8; November 26; December 23.
1847: February 24; March 1, 19; April 3, 7, 21; May 7; June 24; September 24, 26, 27; October 22, 23, 24, 25; November 22; December 17, 18, 19, 20.

The differences between the means at different hours and the mean of the twelve two-hourly means, which have been actually used in the formation of the curves in Plates XVI. and XX., are the following:—

		Declination.	Horizontal Force.	Vertical Force.
	h	′		
1841.	0	+3·7	−0·00123	
	2	+5·9	− 53	
	4	+3·7	+ 10	
	6	+0·7	+ 32	
	8	−1·3	+ 52	
	10	−2·5	+ 42	
	12	−2·8	+ 39	
	14	−2·2	+ 33	
	16	−1·5	+ 28	
	18	−1·5	+ 43	
	20	−1·9	− 03	
	22	−0·4	− 105	
1842.	0	+3·9	−0·00102	+0·00013
	2	+5·6	− 23	+ 33
	4	+3·4	+ 33	+ 40
	6	+0·8	+ 45	+ 24
	8	−0·9	+ 58	+ 11
	10	−2·3	+ 36	− 7
	12	−2·6	+ 38	− 27
	14	−2·1	+ 12	− 34
	16	−1·9	− 04	− 28
	18	−1·8	+ 10	− 20
	20	−2·0	− 02	− 10
	22	0·0	− 92	+ 3
1843.	0	+4·1	−0·00087	−0·00008
	2	+5·9	+ 02	+ 20
1843.	4	+3·4	+0·00042	+0·00043
	6	+0·7	+ 53	+ 38
	8	−0·7	+ 52	+ 29
	10	−2·0	+ 36	+ 8
	12	−2·3	+ 25	− 9
	14	−1·8	+ 01	− 22
	16	−1·9	− 03	− 26
	18	−2·1	+ 01	− 26
	20	−2·3	− 29	− 25
	22	−0·8	− 110	− 17
1844.	0	+4·2	−0·00092	−0·00022
	2	+5·5	+ 06	+ 18
	4	+2·9	+ 53	+ 55
	6	+0·5	+ 80	+ 55
	8	−1·1	+ 78	+ 37
	10	−2·1	+ 49	+ 15
	12	−2·3	+ 27	− 5
	14	−1·8	− 03	− 19
	16	−1·5	− 13	− 26
	18	−1·9	− 13	− 30
	20	−2·1	− 51	− 36
	22	−0·3	− 108	− 33
1845.	0	+3·9	−0·00103	−0·00008
	2	+5·8	+ 10	+ 23
	4	+3·3	+ 55	+ 46
	6	+0·6	+ 62	+ 43

TABLE (continued).

		Declination.	Horizontal Force.	Vertical Force.			Declination.	Horizontal Force.	Vertical Force.
	h	′				h	′		
1845.	8	−0·7	+0·00061	+0·00026	1846.	16	−1·9	+0·00002	−0·00038
	10	−1·6	+ 43	+ 5		18	−2·3	+ 08	− 45
	12	−1·7	+ 31	− 13		20	−2·4	− 39	− 39
	14	−1·9	+ 01	− 25		22	−1·1	− 133	− 32
	16	−1·8	− 03	− 27	1847.	0	+4·4	−0·00119	+0·00015
	18	−2·1	+ 01	− 25		2	+6·9	− 16	+ 35
	20	−2·6	− 40	− 20		4	+3·8	+ 41	+ 37
	22	−1·1	− 118	− 19		6	+1·0	+ 67	+ 27
1846.	0	+4·0	−0·00114	−0·00014		8	−0·4	+ 73	+ 9
	2	+6·2	− 12	+ 26		10	−1·6	+ 52	− 3
	4	+3·7	+ 53	+ 58		12	−2·7	+ 34	− 33
	6	+0·9	+ 83	+ 63		14	−2·5	+ 03	− 33
	8	−1·1	+ 72	+ 49		16	−2·5	+ 08	− 32
	10	−2·0	+ 48	+ 21		18	−2·9	+ 08	− 17
	12	−2·1	+ 29	− 8		20	−2·9	− 24	− 3
	14	−2·0	+ 13	− 27		22	−0·9	− 120	+ 13

The differences of the means for the separate hours from the mean of the twelve two-hourly means, in the aggregates of the numbers for the same nominal month in different years, through the periods 1841–1847 for Declination and Horizontal Force, and 1842–1847 for Vertical Force, are the following:—

		Declination.	Horizontal Force.	Vertical Force.			Declination.	Horizontal Force.	Vertical Force.
	h	′				h	′		
January.	0	+2·9	−0·00047	+0·00006	March.	12	−2·8	+0·00030	−0·00017
	2	+3·8	+ 5	+ 18		14	−1·9	− 1	− 27
	4	+1·7	+ 17	+ 34		16	−2·0	− 11	− 30
	6	0·0	+ 8	+ 22		18	−1·9	+ 7	− 30
	8	−0·9	+ 2	+ 14		20	−1·9	− 4	− 23
	10	−2·5	− 2	+ 6		22	−0·6	− 101	− 10
	12	−2·8	− 17	− 4	April.	0	+4·3	−0·00160	0·00000
	14	−1·4	− 20	− 14		2	+7·9	− 39	+ 33
	16	−1·0	− 13	− 16		4	+4·6	+ 56	+ 67
	18	−0·6	+ 35	− 22		6	+1·1	+ 84	+ 58
	20	0·0	+ 43	− 18		8	−1·2	+ 77	+ 37
	22	+0·8	− 2	− 10		10	−1·9	+ 50	+ 3
February.	0	+3·6	−0·00046	+0·00003		12	−2·7	+ 44	− 23
	2	+4·9	+ 7	+ 27		14	−2·5	+ 30	− 38
	4	+2·7	+ 26	+ 40		16	−2·6	+ 10	− 47
	6	+0·6	+ 29	+ 27		18	−2·3	+ 21	− 40
	8	−1·2	+ 29	+ 13		20	−3·3	− 19	− 28
	10	−2·3	+ 4	0		22	−1·3	− 164	− 12
	12	−2·8	+ 1	− 13	May.	0	+4·4	−0·00134	−0·00005
	14	−2·3	− 26	− 18		2	+6·7	− 9	+ 27
	16	−1·6	− 17	− 22		4	+4·2	+ 66	+ 42
	18	−1·4	+ 17	− 18		6	+1·2	+ 113	+ 50
	20	−0·7	+ 27	− 20		8	−0·1	+ 119	+ 30
	22	+0·3	− 30	− 10		10	−1·1	+ 70	+ 2
March.	0	+4·0	−0·00109	−0·00002		12	−1·6	+ 43	− 20
	2	+6·7	− 3	+ 27		14	−2·1	+ 14	− 33
	4	+3·9	+ 56	+ 50		16	−2·1	− 6	− 30
	6	+0·5	+ 47	+ 40		18	−3·3	− 23	− 22
	8	−1·2	+ 47	+ 20		20	−4·6	− 86	− 8
	10	−2·8	+ 30	0		22	−1·7	− 167	− 8

TABLE (continued).

		Declination.	Horizontal Force.	Vertical Force.
	h	′		
June.	0	+4·0	−0·00130	−0·00005
	2	+6·4	− 10	+ 28
	4	+4·8	+ 57	+ 48
	6	+1·8	+ 101	+ 53
	8	+0·3	+ 110	+ 42
	10	−0·7	+ 71	+ 5
	12	−1·9	+ 51	− 18
	14	−2·2	+ 26	− 30
	16	−2·4	+ 11	− 37
	18	−3·3	− 11	− 33
	20	−4·4	− 106	− 28
	22	−2·0	− 173	− 17
July.	0	+4·1	−0·00146	−0·00008
	2	+6·7	− 20	+ 25
	4	+4·8	+ 73	+ 52
	6	+2·0	+ 109	+ 62
	8	+0·3	+ 120	+ 47
	10	−1·2	+ 77	+ 13
	12	−2·0	+ 54	− 13
	14	−2·5	+ 14	− 35
	16	−2·7	− 3	− 30
	18	−3·7	− 11	− 38
	20	−4·1	− 90	− 30
	22	−1·6	− 170	− 25
August.	0	+5·1	−0·00141	−0·00015
	2	+7·7	− 11	+ 27
	4	+4·6	+ 66	+ 52
	6	+0·9	+ 89	+ 60
	8	−0·7	+ 99	+ 40
	10	−1·9	+ 83	+ 13
	12	−2·4	+ 59	− 20
	14	−2·5	+ 33	− 32
	16	−2·5	+ 11	− 42
	18	−3·6	− 14	− 33
	20	−3·9	− 83	− 28
	22	−1·0	− 190	− 27
September.	0	+5·6	−0·00141	−0·00013
	2	+6·9	16	+ 28
	4	+3·5	+ 39	+ 57
	6	+0·4	+ 53	+ 50
	8	−2·0	+ 71	+ 30
	10	−2·7	+ 63	+ 7
September.	12	−2·5	+0·00054	−0·00018
	14	−2·2	+ 37	− 32
	16	−2·6	+ 23	− 35
	18	−2·2	+ 20	− 32
	20	−2·4	− 49	− 27
	22	+0·1	− 170	− 27
October.	0	+4·4	−0·00120	−0·00007
	2	+5·9	− 27	+ 23
	4	+3·2	+ 17	+ 48
	6	+0·1	+ 37	+ 33
	8	−1·4	+ 50	+ 18
	10	−2·5	+ 53	+ 2
	12	−2·5	+ 41	− 13
	14	−1·8	+ 21	− 25
	16	−1·4	+ 27	− 27
	18	−1·2	+ 26	− 23
	20	−1·5	0	− 18
	22	−1·0	− 121	− 12
November.	0	+3·5	−0·00063	−0·00002
	2	+4·4	− 14	+ 23
	4	+2·2	+ 11	+ 42
	6	+0·3	+ 33	+ 28
	8	−1·2	+ 26	+ 17
	10	−2·5	+ 13	+ 5
	12	−2·2	+ 10	− 12
	14	−1·8	0	− 20
	16	−0·9	+ 1	− 23
	18	−0·9	+ 19	− 20
	20	−0·7	+ 19	− 15
	22	−0·1	− 51	− 10
December.	0	+2·5	−0·00043	−0·00002
	2	+3·7	− 10	+ 22
	4	+1·6	+ 13	+ 25
	6	+0·1	+ 21	+ 17
	8	−1·1	+ 14	+ 15
	10	−2·1	+ 9	+ 7
	12	−2·1	+ 4	− 5
	14	−1·2	− 29	− 12
	16	−0·6	− 14	− 13
	18	−0·6	+ 19	− 15
	20	−0·2	+ 26	− 17
	22	+0·2	− 6	− 8

These means are used in forming the curves of Plates XVIII. and XXII.

For the observations from 1848 to 1857, the details of the record (in the form of measures of the ordinates of every salient point of the photographic curve) will be found in the 'Greenwich Observations' for each year—a few being omitted in the earlier portion of the period. These numbers, however, have not actually been used in forming the means. For that purpose (as is explained in the Reductions printed in the 'Greenwich Observations, 1859') curves have been traced by hand upon the photographic sheets, smoothing down their most rapid inequalities; and the hourly ordinates of these curves have been measured upon the sheets. The means of these are given in the 'Greenwich Observations, 1859;' they are used without alteration here.

The list of days omitted in the period 1848–1857 will be found in the volume for 1859. It may be interesting to collect here the numbers of omitted days for the several years of the entire period from 1841 to 1857.

1841	9	1847	20	1853	18
1842	10	1848	20	1854	13
1843	7	1849	2	1855	4
1844	5	1850	6	1856	0
1845	5	1851	13	1857	10
1846	17	1852	17		

These numbers, as I believe, give a very fair measure of great magnetic disturbances in each year. There is no appearance of decennial cycle in their recurrence. Nor does the number of disturbed days appear to have any distinct relation to the magnitude of diurnal change, as will be seen on comparing the list of omitted days with the curves at the end of this memoir.

I trust to have another opportunity of explaining more fully the reasons which have induced me to separate entire days of disturbed observations from the general mass, instead of separating special observations on every day when their departure from the mean exceeds a previously-defined limit, as has usually been done in late years. For the present, I will only remark that every digest may be considered in some measure satisfactory which actually renders account of the influence of every observation, but that the method which I have followed, and which puts it in my power completely to dissect the whole storm occurring on each disturbed day, appears to me much more satisfactory than any other.

Reverting now to the reductions which form the special subject of this memoir, I will first state that the curves which occupy the four Plates XVI.–XIX. are formed from the means to which I have referred, by comparing the mean for each hour with the mean for the twenty-four hours, and using their difference to form one of the coordinates,—the horizontal ordinate to the left being the measure of hourly westerly declination (as compared with the mean for the twenty-four hours) of the needle's north end, expressed in terms of the whole horizontal force for the year by dividing its measure in minutes of arc by 3438; and the vertical ordinate upwards being the measure of hourly horizontal force (as compared with the mean for the twenty-four hours) acting in the magnetic northerly direction on the needle's north end, expressed in terms of the same horizontal force. The origin of coordinates (the intersection of the straight lines in each diagram), from the nature of the process, necessarily represents the mean declination and mean horizontal force in each month.

Now the means for each month are themselves subject to an annual inequality, which, it seems probable, does not depend on the same causes that produce the secular changes. From 1841 to 1847 the mean secular change of western declination appears to proceed at the rate of $-4'{\cdot}2$ nearly per annum; and from 1848 to 1857 the rate is about $-7'{\cdot}9$ per annum. Applying the proportional parts of these, with changed sign, to the mean of the determinations for months of the same name through their proper periods, comparing each so corrected result with the mean of all, and converting the difference into parts of horizontal force, the following excess for each month is found:—

Annual Inequality of Western Declination.

	Period 1841–1847.	Period 1848–1857.
January . . .	−0·0007	−0·0002
February . .	−0·0004	−0·0003
March . . .	−0·0006	−0·0001
April	−0·0007	+0·0001
May	−0·0004	+0·0001
June	+0·0001	0·0000
July	+0·0002	+0·0003
August . . .	+0·0004	+0·0002
September . .	+0·0010	0·0000
October . . .	+0·0005	−0·0001
November . .	+0·0006	−0·0001
December . .	0·0000	+0·0001

Treating the Horizontal Force in the same way, it is necessary to observe that, for the first period, the secular change can be derived only from the monthly means of Horizontal Force (as the Deflection Apparatus was not used in the earlier years), and that for this purpose several corrections must be made to the printed numbers, either for changes in the position of the scale or mirror, or for the omission of constants in the scale reading (as will be fully explained in the Greenwich Observations, 1862). The annual rate is +0·0012. For the second period (Greenwich Observations, 1859), the annual rate is +0·0022. The year 1843 is omitted because adjustments were changed in the middle of the year, and 1847 because one month is defective. Thus we obtain—

Annual Inequality of Northern Horizontal Force.

	Period 1841–1846.	Period 1848–1857.
January . . .	+0·0004	+0·0004
February . .	−0·0006	+0·0003
March . . .	−0·0006	0·0000
April	+0·0005	+0·0002
May	−0·0013	−0·0004
June	+0·0002	0·0000
July	−0·0001	−0·0004
August . . .	−0·0001	−0·0006
September . .	−0·0002	−0·0010
October . . .	+0·0001	+0·0002
November . .	+0·0005	+0·0005
December . .	+0·0008	+0·0008

Although there are irregularities, the general law of these numbers is sufficiently distinct. There is nothing surprising in the slight diminution of the numbers in the second period, as compared with those of the first; for, as we shall see, every inequality of Declination and Horizontal Force is much larger in the period 1841–1847 than in the period 1848–1857. Some great cosmical change seems to have come upon the earth, affecting in a remarkable degree all the phenomena of terrestrial magnetism.

If now we desired to refer the hourly state of magnetism to the state corresponding to a uniform secular progression through the course of each year, we must apply the numbers just found (their irregularities being first smoothed down), with changed signs, as ordinates from the intersections of lines in the diagrams; and we should so obtain the new point of reference for all the hourly points in each month-diagram. No change is produced in the year-diagrams. It does not appear, so far as I can see, that anything is gained by this. I should have been glad to find that my new point of reference was so related to some one of the hourly points that I could be justified in fixing on that hourly point as a magnetic state which is independent of the periodical daily disturbances. For instance, if the new point of reference bore a constant relation to the point corresponding to 12^h, I should have concluded that there is no diurnal disturbance at 12^h. I have not, however, succeeded in finding a point which possesses this property.

I have now to call attention to the remarkable change in the magnitude and form of the diurnal curves representing the hourly magnetic forces in the horizontal plane. From 1841 to 1848 (see Plates XVI. and XVII.) their magnitude very slowly increases, with a small change of form. From 1848 to 1857 (see Plate XVII.) their magnitude very rapidly diminishes, with a great change of form. Possibly one step in the physical explanation of the change may be made by comparing the change from 1848 to 1857 (in Plate XVII.) with the change from the summer months to the winter months (in Plate XIX.). It would seem that the later years have become entirely winter years; and this seems to imply that the magnetic action of the sun on the earth's southern hemisphere has remained nearly unaltered, while that on the northern hemisphere has undergone a great diminution.

I will now allude to the curves representing the hourly state of Vertical Force, as referred to the mean on each day. The force in these is represented by a simple ordinate, the numerical value of which will be found, either in the preceding pages of this paper, or in the printed books to which I have already referred. On examining the curves in the separate months, Plates XXII. and XXIII., it will be seen that there is considerable difference between those of the first period and those of the second period, both in the place of "node" (or intersection of the curve with the mean line) and in the magnitude of ordinates; also that in the first period there is a sensible difference of magnitude of ordinates between summer and winter, and in the second period a sensible difference in the place of the "node" between summer and winter. On referring to the curves for the different years, a very great change will be found. From 1847 to 1849 the magnitude of the ordinates has somewhat increased; from 1849 to 1850 it has increased still more; and no diminution follows. And on observing the place of the node, a still more remarkable change will be seen. In 1846 the descending node is at

$11\frac{3}{4}^{h}$ nearly; in 1847 it is at 9^{h} nearly; in 1849 at 7^{h} nearly; in 1850 at 5^{h}; in 1851 at 4^{h}; and there it continues with little alteration. (The loss of the observations of 1848 is here unfortunate.) It is important to observe that, though the instrument was changed in 1848, the change in the place of the node did not then occur suddenly; it had begun with the old instrument, and continued to advance gradually for several years with the new instrument.

I have sought for collateral evidence of this remarkable change, but hitherto without success. I have received observations which support the determinations for the earlier period, but I have not yet found any corresponding in date with those of the later period. I have no reason, however, to believe in the possibility of any error. And the change in magnitude is not greater (though in reverse order) than that for the forces in the horizontal plane; and the change of law is not more striking.

These are the principal results that I have yet obtained from discussion of the observations on the less disturbed days. A reduction of the observations on the more disturbed days is far advanced, and may be the subject of another communication.

THE FOLLOWING ARE THE SUBJECTS OF PLATES XVI.–XXIII.

Diurnal Curves of Combination of Declination and Horizontal Force.

Plate XVI.	Mean of every day in each year, 1841 to 1847.
Plate XVII.	Mean of every day in each year, 1848 to 1857.
Plate XVIII.	Mean of every day in each nominal month through the period 1841 to 1847.
Plate XIX.	Mean of every day in each nominal month through the period 1848 to 1857.

And

Diurnal Curves of Vertical Force.

Plate XX.	Mean of every day in each year, 1841 to 1847.
Plate XXI.	Mean of every day in each year, 1849 to 1857.
Plate XXII.	Mean of every day in each nominal month through the period 1841 to 1847.
Plate XXIII.	Mean of every day in each nominal month through the period 1849 to 1857.

XIV. *Researches on the Refraction, Dispersion, and Sensitiveness of Liquids.*

By J. H. Gladstone, *Ph.D., F.R.S., and the Rev.* T. P. Dale, *M.A., F.R.A.S.*

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In a previous paper "On the Influence of Temperature on the Refraction of Light*," we started some inquiries which have been since pursued, and we now lay before the Royal Society some of the later results.

The same apparatus has been employed, with a hollow prism of 61° 0′ angle, and the method of observation has been essentially the same. But experience has led to some modifications, the most important of which is this: instead of attempting to take the angular measurements at certain foredetermined temperatures, as 10° C., 20° C., they were taken first at the temperature of the room, whatever that might be, and then at such other temperatures as seemed to offer the most trustworthy results. This involved more calculation, but it still saved time, and secured greater accuracy. The plan of measuring to 10″ was abandoned as a useless nicety; but, as a rule, two or more observations of each fixed line at each temperature were taken, and if they differed slightly the mean was adopted, but if the discrepancy amounted to 2′ or 3′ the observation was repeated. The average of these observations of the lines A, D, and H at different temperatures gave the refractive indices which are placed together in the Table that constitutes Appendix I., and they afford the data for nearly all the comparisons about to be instituted. Appendix II. contains the mean determinations made of the refractive indices of some of these liquids for a larger number of the lines at the temperature of the room. To it have been added some observations on other liquids, and determinations published in our former papers, so as to render it as complete as possible for any who may desire to investigate the irrationality of the spectrum, or the truth of the formulæ of Cauchy.

An attempt has been made to determine the amount of probable error, not so much absolutely as with reference to the different purposes for which the observations have been made. The conclusions arrived at are as follows:—

Where the refraction of different fixed lines at the same temperature is compared, the probable error is very small. The measurements may be easily obtained accurate to ±1′, corresponding to about ±0·0002 in the refractive index, and thus the relative refraction of A, D, and H in Appendix I., or of all the lines in Appendix II. for any one substance will rarely differ from the truth by more than that amount.

When the refraction of a substance at one temperature is compared with its refraction at another temperature, there exists a source of error in the determination of the precise temperature of that part of the liquid through which the solar beam is passing at the time when the measurement is taken. It is difficult to avoid this error, or to estimate its amount. It is, as may be supposed, generally greatest at the temperatures furthest removed from that of the surrounding objects, and in these cases there is reason to fear that it not unfrequently amounts to 1 or 2 degrees Centigrade. Even at the ordinary temperature an error may arise from the heating power of the sunbeam that passes through the liquid, and which may not affect the thermometer equally with the substance whose refraction is measured. In some of our more exact and our later determinations a strong solution of alum in a flat-sided glass was interposed in the path of the ray to reduce its heating power.

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* Philosophical Transactions, 1858, p. 887.

Where the refraction of one substance is compared with that of another, error may also arise from inaccuracy in obtaining the minimum deviation. Though several adjustments have to be made, the error from this source is practically confined within very narrow limits, and rarely if ever passes beyond the fourth place of decimals even with very dispersive substances. This error was not so well guarded against in the observations recorded in our previous paper; and it may also affect the determination of the sensitiveness of a few substances, namely those where a different adjustment of the prism was made at different temperatures; but these are easily known, as that was only done for low temperatures such as 8° C., and they are all marked in Appendix I. with an asterisk. In order to be rigidly correct, the hollow prism ought to have been adjusted afresh for minimum deviation in the case of each line and at each temperature, but the movement of the apparatus necessitated by this would practically have introduced greater errors than resulted from the neglect of it. Yet this has an appreciable effect on the length of the spectrum in highly dispersive substances; and in order to obviate the error as much as possible in the later measurements of such substances, care was taken to fix the minimum deviation not for either of the extremities, but for the middle of the spectrum. It would not have been difficult to make a correction by the usual formula for a small deviation from the minimum angle, but we doubted whether practically anything would be gained, considering the greater complexity of the calculation.

If the indices of refraction were to be considered not relatively, but absolutely, other sources of error would have to be taken into account; for instance, inaccuracy in the determination of the prism-angle, faults of workmanship in the apparatus. For these it is more difficult to assign a limit: they may even affect the third place of decimals, whereas the combined errors from all the other sources are probably confined to the fourth place. But the *absolute* accuracy of an index is of minor importance in the present research.

The purity of the liquids experimented on is of course a matter of the utmost consequence. When commercial specimens were employed they were always purified, or their purity ascertained. Many of the liquids were prepared in Dr. GLADSTONE'S laboratory with special reference to this inquiry, and many others were kindly placed at our disposal by those chemists who had paid special attention to them, and we have generally taken their word for the purity of the specimen. In this way we are under obligations to Professor WILLIAMSON, Professor HOFMANN, Professor FRANKLAND, Dr. WARREN DE LA RUE and Dr. HUGO MÜLLER, Mr. BUCKTON, Dr. ODLING, Mr. A. H. CHURCH, Mr. GREVILLE WILLIAMS, and Mr. PIESSE, to whom we return our best thanks.

The present paper takes up five points.

I. The relation between sensitiveness and the change of volume by heat.

II. The refraction and dispersion of mixed liquids.

III. The refraction, dispersion, and sensitiveness of different members of homologous series.

IV. The refraction, dispersion, and sensitiveness of isomeric liquids.

V. The effect of chemical substitution on these optical properties.

SECTION I.—*The relation between Sensitiveness and the Change of Volume by Heat.*

Having examined now about ninety different liquids, we have uniformly found that the refraction diminishes as the temperature increases. This property we have already named "sensitiveness."

We have uniformly found also that the spectrum diminishes in length as the temperature increases. In a very few instances this diminution is lost within the limits of errors of observation, but we believe it always occurs.

This diminution in length is progressive, the different rays being more sensitive in the order of their refrangibility. The following observations on a most dispersive and sensitive substance exhibit this:—

Substance.	Temp.	Refractive indices.						
		A.	B.	D.	E.		G.	H.
	°C.							
Bisulphide of Carbon.........	11	1·6142	1·6207	1·6333	1·6465	1·6584	1·6836	1·7090
Bisulphide of Carbon.........	36·5	1·5945	1·6004	1·6120	1·6248	1·6362	1·6600	1·6827
Difference	......	0·0197	0·0203	0·0213	0·0217	0·0222	0·0236	0·0263

That there is some intimate connexion between the sensitiveness of a liquid and its change of volume by heat was pointed out in our former paper; and our subsequent experiments only confirmed this opinion.

It became therefore a matter of interest to determine, if possible, what this relation is. The determinations of the sensitiveness of bisulphide of carbon, water, benzole, alcohol, wood-spirit, fousel-oil, ether, acetone, acetic acid, formic, acetic, and butyric ethers, and the iodides of methyl and ethyl afforded an opportunity of examining the matter, since the alteration of their volume by heat has been very accurately determined by KOPP and others; cumole, xylole, nitrobenzole, hydrate of phenyl, oil of turpentine, rectified oil of Portugal, eugenic acid, bromoform, and salicylate of methyl also answered the same purpose, since we determined the expansibility of the specimens employed for measuring the refractive indices at different temperatures.

In the case of every one of these liquids the refractive index of any ray alters less rapidly than the volume; but it was found that the refractive index minus unity, multiplied by the volume, gives nearly a constant.

It is otherwise with the contraction of the spectrum itself. In some cases, as bisulphide of carbon, it contracts much more rapidly than the volume increases, and in other cases, as ether, much less rapidly.

Here it must be borne in mind that every refractive index contains at least two coefficients. Whatever may be the physical reason, and to whatever extent we may accept such theoretical explanations as those given by CAUCHY, LUBBOCK, Sir WILLIAM HAMILTON, B. POWELL, and others, the formula $\mu=A+\frac{B}{\lambda^2}+\frac{C}{\lambda^4}+\ldots$ does certainly give results very near the truth, μ being the refractive index, λ the length of an undulation, and A, B, C coefficients depending on the nature of the medium. As we must employ A, B, C for the fixed lines of the spectrum so designated by FRAUNHOFER, we shall write the above formula for the future $\mu=\nu+\frac{\varkappa}{\lambda^2}+\frac{\varkappa'}{\lambda^4}+\ldots$ and shall suppose $\varkappa'$ and all subsequent coefficients too small to be sensible within the limits of error. Hence we have ν the coefficient of refraction, and $\varkappa$ the coefficient of dispersion; and ν may evidently be considered the refractive index of any substance freed from the influence of dispersion. As it appears that the function $\mu-1$ is of peculiar interest in these investigations, we propose giving it a distinct name, that of "*refractive energy*," this number really representing the influence of the substance itself on the rays of light. $(\mu-1)\times$ vol., or, which is the same thing, $\frac{(\mu-1)}{\text{density}}$, we propose calling the "*specific refractive energy.*"

As the value of μ for any particular luminous ray is affected by the dispersion, it was clearly desirable to calculate ν in certain cases, and see whether $(\nu-1)\times$ vol. would give a constant. Some doubt rests on the position of this theoretical limit; but its value was calculated by the formula given on pages 82 and 132 of BADEN POWELL's treatise 'On the Undulatory Theory as applied to Dispersion.' It will easily be seen by referring to the example on p. 132, that, in consequence of an accidental relation between the coefficients, $\nu=\mu_H-3(\mu_F-\mu_B)$ to very considerable exactness. This formula has been used by us, but in all cases given below the results have been verified by the accurate one.

Bisulphide of carbon and water were the liquids chosen, being very definite substances and extremely different in their degree of expansibility, water also having the advantage of a very irregular rate of change of volume. The refractive indices of the fixed lines B, F, and H (on which the calculation of ν depends) were determined at different temperatures with every precaution*.

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* The determinations for water in the accompanying Table were substituted during the printing for less accurate numbers.

Substance.	Temp.	Refractive indices.		
		B.	F.	H.
Bisulphide of Carbon...	11°	1·6207	1·6584	1·7090
Bisulphide of Carbon...	22·5	1·6116	1·6484	1·6972
Bisulphide of Carbon...	36·5	1·6004	1·6362	1·6827
Water	1	1·33005	1·33685	1·3431
Water	15·5	1·3298	1·3364	1·3426
Water	27·5	1·3289	1·3355	1·3416
Water	48	1·32595	1·33245	1·3387

The subjoined Table contains the calculations founded on these numbers. Column I. gives the refractive index of the theoretical limit, or ν. Column II. the specific refractive energy for this limit, or $(\nu-1)$ vol. Column III. the specific refractive energy for the line B, or (μ_B-1) vol. Column IV. the same for H, or (μ_H-1) vol. Column V. gives what Newton called the "absolute refractive power" reckoned for the limit, or (ν^2-1) vol.

Substance.	Temp.	Volume.	I.	II.	III.	IV.	V.
Bisulphide of Carbon...	11°	0·9554	1·5960	0·5694	0·5930	0·6773	1·4782
Bisulphide of Carbon...	22·5	0·9685	1·5865	0·5680	0·5923	0·6752	1·4714
Bisulphide of Carbon...	36·5	0·9854	1·5753	0·5669	0·5916	0·6727	1·4599
Extreme difference......	25·5	0·0300	0·0207	0·0025	0·0014	0·0046	0·0183
Water	1	0·9999	1·3227	0·3227	0·3300	0·3431	0·7495
Water	15·5	1·0007	1·3228	0·3230	0·3300	0·3429	0·7497
Water	27·5	1·0034	1·3216	0·3227	0·3300	0·3428	0·7492
Water	48	1·0109	1·3193	0·3227	0·3295	0·3429	0·7486
Extreme difference......	47	·0110	0·0035	0·0003	0·0005	0·0003	0·0011

It thus appears that the specific refractive energy is nearly a constant, whether we take the limit ν or the line B as the basis of calculation. The "absolute refractive power" is evidently not a constant.

The following Table exhibits the specific refractive energy at various temperatures for some of the other liquids mentioned above, the selection being made not of those which give the most accordant results, but of those which may be considered representative bodies, or of which we happen to possess observations at the longest range of temperature. The columns are numbered as before, the only difference being that in Column III. the line A is taken instead of B. The refractive indices observed will be found in Appendix I., or in our previous paper.

Substance.	Temp.	Volume.	I.	II.	III.	IV.
Alcohol	0°	0·9132	1·3598	0·3286	0·3340	0·3480
Alcohol	20	0·9326	1·3518	0·3280	0·3337	0·3478
Alcohol	40	0·9534	1·3435	0·3275	0·3332	0·3473
Alcohol	60	0·9762	1·3347	0·3268	0·3326	0·3473
Difference	......	+0·0630	−0·0251	−0·0018	−0·0014	−0·0007
Formic Ether	22	1·0305	1·3476	0·3582	0·3650	0·3807
Formic Ether	31	1·0436	1·3434	0·3584	0·3653	0·3811
Formic Ether	40	1·0573	1·3390	0·3584	0·3654	0·3815
Difference	......	+0·0268	−0·0086	+0·0002	+0·0004	+0·0008
Iodide of Ethyl	23·5	0·9440	1·4878	0·4604	0·4720	0·5116
Iodide of Ethyl	36	0·9583	1·4795	0·4595	0·4712	0·5103
Iodide of Ethyl	48	0·9730	1·4718	0·4590	0·4710	0·5108
Difference	......	+0·0290	−0·0160	−0·0014	−0·0010	−0·0008
Acetic Acid	20·5	1·0228	1·3656	0·3739	0·3794	0·3969
Acetic Acid	28·5	1·0305	1·3624	0·3734	0·3792	0·3967
Acetic Acid	40	1·0432	1·3579	0·3733	0·3791	0·3964
Acetic Acid	47·5	1·0517	1·3543	0·3726	0·3786	0·3963
Difference	......	+0·0289	−0·0113	−0·0013	−0·0008	−0·0006
Benzole	10·5	1·0125	1·4777	0·4836	0·4940	0·5371
Benzole	23	1·0278	1·4704	0·4834	0·4939	0·5370
Benzole	39	1·0481	1·4601	0·4822	0·4929	0·5353
Difference	......	+0·0356	−0·0176	−0·0014	−0·0011	−0·0018
Oil of Turpentine	24	1·1621	1·4521	0·5253	0·5341	0·5630
Oil of Turpentine	41	1·1778	1·4449	0·5240	0·5323	0·5611
Oil of Turpentine	47	1·1831	1·4414	0·5222	0·5308	0·5594
Difference	......	+0·0210	−0·0107	−0·0031	−0·0033	−0·0036
Eugenic Acid	18	0·9349	1·5159	0·4818	0·4942	0·5403
Eugenic Acid	27·5	0·9412	1·5119	0·4817	0·4934	0·5383
Difference	......	+0·0063	−0·0040	−0·0001	−0·0008	−0·0020

These results suffice to show that any refractive index minus unity, multiplied into the volume or divided by the density, gives nearly a constant. Indeed the numbers generally fall within the limits of experimental error. It is worthy of notice, too, that in the majority of cases, as bisulphide of carbon or alcohol, the products show a tendency to diminish as the temperature rises; but there are other cases, as formic ether, where the tendency seems to be to increase. Again, in some cases $(\nu-1)$ vol. gives the most accordant results; in other cases (μ_H-1) vol.

Supposing this true of the coefficient of refraction, does the law equally hold good of the coefficient of dispersion? It is evident from the formula $\mu=\nu+\frac{\kappa}{\lambda^2}$ that in the difference of any μ and ν, or of the refractive indices of any two rays, we have a measure of the coefficient of dispersion $\varkappa$. For convenience sake we adopt $\mu_H-\mu_A$ as this measure; and this is what is headed "Dispersion" in many subsequent tables. It is the same as "Length of Spectrum" in our former paper. This, multiplied by the volume, or $(\mu_H-\mu_A)$ vol., we call "Specific Dispersion." But, as already stated, there is no simple relation holding good for different liquids between the increase of volume and the decrease of dispersion by heat. The phenomena seem independent.

We therefore arrive at the empirical law, that *the refractive energy of a liquid varies directly with its density under the influence of change of temperature, or, in other words, that the specific refractive energy of a liquid is a constant not affected by temperature.* But in concluding thus, we wish it to be borne in mind that there is some influence, arising wholly or partially from dispersion, which we have not been able to take into account, but which gives rise to the slight progression of most of the calculated products, and perhaps to the non-inversion of the sensitiveness of water at 4° C., remarked on already by JAMIN and ourselves.

SECTION II.—*The Refraction and Dispersion of Mixtures of Liquids.*

This subject engaged the attention of M. DEVILLE as far back as 1842[*]; and of late years Messrs. HANDL and A. and E. WEISS[†] have published elaborate papers on it, but without arriving at a solution of the question. M. HOEK[‡], however, proceeding on the assumption of FRESNEL, that the density of the ether enclosed in a medium is μ^2-1 if the density of the ether in space is 1, found that the formula deduced from it gave numbers closely agreeing with those found experimentally by DEVILLE for mixtures of alcohol and water, or wood-spirit and water. Yet it happens that these results can equally well be explained on the supposition that the specific refractive power of a mixture is the mean of the specific refractive power of its components. And this supposition seemed also warranted by most of the results of Messrs. WEISS, and by several that we ourselves obtained.

It was clearly desirable to test these two, or any other suppositions, in a case where the refractive indices of the liquids mixed were very wide apart. Fortunately bisulphide of carbon and ether, substances almost at the opposite limits of the scale, were found to mix, and that without perceptible condensation, not indeed in equal volumes, but in the proportion of three volumes of ether to one of the bisulphide at low temperatures, and in the proportion of two to one at 20° C.

Two experiments were made at different seasons on mixtures of commercially pure specimens of these substances. The greatest care was taken to prevent evaporation as far as possible during the progress of the experiments.

It will be seen that in a case such as this, where there is no condensation on mixture, the calculation is much simplified, since for the specific refractive powers we may substitute the refractive indices themselves, and the supposition will stand thus: the refractive index of a mixture is the mean of the refractive indices of its components. And in such a case Hoek's formula resolves itself into the mean of μ^2-1.

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[*] Ann. de Chim. et de Phys. (sér. 3) tome v. p. 129.

[†] Wien. Ber. xxv. xxx. xxxi. and xxxiii. 589–656.

[‡] Poggendorff's Annalen, cxii.

Liquid.	Temperature.	Specific gravity.	Refractive index.		
			A.	D.	H.
Bisulphide of Carbon.........	8° C.	1·2790	1·6184	1·6366	1·7093
Ether	8	0·7374	1·3542	1·3575	1·3692
Mixture of 1 vol. Bisulph. and 3 vols. Ether	8	0·8710	1·4165	1·4235	1·4480
Mean	8	....	1·4202	1·4272	1·4542
Hoek's theory	8	....	1·4247	1·4323	1·4619
Bisulphide of Carbon.........	20	1·2685	1·6121	1·6299	1·7008
Ether	20	0·7246	1·3487	1·3525	1·3636
Mixture of 1 vol. Bisulph. and 2 vols. Ether	20	0·9059	1·4305	1·4390	1·4686
Mean	....	....	1·4365	1·4450	1·4760
Hoek's theory	....	....	1·4417	1·4509	1·4845

These two experiments confirm one another, but they fail to support either hypothesis. The calculation founded on μ^2-1 gives numbers which are far too high; and though the mean of the indices is certainly much nearer to the calculated numbers, the discrepancy in each case is beyond the limits of probable error. The calculation for A is certainly nearer than that for H, but evidently not much would be gained by assuming the theoretical limit as the basis of calculation.

Similar experiments were made by mixing aniline and alcohol of 90 per cent. together in equal volumes, but in this case a slight condensation ensues.

Liquid.	Temperature.	Specific gravity.	Refractive indices.		
			A.	D.	H.
Aniline	23·5° C.	1·0073	1·5642	1·5772	1·6263
Alcohol, 90 per cent..........	23·5	0·8154	1·3576	1·3614	1·3729
Mixture of equal vols., mean of two experiments	23·5	0·9167	1·4621	1·4707	1·5018
Mean deduced from specific refractive powers...	23·5	....	1·4636	1·4721	1·5025
Hoek's theory	....	....	1·4668	1·4754	1·5070

This shows precisely the same thing as the previous mixture; and, as in that case, the experimental numbers are slightly below those deduced from the mean of the specific refractive powers. This is also the case in other mixtures examined; yet no other simple formula gives numbers so closely approaching those obtained by experiment. The hypothesis that *the specific refractive power of a mixture of liquids is the mean of the specific refractive powers of its constituents* must therefore stand as the nearest approximation to the truth.

In one or two cases, as in the mixtures of sulphuric acid and water examined by Messrs. Weiss, the refraction is not at all in accordance with the above theory. This probably arises from some chemical combination between the two substances, different hydrates being formed.

We hope to revert to this subject more fully on some future occasion, when we propose extending our inquiry to solutions of solids.

SECTION III.—*The Refraction, Dispersion, and Sensitiveness of different members of Homologous Series.*

In our paper on the influence of temperature we remarked an advance in refraction and dispersion with each increment of C_2H_2 in the alcohol series. This has been examined more carefully, and the investigation has been carried much further in the same direction. The new data for the comparisons are given in Appendix I., from which the subsequent Tables are calculated, a reduction of the indices to 20° C. of temperature being always made, and the sensitiveness being calculated for 10 degrees rising from that temperature. The length of the spectrum, or the dispersion, is also reckoned at 20° C. The refractive index for only one line is given, in order to save space; and A is the line chosen, as it is least affected by dispersion. Where two specimens of the same substance have been examined, the mean of the observations has usually been adopted.

The Alcohol Series.

Liquid.	Formula.	Refractive index of A at 20° C.	Length of spectrum or dispersion.	Sensitiveness for 10° C.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Methylic Alcohol	$C_2H_4O_2$	1·3268	0·0128	0·0036	0·4105	0·0163	0·0045
Ethylic Alcohol.........	$C_4H_6O_2$	1·3578	0·0151	0·0041	0·4482	0·0190	0·0052
Amylic Alcohol.........	$C_{10}H_{12}O_2$	1·4005	0·0174	0·0039	0·4895	0·0212	0·0047
Caprylic Alcohol	$C_{16}H_{18}O_2$	1·4186	0·0195	0·0042	0·5096	0·0237	0·0051

From this it is evident that on ascending the series the refraction increases, the dispersion more rapidly still, while the sensitiveness remains nearly the same.

It should be borne in mind that on account of the small numbers by which the sensitiveness is expressed, and the serious source of error arising from the difficulty of determining the temperature with accuracy, comparisons of the sensitiveness of different liquids cannot be so satisfactory as comparisons of their refractive indices, or the length of the spectrum. As all the degrees of sensitiveness at 20° C. known to us lie between 0·0007 and 0·0074, we propose in future omitting the zeros, and simply stating that the sensitiveness of methylic alcohol for instance is 36. We shall omit the zeros also in the last three columns.

As we have already learnt the importance of comparisons of specific refractive energy, we have added in the three last columns the refractive energy $(\mu_A - 1)$, the dispersion $(\mu_H - \mu_A)$, and the sensitiveness (μ_A at 20° C. $-\mu_A$ at 30° C.), all divided by the density. It will be seen that the progression is maintained.

But some might prefer that the different alcohols should be compared, not at the same absolute temperature, but at the same distance from their boiling-points. This is attempted in the following Table for 82° below the boiling-points, but as the observations do not extend nearly to that in the case of methylic or caprylic alcohols, there is more left for calculation than is desirable.

Liquid.	Temperature.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refraction.	Specific dispersion.	Specific sensitiveness.
Methylic Alcohol	− 22°	1·3410	0·0135	33	4079	161	40
Ethylic Alcohol.........	− 4	1·3674	0·0154	40	4515	189	49
Amylic Alcohol.........	+ 50	1·3888	0·0167	40	4914	211	50
Caprylic Alcohol	100	1·3807	0·0180	51	5123	242	68

Here the advance of the refraction and dispersion with each addition of $C_2 H_2$ appears again (with one exception), and the sensitiveness advances likewise; and this is still more evident when the numbers are divided by the density.

It was a matter of interest to compare with these results the refractive indices of other homologous series belonging to the same group.

Iodide of Methyl Series.

Substance.	Formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Iodide of Methyl	$C_2 H_3 I$	1·5171	0·0460	73	2359	209	33
Iodide of Ethyl.........	$C_4 H_5 I$	1·5026	0·0420	66	2614	218	34
Iodide of Propyl	$C_6 H_7 I$	1·4934	0·0408	63	2882	235	36
Iodide of Amyl	$C_{10} H_{11} I$	1·4804	0·0335	50	3213	224	33

In this case the refraction, dispersion, and sensitiveness are also progressive, but in the opposite direction, for they all decrease as we ascend the series, instead of increasing, as was the case with the alcohols. This may be attributed to the larger proportion of iodine which the earlier members of the series contain, for iodine has a very great influence on the rays of light. If the numbers be divided by the specific gravity, the progression becomes in the direction of increase as with the alcohols, both in regard to refraction and dispersion, while in regard to sensitiveness the four members give nearly the same number, as was also the case in the series of alcohols.

Formic Ether Series.

Substance.	Formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Formic Ether	$C_4 H_5 O, C_2 H O_3$	1·3549	0·0154	44	3905	168	48
Acetic Ether	$C_4 H_5 O, C_4 H_3 O_3$	1·3659	0·0157	48	4152	178	55
Propionic Ether.........	$C_4 H_5 O, C_6 H_5 O_3$	1·3707	0·0164	44	4333	191	51
Butyric Ether	$C_4 H_5 O, C_8 H_7 O_3$	1·3864	0·0168	48	4402	191	54
Valerianic Ether	$C_4 H_5 O, C_{10} H_9 O_3$	1·3908	0·0172	42	4502	198	48

Here, as in the case of the alcohols, there is a progressive increase of refraction, dispersion, and specific energy. The numbers representing the sensitiveness appear rather irregular, but it is difficult to say how far this may be due either to impurity of specimens or to errors of observation.

During the progress of these experiments we found Professor Delffs has preceded us in examining the refraction of members of the formic ether series*. He gives as the indices of the red ray—

Formic Ether	1·3570
Acetic Ether	1·3672
Butyric Ether	1·3778
Valerianic Ether . . .	1·3904
Œnanthylic Ether . .	1·4144
Laurostearic Ether . .	1·4240

He does not note the temperature. His conclusion is, that "the indices of refraction of the compound ethers increase with their equivalents." His experiments afforded him no means of drawing a conclusion in regard to the dispersion; and the sensitiveness was a property not fully recognized at that time.

Acetate of Ethyl Series.

Substance.	Formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Acetate of Ethyl	$C_4 H_5 O, C_4 H_3 O_3$	1·3659	0·0157	48	4152	178	55
Acetate of Amyl	$C_{10} H_{11} O, C_4 H_3 O_3$	1·3911	0·0172	43	4506	198	49
Acetate of Capryl	$C_{16} H_{17} O, C_4 H_3 O_3$	1·4088	0·0211	58	......	......	...

This resembles the preceding series, or that of the alcohols, as might be anticipated. Professor Delffs in his second paper gives

Acetate of Methyl	1·3576
Acetate of Ethyl	1·3672
Acetate of Amyl	1·3904

He also gives the following indices, which bear similar witness:—

Butyrate of Methyl . . .	1·3752
Butyrate of Ethyl . . .	1·3778
Butyrate of Amyl . . .	1·4024
Oxalate of Ethyl	1·3803
Oxalate of Amyl	1·4168
Formiate of Ethyl . . .	1·3570
Formiate of Amyl . . .	1·3928

Poggendorff's Annalen, lxxxi. 470.

Hydride Series.

Liquid.	Formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Hydride of Œnanthyl	$C_{14}H_{15}H$	1·3898	0·0172	55	5499	242	77
Hydride of Capryl	$C_{16}H_{17}H$	1·3971	0·0170	47	5522	236	65

This also bears similar evidence.

Mercuric and Stannic Series.

Through the kindness of Mr. Buckton and Dr. Frankland, we have been able to examine some of the combinations of the metals with the compound radicals. Unfortunately the specimens had all suffered a partial decomposition on standing, and thus the results are not so trustworthy as might be desired.

Substance.	Formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Mercuric Methyl	C_2H_3Hg	1·5241	0·0431	43	1707	140	14
Mercuric Ethyl	C_4H_5Hg	1·5162	0·0416	?	2112	170	?
Stannic Ethyl-methyl ...	$\left.\begin{matrix}C_2H_3\\C_4H_5\end{matrix}\right\}Sn$	1·4550	0·0313	50	3727	256	41
Stannic Ethyl	$(C_4H_5)_2Sn$	1·4621	0·0301	50	3876	268	42

The specific index here, as in every preceding case, increases with the addition of C_2H_2; the great absolute influence of mercury on the rays of light makes itself manifest, as iodine did, in the inversion of the order of progress in regard to actual refraction and dispersion; it should be remembered that mercuric methyl contains close upon 87 per cent. of mercury.

It is worthy of notice that in the two series last given there occur the heaviest and about the lightest known liquid in the whole range of organic chemistry; and the light hydride of œnanthyl has a very high, and the heavy mercuric methyl a very low specific refractive energy.

All these series containing the compound radicals methyl and its congeners, agree in exhibiting a progressive change in refraction and dispersion with the advancing members of the series; but in which direction and to what extent depend on the other substances with which the radical is combined. Yet, if we regard not the actual indices, but these minus unity, divided by the specific gravity, we find an invariable increase as the series advances. The following Tables exhibit this:—

Specific Refractive Energy.

Radical.	Formula.	Alcohol.	Iodide.	Ether of acid.	Formiate.	Acetate.	Butyrate.	Oxalate.	Mercuric compound.	Stannic compound.	Hydride.
Methyl	$C_2 H_3$	4105	2359	3905	...	...	...	...	1707	3727*	...
Ethyl	$C_4 H_5$	4482	2614	4152	3905	4152	4402	3502	2112	3876	...
Propyl	$C_6 H_7$	...	2882	4333	...	...	...	...	...	...	...
Butyl	$C_8 H_9$	...	...	4402	...	...	...	...	...	...	...
Amyl	$C_{10} H_{11}$	4895	3213	4502	4432	4506	4724	4306	...	...	...
Œnanthyl ...	$C_{14} H_{15}$	...	...	4750	...	...	...	...	...	...	5499
Capryl	$C_{16} H_{17}$	5096	...	...	...	...	...	...	...	...	5522
Laurostearyl .	$C_{24} H_{25}$	...	...	4890	...	...	...	...	...	...	...

Specific Dispersion.

Radical.	Alcohol.	Iodide.	Ether of acid.	Acetate.	Mercuric compound.	Stannic compound.	Hydride.
Methyl	163	209	168	...	140	256*	...
Ethyl	190	218	178	178	170	268	...
Propyl	...	235	191	...	...	...	...
Butyl	...	...	191	...	...	...	...
Amyl	212	224	198	198	...	...	...
Œnanthyl	...	...	...	...	...	...	242
Capryl	237	...	...	...	...	...	236

Other Homologous Series.

It seemed desirable to examine other groups of homologous bodies in order to see whether there existed in them the same progressive change in the optical properties answering to the progressive additions of the increment $C_2 H_2$. Through the kindness of Mr. CHURCH and others we were able so to test the benzole, the pyridine, and the chinoline series.

Benzole Group.

Substance.	Formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Benzole	$C_{12} H_6$	1·4823	0·0419	60	5564	483	69
Toluole	$C_{14} H_8$	1·4835	0·0402	55	5584	464	63
Xylole...............	$C_{16} H_{10}$	1·4835	0·0408	58	5583	472	67
Cumole	$C_{18} H_{12}$	1·4819	0·0377	52	5547	425	60
Cymole	$C_{20} H_{14}$	1·4696	0·0312	53	5454	362	61

The first four members of this series, all of which were derived from coal-tar, bear a close resemblance to one another, instead of showing that progression in refractive and dispersive properties which marks all the series of the preceding group. Cymole gives lower numbers; but the difficulties arising from isomerism, which we shall shortly advert to, render any deduction from this group very doubtful.

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* This compound contains both methyl and ethyl.

Hydrate of Phenyl Series.

Allied to benzole and toluole are the two main constituents of creasote.

Substance.	Formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Hydrate of Phenyl......	$C_{12}H_5O, HO$	1·5344	0·0503	46	5034	475	43
Hydrate of Cresyl......	$C_{14}H_7O, HO$	1·5319	0·0467	33	5122	450	32

Pyridine Group.

Substance.	Formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Pyridine..................	$C_{10}H_5N$	1·4948	0·0447	55	5081	458	56
Picoline	$C_{12}H_7N$	1·4902	0·0427	56	5132	446	57
Lutidine..................	$C_{14}H_9N$	1·4909	0·0416	?	5244	448	?
Collidine..................	$C_{16}H_{11}N$	1·4946	0·0404	51	5370	444	53

In this series the actual refractive indices are nearly the same, but somewhat irregular; yet the density is progressive, and in such a manner that when the refractive power is divided by it, a series of increasing numbers is obtained. The dispersion decreases regularly and more rapidly than the density does, so that an addition of C_2H_2 yields a lower number in regard to specific dispersion, though a higher one in regard to specific refractive energy.

Chinoline Group.

Substance.	Formula.	Refractive index A.	Dispersion	Sensitiveness.	Specific refractive energy.	Specific dispersion	Specific sensitiveness.
Chinoline	$C_{18}H_7N$	1·5590	0·0631	55	5170	583	50
Lepidine	$C_{20}H_9N$	1·6045	0·0783	58	5639	730	54

In this case, unlike the pyridine group, which it so closely resembles (chemically speaking), the refraction and dispersion increase rapidly, whether we consider the absolute numbers or these divided by the specific gravity.

Lepidine, kindly given by its discoverer Mr. C. GREVILLE WILLIAMS, proves to be the most refractive organic liquid known, very nearly equalling bisulphide of carbon.

This examination of other homologous groups shows that *the influence of each addition of C_2H_2, which was observable throughout the series of the methyl group, does not necessarily hold good when we pass to substances of quite another type.*

POSTSCRIPT TO SECTION III., *February* 26, 1863.—A few days after the above was presented to the Royal Society, we observed, on taking up the last number of Poggendorff's

Annalen (cxvii. 353), a paper by M. LANDOLT "On the Refractive Indices of Fluid Homologous Compounds." He has examined, evidently with great care, the acids of the $C_n H_n O_4$ type, and finds that on ascending the series the refraction and dispersion increase, and the sensitiveness very slightly diminishes, with the exception of formic acid, which appears unconformable. This, however, is clearly due to the high density of that acid; and if we divide the numbers of LANDOLT by the densities, the anomaly disappears, and we obtain a series of valuations confirmatory in every way of those drawn out in the preceding Tables. LANDOLT measured, not A and H, but α and γ of the hydrogen light, which are nearly coincident with C and G of the solar spectrum.

Liquid.	Formula.	Specific refractive energy $(\mu_\alpha - 1) \div$ density.	Specific dispersion $(\mu_\gamma - \mu_\alpha) \div$ density.
Formic Acid	$C_2 H_2 O_4$	3024	91
Acetic Acid	$C_4 H_4 O_4$	3517	98
Propionic Acid	$C_6 H_6 O_4$	3860	105
Butyric Acid	$C_8 H_8 O_4$	4115	114
Valeric Acid	$C_{10} H_{10} O_4$	4318	121
Caproic Acid	$C_{12} H_{12} O_4$	4449	125
Œnanthic Acid	$C_{14} H_{14} O_4$	4569	129

This also shows, what is apparent both in our Tables given above and in some in Section V., that the amount of optical change is less between the higher than between the lower members of the series.

SECTION IV.—*The Refraction, Dispersion, and Sensitiveness of Isomeric Liquids.*

There are some isomeric bodies which we know differ from one another in their chemical constitution, while there are others to which we cannot yet assign any different arrangement of their elements. We have examined instances of both these classes.

Benzole Group.—This group offers a remarkable number of isomeric bodies differing slightly in their physical and chemical characters:—

Substance.	Boiling-point.	Density.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Benzole	80°·8	·8667	1·4823	0·0419	60	5564	483	69
Parabenzole	97·5	·8469	1·4814	0·0402	...	5684	474	...
Toluole	103·7	·8650	1·4739	0·0377	58	5478	435	67
Paratoluole	119·5	·8333	1·4715	0·0363	59	5658	435	70
Toluole (2nd specimen)	113	·8658	1·4835	0·0402 ?	55	5584	464 ?	63
Cumole (from Cuminic Acid)...	148·4	·8710	1·4825	0·0372	56	5547	427	65
Cumole (from Wood-spirit) ...	149·5	·8580	1·4631	0·0311	51	5400	363	59
Pseudo-cumole (from Coal-tar)	140·5	·8692	1·4819	0·0370	52	5544	425	60
Cymole (from Oil of Cumin)...	171	·8600	1·4696	0·316	53	5460	367	61
Cymole (from Camphor)	170	·8565	1·4693	0·317	48	5478	370	56

Here we have a variety of results:—isomeric bodies probably identical in refractive index, specific energy, and dispersion (cumole from cuminic acid, and Dr. H. MÜLLER'S pseudocumole, and the two cymoles); isomeric bodies nearly identical in their actual optical properties, but, on account of a difference in their densities, differing in their specific refractive energy (benzole and parabenzole, toluole and paratoluole); isomeric bodies identical in density, but differing in optical properties (two toluoles); isomeric bodies differing in density, and in each of the optical properties (two cumoles).

Essential Oil Group.—There are a large number of essential oils which consist of carbon and hydrogen in the proportion of 5 equivs. of the first to 4 equivs. of the second, and which differ from one another slightly in physical characters. Through Mr. PIESSE we obtained pure specimens of the crude oils, from which many of these hydrocarbons were prepared, carefully purified, and examined. They are arranged in the following Table according to their boiling-points. When two or more from different plants appeared to be identical, or nearly so, in all their known physical properties except odour, only one is given:—

Substance.	Boiling-point.	Specific gravity.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Hydrocarbon from Turpentine	160°	·8644	1·4612	0·0250	47	·5319	289	55
„ „ Anise	160	·8580	1·4608	0·0269	48	·5370	313	56
„ „ Thyme ...	160	·8635	1·4617	0·0262	48	·5346	326	56
„ „ Carraway ..	160	·8530	1·4610	0·0261	48	·5391	305	56
„ „ Bergamot..	173	·8467	1·4619	0·0297	49	·5456	350	57
„ „ Bay	173	·8510	1·4542	0·0257	47	·5337	302	55
„ „ Cloves ...	252	·9041	1·4898	0·0284	45	·5417	314	50
„ „ Cubebs ...	259	·9260	1·4950	0·0302	41	·5345	326	44

Here, as in the case of parabenzole and paratoluole, we have five isomeric bodies with sensibly the same refraction 1·461, although there are slight differences in the density and other properties, differences that seem to be real. The dispersion varies considerably; and the difference between turpentine, the least, and bergamot, the most dispersive, is only increased when the difference of density enters into the calculation. The sensitiveness seems the same in each of the five. The hydrocarbon from bay seems slightly lower in refraction; while those from cloves and cubebs, with much higher boiling-points and densities, are much higher in refraction and dispersion, and lower in sensitiveness. The specific refractive energies of the whole group do not differ widely.

Sugars.—We do not propose entering now on the subject of solutions, but we may state that solutions of cane-, grape- and honey-sugar, and gum, of the same strength, gave the same, or very nearly the same, amount of refraction and dispersion.

Compound Ethers.—It is well known that among the compound ethers pairs exist which have the same ultimate composition and similar density, but which are broken up by alkalies into different acids and alcohols. If it does not matter, as to its action on light, whether the increment $C_2 H_2$ be in the electro-positive or electro-negative element, these pairs will present an identity in refraction, dispersion, and sensitiveness. Such indeed seems to be the case.

Liquid.	Formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness
Valerianic Ether ...	$C_4 H_5 O, C_{10} H_9 O_3$	1·3908	0·0173	42	4502	199	48
Acetate of Amyl ...	$C_{10} H_{11} O, C_4 H_3 O_3$	1·3911	0·0172	43	4506	198	49

In this also we find that Professor DELFFS has preceded us, as far as refraction is concerned. He gives

Formic Ether	1·3570
Acetate of Methyl . .	1·3576

and

Valerianic Ether . . .	1·3904
Acetate of Amyl . . .	1·3904

Aniline and Picoline.—These two bodies have each the ultimate composition $C_{12} H_7 N$, but the action of chemical reagents proves that they are constructed very differently. The following comparison will show that they differ widely in refraction, dispersion, and sensitiveness, even when the difference in specific gravity is taken into account.

Liquid.	Rational formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Aniline	$\left.\begin{matrix} C_{12} H_5 \\ H_2 \end{matrix}\right\} N$	1·5650	0·0653	47	550	635	47
Picoline	$C_{12} H_7''' N$	1·4902	0·0428	56	513	448	59

It thus appears that *isomeric bodies are sometimes widely different in these optical properties; but in many cases, especially when there is close chemical relationship, there is identity also in this respect.*

SECTION V.—*The Effect of Chemical Substitution.*

The doctrine of types and substitution is fully recognized, at least by all the students of organic chemistry. It becomes a matter of interest to determine the amount of change in the optical properties which results from a replacement of one element by another, the type remaining the same. By this means we may attain to a knowledge of the influence of the individual elements on the rays of light transmitted by them. The data for such an inquiry ought to be very numerous, but those which we have already collected point to some conclusions.

The Substitution of Hydrogen by an Organic Radical.

In the following instance the substitution of amyl has not produced much change when the refractive power is divided by the density.

Liquid.	Formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Aniline	$\left.\begin{matrix}C_{12}H_5\\H\\H\end{matrix}\right\}N$	1·5650	0·0653	47	550	635	45
Amyl-aniline ...	$\left.\begin{matrix}C_{12}H_5\\C_{10}H_{11}\\H\end{matrix}\right\}N$	1·5130	0·0508	46	559	554	49

But a more interesting case is that of ether, alcohol, and water, which, according to WILLIAMSON, are of the same type,—a theory which the subjoined numbers favour, as the optical properties of the three are analogous, and those of alcohol are intermediate between those of water and ether when the proper allowance is made for the difference of specific gravity. The comparison is made, not at the same temperature, but at 30° below their respective boiling-points.

Liquid.	Williamson's formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness
Water............	$\left.\begin{matrix}H\\H\end{matrix}\right\}\Theta$	1·3203	143	24	·3272	145	25
Alcohol	$\left.\begin{matrix}Ȼ_2H_5\\H\end{matrix}\right\}\Theta$	1·3460	148	44	·4499	192	57
Ether	$\left.\begin{matrix}Ȼ_2H_5\\Ȼ_2H_5\end{matrix}\right\}\Theta$	1·3575	154	52	·4888	210	70

The Substitution of Hydrogen by Oxygen.

Of the effect of this replacement we have the following instances:-

Liquid.	Formula.	Refractive index A.	Dispersion.	Sensitiveness	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Alcohol	$\left.\begin{matrix}C_4H_5\\H\end{matrix}\right\}O_2$	1·3578	0·0151	41	·4482	190	52
Acetic Acid	$\left.\begin{matrix}C_4H_3O_2\\H\end{matrix}\right\}O_2$	1·3690	0·0172	37	·3483	162	35
Ether	$\left.\begin{matrix}C_4H_5\\C_4H_5\end{matrix}\right\}O_2$	1·3487	0·0149	51	·4868	208	71
Acetic Ether......	$\left.\begin{matrix}C_4H_5\\C_4H_3O_2\end{matrix}\right\}O_2$	1·3659	0·0157	48	·4152	178	55
Carvene	$C_{20}H_{16}$	1·4610	0·0261	48	·5391	305	56
Carvole	$C_{20}H_{14}O_2$	1·4886	0·0345	46	·5126	362	48
Eugenic Acid ...	$C_{20}H_{12}O_4$	1·5277	0·0495	42	·4945	463	39

In all these cases the replacement of hydrogen by oxygen has increased the actual refraction and dispersion, but decreased the specific refractive energy.

The two pairs, alcohol and acetic acid, ether and acetic ether, are interesting for comparison, since the chemical change is the same in the two, and it will be observed that the optical change is nearly the same also.

The three last bodies occur together, or under similar circumstances in nature, and form a series of which the second (carvole) is precisely intermediate in composition. It is also intermediate in optical properties.

The Substitution of Hydrogen by Peroxide of Nitrogen.

Of this the following instances have been examined:—

Substance.	Formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Benzole	$C_{12}H_6$	1·4823	0·0323*	60	5671	380*	70
Nitrobenzole	$C_{12}H_5(NO_4)$	1·5356	0·0501*	50	4610	423*	43
Dinitrobenzole † ..	$C_{12}H_4(NO_4)_2$	1·5486	0·0570*	48	3880	404*	34
Glycerine	$C_6H_8O_6$	1·4659	0·0191	25	3690	151	19
Nitroglycerine ...	$C_6H_5(NO_4)_3O_6$	1·4654	0·0264	45	2909	165	28
Amylic Alcohol...	$C_{10}H_{12}O_2$	1·4005	0·0174	39	4895	212	47
Nitrate of Amyl...	$C_{10}H_{11}(NO_4)O_2$	1·4065	0·0210	45	4061	202	44

Here we observe in the case of the benzole compounds a considerable increase of actual refraction, but in those of glycerine and amylic alcohol little change, while in each case there is a very marked increase of dispersion; yet when the numbers are divided by the density the refraction at least shows a decrease. The sensitiveness is greatly diminished by the substitution in the benzole compounds.

The Substitution of Hydrogen by Chlorine.

Substance.	Formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Benzole	$C_{12}H_6$	1·4823	0·0419	60	5564	483	69
Chlorobenzole	$C_{12}H_5Cl$	1·5135	0·0437	53	4634	394	48
Trichlorobenzole ...	$C_{12}H_3Cl_3$	1·5563	0·0502	46	3836	346	31

In this case there is also an increase both of actual refraction and dispersion, and a decrease of sensitiveness; but when the density of the chlorinated products is taken into account, the result is a great diminution in each of the optical properties.

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* In these cases $\mu_G-\mu_A$ is taken as the measure of dispersion, since H was invisible through nitrobenzole.

† Calculated on the assumption that the specific refractive energy of a mixture is the mean of the specific refractive energies of its constituents.

The Substitution of Chlorine by Bromine.

Of this we have the following instances:—

Substance.	Formula.	Refractive index A.	Dispersion.	Sensitiveness.	Specific refractive energy.	Specific dispersion.	Specific sensitiveness.
Terchloride of Phosphorus ...	PCl_3	1·5062	394	58	3489	271	40
Terbromide of Phosphorus ...	PBr_3	1·6730	808	64	2338	280	22
Chloroform	$C_2 HCl_3$	1·4400	220	54	2949	148	37
Bromoform	$C_2 HBr_3$	1·5554	418	55	2107	158	21
Bichloride of Chlorethylene...	$C_4 H_3 Cl, Cl_2$	1·4619	228	59	3259	160	42
Bibromide of Chlorethylene...	$C_4 H_3 Cl, Br_2$	1·5430	354	56	2415	157	25
Bibromide of Bromethylene...	$C_4 H_3 Br, Br_2$	1·5809	430	50	2220	164	19

Here in each case the bromine has greatly increased the refraction; but that this is owing to its great weight is evident from the fact that the specific refractive energy is much diminished. The dispersion is increased, but this is very nearly counterbalanced by the increase of weight. The sensitiveness is diminished, at least in the ethylene group.

It will be observed that, in each of the five cases mentioned in this section where there are two substitution products, the lower one is intermediate between the original substance and the higher product.

These observations put us in a position to consider the question, Does an element retain its special influence on the rays of light with whatever other elements it may be combined?

As the specific refractive energy of a mixture, or a feeble combination such as alcohol and water, is approximately the mean of the specific refractive energies of its constituents, we are prepared to find the rule holding good in more distinct chemical compounds. In the only case in which we have been able to try it among liquid elements, namely terbromide of phosphorus, the result was pretty near; but there is no doubt that chemical combination often greatly changes the optical as it does the other properties of elementary bodies.

Yet it is quite conceivable that an element may retain a specific influence on the rays of light through many if not all its compounds; and this view certainly finds some support in our experiments. Witness the fact of the great increase both of refraction and dispersion caused by the addition of nitrogen, whether combined with oxygen or not, to compounds of carbon and hydrogen (see Appendix I., Nos. 32, 52–55, 57–62, 75, 76). But when we look more narrowly at the numbers, we find this general permanence of special optical properties subject to much modification. Thus the difference in the optical properties of some isomeric bodies shows that such a generalizatien cannot be strictly true. Again, we may examine the different cases of replacement of hydrogen by oxygen mentioned above; and as in each case the atomic volume of the substitution-product is the same, or nearly so, as the atomic volume of the primary compound, the comparison is peculiarly legitimate. We infer at once that oxygen in combination is actually more refractive and dispersive than hydrogen, but that, if we take into account its much higher density, its specific refractive energy is less. But when we come to compare the different cases quantitatively, we see that a good deal depends on the peculiar nature of the compound. In the following Table the effect of the replacement of two equivalents of hydrogen by two of oxygen is given both with respect to refraction and dispersion. The specific refractive energy of ν is taken as the best exponent of the influence of refraction, and $\mu_H - \mu_A$, divided by density, is assumed, as before, for the specific dispersion.

Substance.	Atomic volume.	Specific refractive energy (ν).	Effect of substitution.	Specific dispersion.	Effect of substitution.
Alcohol	58	444	......	190	......
Acetic Acid	57	344	−100	162	− 28
Carvene	62	530	......	305	......
Carvole	63	501	− 29	362	+ 57
Eugenic Acid	64	474	− 27	463	+101

Hence we find that the substitution of two equivalents of oxygen for two of hydrogen has produced a far greater reduction in specific refractive energy in the case of alcohol than in that of the essential oil; while in specific dispersion it has produced a reduction in the one case, and an augmentation in the other.

As the main conclusions have been marked by italics under each head as they were arrived at, they are not recapitulated; but the following may be taken as a larger generalization deduced from them, and approximately if not absolutely true. *Every liquid has a specific refractive energy composed of the specific refractive energies of its component elements, modified by the manner of combination, and which is unaffected by change of temperature, and accompanies it when mixed with other liquids.* The product of this specific refractive energy, and the density, at any given temperature, is, when added to unity, the refractive index.

APPENDIX I.

Table of refractive indices of the lines A, D, H at different temperatures.

The initials in the column headed "From whose laboratory," are those of Messrs. G. B. BUCKTON, A. H. CHURCH, WARREN DE LA RUE, E. FRANKLAND, J. H. GLADSTONE, A. W. HOFMANN, W. ODLING, C. GREVILLE WILLIAMS, and A. W. WILLIAMSON.

The sign ? attached to a liquid denotes that the purity of the specimen is doubted.

An asterisk * attached to a degree of temperature signifies that the observations at that temperature were made on a different occasion to the observations at other temperatures.

Specific gravities not determined from the specimens examined are included in brackets.

No.	Liquid.	From whose laboratory.	Specific gravity.	Temperature of observation.	Refractive indices.		
					A.	D.	H.
1.	Methylic Alcohol	E. F.	0·7972 at 20 °C.	20°	1·3264	1·3299	1·3395
				37	1·3205	1·3238	1·3330
2.	Ditto from oxalate	J. H. G.	0·796 at 20	20	1·3268	1·3297	1·3396
				29·5	1·3230	1·3262	1·3359
3.	Amylic Alcohol	J. H. G.	0·8179 at 15·5	24·5	1·3988	1·4030	1·4161
				41	1·3924	1·3966	1·4093
4.	Caprylic Alcohol................	C. G. W.	0·8214 at 15·5	27	1·4157	1·4202	1·4351
				47	1·4073	1·4118	1·4266
5.	Iodide of Methyl................	A. W. W.	2·1912 at 20	23·5	1·5203	1·5307	1·5670
				29·5	1·5104	1·5202	1·5549
6.	Iodide of Ethyl	A. W. W.	1·9228 at 20	23·5	1·5003	1·5095	1·5420
				36	1·4918	1·5006	1·5326
				48	1·4841	1·4934	1·5250
7.	Iodide of Propyl................	A. W. H.	1·7117 at 20	8·5	1·5001	1·5095	1·5418
				20	1·4934	1·5024	1·5342
				30	1·4871	1·4963	1·5272
8.	Iodide of Amyl	J. H. G.	1·4950 at 20	17·5	1·4816	1·4892	1·5149
				37	1·4720	1·4797	1·5046
9.	Formic Ether	A. W. W.	0·9088 at 20	22	1·3540	1·3582	1·3694
				31	1·3500	1·3540	1·3652
				40	1·3456	1·3494	1·3608
10.	Acetic Ether	A. W. W.	0·8648 at 20	20	1·3645	1·3685	1·3798
				28	1·3606	1·3644	1·3755
11.	Acetic Ether	J. H. G.	0·8972 at 20	23·5	1·3653	1·3692	1·3809
				33	1·3606	1·3643	1·3757
				41	1·3563	1·3602	1·3711
12.	Propionic Ether	A. W. W.	0·8555 at 20	22·5	1·3696	1·3736	1·3860
				32	1·3657	1·3698	1·3819
				42	1·3610	1·3651	1·3771
13.	Butyric Ether................	A. W. W.	0·8778 at 20	23	1·3850	1·3888	1·4018
				40	1·3768	1·3808	1·3933
14.	Valerianic Ether................	J. H. G.	0·868 at 20	18	1·3916	1·3958	1·4089
				32·5	1·3856	1·3898	1·4024
15.	Acetate of Amyl................	J. H. G.	0·8680 at 20	24·5	1·3910	1·3950	1·4081
				34·5	1·3867	1·3905	1·4037
				44	1·3817	1·3859	1·3985
16.	Ditto, second specimen	J. H. G.	....	8·5	1·3944	1·3988	1·4113
				21·5	1·3886	1·3928	1·4058
				35	1·3820	1·3866	1·3990
17.	Acetate of Capryl?................	C. G. W.	....	27·5	1·4045	1·4092	1·4255
				40	1·3972	1·4020	1·4181

TABLE (continued).

No.	Liquid.	From whose laboratory.	Specific gravity.	Temperature of observation.	Refractive indices.		
					A.	D.	H.
				°C.			
18.	Hydride of Œnanthyl	J. H. G.	0·7090 at 20	9·5	1·3956	1·3996	1·4135
				22	1·3888	1·3931	1·4059
				36	1·3811	1·3854	1·3976
19.	Hydride of Capryl	J. H. G.	0·7191 at 20	9	1·4022	1·4065	1·4197
				28·5	1·3931	1·3972	1·4097
				41	1·3870	1·3911	1·4032
20.	Mercuric Methyl?	G. B. B.	(3·069)	8·5*	1·5274	1·5378	1·5726
				15*	1·5262	1·5355	1·5694
				26·5	1·5197	1·5296	1·5626
21.	Mercuric Ethyl?	G. B. B.	(2·444)	8·5*	1·5300	1·5397	1·5729
				24·5	1·5124	1·5217	1·5538
22.	Stannic Ethyl-methyl?	E. F.	1·222 at 20	19	1·4555	1·4625	1·4868
				34·5	1·4479	1·4747	1·4783
23.	Stannic Ethyl?	G. B. B.	1·192 at 20	23	1·4606	1·4673	1·4905
				35	1·4551	1·4621	1·4844
				48	1·4481	1·4549	1·4769
24.	Triethylarsine	A. W. H.	....	19·5	1·4598	1·4669	1·4919
				26·5	1·4588	1·4657	1·4906
25.	Acetic Acid	A. H. C.	1·0592 at 20	24	1·3674	1·3718	1·3846
				34·5	1·3635	1·3680	1·3803
				45	1·3596	1·3634	1·3757
26.	Acetone	J. H. G.	0·8117 at 15·5	25·5	1·3540	1·3580	1·3706
				40	1·3469	1·3512	1·3631
27.	Amylene	A. W. W.	0·7151 at 20	23	1·3832	1·3878	1·4028
				35	1·3786	1·3834	1·3982
28.	Carbonic Ether	E. F.	0·972 at 20	22	1·3773	1·3810	1·3936
				31	1·3746	1·3787	1·3898
				40	1·3692	1·3734	1·3846
29.	Boracic Ether	E. F.	0·876 at 20	22·5	1·3664	1·3698	1·3815
				40·5	1·3578	1·3604	1·3724
30.	Silicic Ether	E. F.	0·932 at 20	20	1·3781	1·3821	1·3940
				33·5	1·3724	1·3768	1·3881
31.	Salicylate of Methyl	J. H. G.	1·176 at 20	21	1·5206	1·5319	1·5810
				37	1·5140	1·5253	1·5735
32.	Nitrate of Amyl	W. D. L. R.	1·0008 at 20	10	1·4109	1·4157	1·4320
				22·5	1·4053	1·4097	1·4256
				36·5	1·3988	1·4035	1·4191
33.	Chloroform	J. H. G.	1·498 at 20	18	1·4411	1·4463	1·4630
				30	1·4346	1·4397	1·4561
				44	1·4253	1·4308	1·4471
34.	Bromoform	J. H. G.	2·636 at 12	15·5	1·5579	1·5674	1·5998
				29	1·5505	1·5598	1·5921
				39	1·5437	1·5531	1·5846
35.	Dutch Liquid	J. H. G.	....	21	1·4175	1·4221	1·4371
				38	1·4082	1·4126	1·4276
36.	Bibromide of Bromethylene	A. W. H.	2·616 at 20	18	1·5819	1·5915	1·6249
				39·5	1·5701	1·5787	1·6112
37.	Bibromide of Chlorethylene	W. D. L. R.	2·2477 at 20	13	1·5477	1·5559	1·5839
				24	1·5413	1·5495	1·5770
38.	Bichloride of Chlorethylene	W. D. L. R.	1·4177 at 20	13	1·4661	1·4714	1·4892
				29·5	1·4563	1·4619	1·4789
39.	Benzole	A. H. C.	0·8667 at 20	10·5	1·4879	1·4975	1·5305
				23	1·4806	1·4900	1·5225
				39	1·4703	1·4793	1·5108
40.	Parabenzole	A. H. C.	0·8469 at 20	20	1·4814	1·4903	1·5216
41.	Toluole	A. H. C.	0·865 at 20	25·5	1·4709	1·4794	1·5090
				32·5	1·4672	1·4755	1·5048
				39	1·4629	1·4710	1·5001

TABLE (continued).

No.	Liquid.	From whose laboratory.	Specific gravity.	Temperature of observation.	Refractive indices.		
					A.	D.	H.
			°C.				
42.	Paratoluole	A. H. C.	0·8333 at 20	28°	1·4667	1·4751	1·5030
				40	1·4590	1·4671	1·4944
43.	Toluole	W. O.	0·8658 at 20	14	1·4869	1·4957	1·5271
				33	......	1·4856	
44.	Xylole	W. O.	0·866 at 20	11	1·4888	1·4982	1·5300
				28	1·4788	1·4879	1·5192
				42	1·4716	1·4805	1·5116
45.	Cumole (from Cuminic Acid)	A. H. C.	0·871 at 20	7	1·4898	1·4983	1·5280
				27·5	1·4783	1·4864	1·5148
46.	Cumole (from impure Wood-spirit)	J. H. G.	0·858 at 20	8·5	1·4687	1·4759	1·5008
				24	1·4608	1·4680	1·4919
				34	1·4555	1·4634	1·4848
47.	Pseudocumole	W. D. L. R.	0·8692 at 20	12·5	1·4843	1·4932	1·5236
				35·5	1·4728	1·4812	1·5093
48.	Cymole	A. H. C.	0·861 at 20	8	1·4760	1·4834	1·5076
				29	1·4648	1·4717	1·4957
49.	Cymole (from Camphor)	W. D. L. R.	0·8565 at 20	12	1·4731	1·4803	1·5050
				26	1·4659	1·4729	1·4975
				36	1·4614	1·4684	1·4927
50.	Chlorobenzole	W. D. L. R.	1·108 at 20	9	1·5194	1·5290	1·5636
				27·5	1·5095	1·5189	1·5528
51.	Trichlorobenzole	A. W. H.	1·450 at 20	20	1·5563	1·5671	1·6065
				37	1·5495	1·5600	1·5983
52.	Nitrobenzole	A. H. C.	1·159 at 20	25	1·5331	1·5465	1·5832 G.
				38	1·5266	1·5399	1·5766 G.
53.	Dinitrobenzole in 2 equivs. of nitrobenzole	A. H. C.	1·267 at 20	23·5	1·5460	1·5600	1·5994 G.
				35	1·5404	1·5542	1·5932 G.
				56	1·5296	1·5425	1·5816 G.
54.	Aniline	A. H. C.	1·027 at 16	21·5	1·5644	1·5784	1·6297
				37	1·5567	1·5701	
				42	1·5537	1·5676	1·6183
				47	1·5520	1·5647	1·6145
55.	Amyl-aniline	A. W. H.	0·9177 at 20	23·5	1·5114	1·5222	1·5622
				42	1·5035	1·5138	1·5532
56.	Hydrate of Cresyl	W. D. L. R.	1·0364 at 20	11·5	1·5341	1·5454	1·5824
				32	1·5281	1·5377	1·5733
57.	Pyridine	A. W. H.	0·9738 at 20	21·5	1·4940	1·5030	1·5387
				36	1·4860	1·4951	1·5301
58.	Picoline	A. H. C.	(0·955)	22·5	1·4888	1·4980	1·5314
				37·5	1·4803	1·4890	1·5213
				52	1·4718	1·4807	1·5122
59.	Lutidine	A. H. C.	(0·936)	8·5*	1·4932	1·5028	1·5353
				22·5	1·4894	1·4987	1·5308
60.	Collidine	A. H. C.	(0·921)	23·5	1·4927	1·5013	1·5329
				45	1·4820	1·4907	1·5210
61.	Chinoline	C. G. W.	(1·081 at 10)	24	1·5567	1·5687	1·6198
				35	1·5466	1·5587	1·6084
				37	1·5496	1·5617	1·6124
62.	Lepidine	C. G. W.	1·072 at 15	21	1·6039	1·6189	1·6822
				47	1·5909	1·6054	1·6473 G.
63.	Hydrocarbon from Anise	A. H. C.	0·858 at 20	11	1·4653	1·4718	1·4921
				30	......	1·4625	
64.	„ Turpentine	J. H. G.	0·8644 at 20	10*	1·4669	1·4734	1·4934
				24	1·4596	1·4653	1·4845
				47	1·4487	1·4545	1·4730
65.	„ Carraway	J. H. G.	0·8529 at 20	14	1·4640	1·4701	1·4901
				37	1·4529	1·4589	1·4783

TABLE (continued).

No.	Liquid.	From whose laboratory.	Specific gravity.	Temperature of observation. °C.	Refractive indices.		
					A.	D.	H.
66.	Hydrocarbon from Thyme.........	J. H. G.	0·8635 at 20	25°	1·4594	1·4652	1·4856
				35·5	1·4545	1·4606	1·4805
67.	" Bay	J. H. G.	0·851 at 20	23	1·4545	1·4610	1·4818
				43	1·4468	1·4528	
68.	" Bergamot ...	J. H. G.	0·8467 at 20	26·5	1·4574	1·4640	1·4865
				38	1·4517	1·4578	1·4800
69.	" Cloves	J. H. G.	0·9041 at 20	17	1·4918	1·4985	1·5209
				28·5	1·4870	1·4936	1·5157
				39	1·4828	1·4892	1·5110
70.	" Cubebs	J. H. G.	0·927 at 20	10·5	1·4988	1·5055	1·5294
				20	1·4950	1·5014	1·5252
				31	1·4905	1·4977	1·5209
71.	Carvole	J. H. G.	0·9530 at 20	12·5	1·4913	1·4992	1·5270
				24·5	1·4862	1·4935	1·5196
				34	1·4812	1·4884	1·5145
72.	Eugenic Acid	J. H. G.	1·064 at 20	18	1·5285	1·5394	1·5780
				27·5	1·5244	1·5347	1·5722
73.	Camphor of Peppermint............	....	0·8786 at 43	30	1·4503	1·4553	1·4703
				43	1·4451	1·4505	1·4653
74.	Glycerine..............................	J. H. G.	1·261 at 17	20	1·4659	1·4705	1·4850
				30	1·4634	1·4680	1·4823
				48	1·4586	1·4631	1·4773
75.	Nitroglycerine?	J. H. G.	(1·60)	13·5	1·4683	1·4749	1·4947
				32·5	1·4596	1·4662	
76.	Nicotine	J. H. G.	1·026 at 18	18	1·5149	1·5234	1·5542
				32	1·5107	1·5194	1·5493
77.	Terbromide of Phosphorus	J. H. G.	2·88 at 20	25	1·6698	1·6866	1·7506
				36	1·6627	1·6792	1·7422
78.	Terchloride of Phosphorus	J. H. G.	1·453 at 20	25·5	1·5030	1·5118	1·5418
				38	1·4957	1·5042	1·5334
79.	Oxychloride of Phosphorus	J. H. G.	1·680 at 20	17	1·4810	1·4882	1·5118
				26	1·4756	1·4832	1·5067

The determinations of iodide of propyl were added, and those of acetic acid and terchloride of phosphorus were altered during the printing of the paper.

APPENDIX II.—Table of Refractive Indices.

The liquids in this Table are arranged according to their power of refracting the line A at 20° C.

Liquid.	Temp. °C.	Refractive indices.							
		A.	B.	C.	D.	E.	F.	G.	H.
Phosphorus..................................	35	2·0389	......	......	2·0746	......	2·1201	2·1710	2·2267?
Phosphorus in Bisulphide of Carbon..	?	1·9209	1·9314	......	1·9527	1·9744	1·9941	2·0361	2·0746
Terbromide of Phosphorus	25	1·6698	1·6752	......	1·6866	......	1·7083	1·7300	1·7506
Bisulphide of Carbon......................	11	1·6142	1·6207	1·6240	1·6333	1·6465	1·6584	1·6836	1·7090
Lepidine.......................................	21	1·6039	1·6094	......	1·6189	......	1·6403	1·6615	1·6822
Bibromide of Bromethylene	18	1·5819	1·5851	......	1·5915	......	1·6037	1·6149	1·6249
Rectified Oil of Cassia	28	1·5649	1·5699	1·5727	1·5801	1·5909	1·6014	1·6244	......
Aniline	21·5	1·5644	1·5684	......	1·5774	......	1·5951	1·6125	1·6297
Chinoline	24	1·5567	1·5617	......	1·5687	......	1·5879	1·6030	1·6198
Trichlorobenzole	20	1·5563	1·5602	......	1·5671	......	1·5809	1·5945	1·6065

TABLE (continued).

Liquid.	Temp.	Refractive indices.							
		A.	B.	C.	D.	E.	F.	G.	H.
	°C.								
Nitrobenzole	25	1·5331	1·5374	1·5398	1·5465	1·5554	1·5643	1·5832	......
Hydrate of Phenyl	13	1·5377	1·5416	1·5433	1·5488	1·5564	1·5639	1·5763	1·5886
Hydrate of Cresyl	11·5	1·5341	1·5377	......	1·5445	......	1·5573	1·5699	1·5813
Eugenic Acid	18	1·5285	1·5321	1·5341	1·5394	1·5464	1·5528	......	1·5780
Mercuric Methyl	26·5	1·5197	1·5232	......	1·5296	......	1·5368	1·5526	1·5626
Salicylate of Methyl	21	1·5206	1·5241	1·5263	1·5319	1·5402	1·5478	1·5640	1·5810
Iodide of Methyl	16	1·5203	1·5234	......	1·5307	1·5377	1·5440	1·5558	1·5670
Mercuric Ethyl	8·5	1·5300	1·5333	......	1·5397	......	1·5518	1·5634	1·5729
Nicotine	18	1·5149	1·5174	......	1·5234	......	1·5346	1·5449	1·5542
Chlorobenzole	9	1·5194	1·5223	......	1·5290	......	1·5418	1·5530	1·5636
Amyl-aniline	23·5	1·5114	1·5150	1·5168	1·5222	1·5292	1·5361	1·5491	1·5622
Terchloride of Phosphorus	23·5	1·5052	1·5088	......	1·5148	......	1·5252	1·5357	1·5446
Iodide of Ethyl	23·5	1·5003	1·5034	......	1·5095	1·5156	1·5214	1·5321	1·5420
Rectified Oil of Santal-wood	25·5	1·4954	1·4977	......	1·5015	......	1·5093	1·5161	1·5223
Hydrocarbon from Cubebs	10·5	1·4988	1·5012	......	1·5055	......	1·5145	1·5227	1·5294
Pyridine	21·5	1·4940	1·4967	......	1·5030	......	1·5155	1·5278	1·5387
Lutidine	22·5	1·4894	1·4924	......	1·4987	......	1·5100	1·5204	1·5308
Collidine	23·5	1·4927	1·4958	......	1·5013	......	1·5127	1·5232	1·5329
Hydrocarbon from Cloves	17	1·4918	1·4944	......	1·4985	......	1·5064	1·5140	1·5209
Pseudocumole	12·5	1·4843	1·4872	......	1·4932	......	1·5040	1·5146	1·5236
Iodide of Amyl	17·5	1·4816	1·4843	......	1·4892	1·4941	1·4987	1·5074	1·5149
Oxychloride of Phosphorus	17	1·4810	1·4840	......	1·4882	......	1·4967	1·5047	1·5118
Benzole	10·5	1·4879	1·4913	1·4931	1·4975	1·5036	1·5089	1·5202	1·5305
Toluole	14	1·4869	1·4898	......	1·4957	......	1·5072	1·5174	1·5271
Cymole	29	1·4648	1·4671	......	1·4717	1·4766	1·4808	1·4866	1·4957
Nitroglycerine	13·5	1·4683	1·4706	......	1·4749	......	1·4824	1·4899	1·4947
Hydrocarbon from Portugal	25	1·4617	1·4640	......	1·4684	......	1·4758	1·4826	1·4894
Cumole (2nd specimen)	8·5	1·4687	1·4709	......	1·4759	......	1·4853	1·4936	1·5008
Stannic Ethyl	23	1·4606	1·4629	......	1·4673	......	1·4758	1·4838	1·4905
Bichloride of Chlorethylene	13	1·4661	1·4680	......	1·4714	......	1·4784	1·4841	1·4892
Hydrocarbon from Turpentine	24	1·4596	1·4616	......	1·4653	1·4691	1·4724	1·4790	1·4845
Hydrocarbon from Carraway	24	1·4594	1·4615	......	1·4652	......	1·4724	1·4789	1·4844
Hydrocarbon from Bergamot	26·5	1·4574	1·4598	......	1·4640	......	1·4721	1·4798	1·4865
Rectified Oil of Citronella	19	1·4598	1·4619	......	1·4655	......	1·4730	1·4795	1·4860
Hydrocarbon from Bay	23	1·4545	1·4567	......	1·4610	......	1·4690	1·4756	1·4818
Stannic Ethyl-methyl	19	1·4555	1·4578	1·4590	1·4625	1·4674	1·4716	1·4795	1·4868
Chloroform	10	1·4438	1·4457	1·4466	1·4490	1·4526	1·4555	1·4614	1·4661
Caprylic Alcohol	9·5	1·4230	1·4246	1·4255	1·4279	1·4309	1·4338	1·4386	1·4429
Nitrate of Amyl	10	1·4109	1·4127	......	1·4157	......	1·4219	1·4274	1·4320
Amylic Alcohol	25	1·3981	1·3999	......	1·4024	......	1·4078	1·4122	1·4161
Hydride of Capryl	9	1·4022	1·4037	......	1·4065	......	1·4076	1·4141	1·4197
Hydride of Œnanthyl	9·5	1·3956	1·3968	......	1·3996	......	1·4045	1·4087	1·4135
Acetate of Amyl	8·5	1·3944	1·3958	......	1·3998	......	1·4035	1·4077	1·4113
Butyric Ether	23	1·3850	1·3864	......	1·3888	......	1·3938	1·3981	1·4018
Amylene	8	1·3850	1·3866	......	1·3896	......	1·3944	1·3992	1·4033
Carbonic Ether	22	1·3773	1·3785	......	1·3810	......	1·3856	1·3896	1·3936
Propionic Ether	22·5	1·3696	1·3713	......	1·3736	......	1·3785	1·3827	1·3860
Boracic Ether	22·5	1·3664	......	......	1·3698	......	1·3742	1·3785	1·3815
Acetic Ether	20	1·3645	1·3658	......	1·3685	......	1·3728	1·3766	1·3798
Alcohol	15	1·3600	1·3612	1·3621	1·3638	1·3661	1·3683	1·3720	1·3751
Acetone	25·5	1·3540	1·3554	......	1·3582	......	1·3629	1·3670	1·3706
Formic Ether	22	1·3540	1·3553	......	1·3582	......	1·3627	1·3666	1·3694
Ether	15	1·3529	1·3545	1·3554	1·3566	1·3590	1·3606	1·3646	1·3683
Water	15	1·3284	1·3300	1·3307	1·3324	1·3347	1·3366	1·3402	1·3431
Methylic Alcohol	20	1·3264	1·3277	......	1·3299	......	1·3330	1·3369	1·3395

P.S. [*Received May* 28.]—It was not till after this paper was read that we became aware of the existence of an elaborate treatise by Dr. SCHRAUF, "On the Dependence of the Velocity of Light on the Density of Bodies," in Poggendorff's Annalen, cxvi. 193, in which he investigates the question mathematically, taking as the basis of his calculations our former experiments, and those of DEVILLE, WEISS, and others. Our own line of thought has many points of analogy with that pursued by him, but there is this difference in the conclusion: he believes that $\frac{A^2-1}{D}$ and $\frac{B}{D^2}$ $\left(\text{or in our notation } \frac{\nu^2-1}{D} \text{ and } \frac{\kappa}{D^2}\text{, D being the density}\right)$ are the constants at all temperatures, and are the functions on which depend the optical properties of mixtures; while we are led by our new experiments to accord that quality rather to $\frac{\nu-1}{D}$, and to doubt any such simple formula as $\frac{\kappa}{D^2}$ for the changes of dispersion. To this point we propose to recur at some future period if we have the opportunity.

There is one point in reference to our method of observation which seems to call for a remark. SCHRAUF thinks that there is a slight change in the refringent angle of our prism on its being heated. Now our hollow prism has glass ends as well as glass sides; but supposing such a change actually occurs, it is evident it will produce a uniform error running through all our observations in Section I. This may be the reason why at high temperatures the observed is almost always less than the calculated index; but as bisulphide of carbon and water agree so closely with either his or our theory, this source of error must be extremely minute.

We await with curiosity the publication of the experiments referred to in Dr. SCHRAUF's short note, "On the Velocity of Light and Chemical Composition," in the April Number of Poggendorff's Annalen.

XV. *Researches into the Chemical Constitution of Narcotine, and of its Products of Decomposition.*—Part I.

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§ I.—COMPOSITION OF NARCOTINE AND COTARNINE.

THE existence of Narcotine was indicated by DEROSNE as early as the year 1803, but its chemical nature remained almost entirely unknown until ROBIQUET *, in 1817, showed that it belonged to the class of vegetable alkaloids. Numerous analyses of narcotine were subsequently published by DUMAS and PELLETIER †, PELLETIER ‡, LIEBIG §, REGNAULT ||, and others; but its composition was first determined to the general satisfaction of chemists by BLYTH ¶, who, in 1844, proposed the formula $C_{46} H_{25} NO_{14}$, supporting it by numerous analyses of the double hydrochlorate of narcotine and platinum, and showing, at the same time, that it accorded well with the most trustworthy results of previous investigators, and also accounted satisfactorily for the formation of the remarkable decomposition-products of narcotine discovered by himself and by WÖHLER **. Since the publication of BLYTH'S investigation, the formula which he proposed has been generally adopted as expressing correctly the composition of this base. More recently, however, WERTHEIM ††, founding his opinion chiefly on the composition of the volatile bases obtained by distilling narcotine with potash, has maintained the existence of two additional varieties of narcotine, homologous with that examined by BLYTH, and represented respectively by the formulæ $C_{44} H_{23} NO_{14}$ and $C_{48} H_{27} NO_{14}$; while HINTERBERGER ‡‡ has analysed a compound of chloride of mercury with what he considers as a fourth variety, still homologous with the preceding, and represented by the formula $C_{42} H_{21} NO_{14}$.

Such being the results of previous investigations, it was plainly necessary to begin any new research into the chemical nature of narcotine by endeavouring to ascertain, by direct analysis, whether there existed in reality more than one kind of narcotine, and,

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* Ann. de Chim. et de Phys. vol. v. p. 275. † Ibid. vol. xxiv. pp. 186 and 191. ‡ Ibid. vol. l. p. 271.

§ Jahresbericht über die Fortschritte der physischen Wissenschaften, von Jacob Berzelius, vol. xi. (1832) p. 231.

|| Ann. de Chim. et de Phys. vol. lxviii. p. 138.

¶ Memoirs and Proceedings of the Chemical Society, vol. ii. p. 167.

** Annalen der Chemie und Pharmacie, vol. l. p. 1. †† Journal für praktische Chemie, vol. liii. p. 431.

‡‡ Annalen der Chemie und Pharmacie, vol. lxxxii. p. 312.

if so, which of these kinds was being operated upon. The following are the results obtained on analysing specimens of narcotine procured from several distinct sources.

A. *Narcotine prepared by* Mr. Morson *from a mixture of various kinds of opium*[*].

I. 0·6311 grm. substance gave 1·4789 grm. carbonic acid and 0·3438 grm. water.
II. ·2372 grm. substance gave ·5552 grm. carbonic acid and ·1234 grm. water.
III. ·2244 grm. substance gave ·5266 grm. carbonic acid and ·1148 grm. water.
IV. ·3258 grm. substance gave ·7576 grm. carbonic acid and ·1682 grm. water.
V. ·7470 grm. substance gave ·1724 grm. platinum.
VI. ·5708 grm. substance gave ·1338 grm. platinum.
VII. ·5555 grm. substance gave ·1304 grm. platinum.
VIII. ·6153 grm. substance gave ·1482 grm. platinum.

B. *Narcotine prepared from Turkish opium by* Dr. G. Merck *of Darmstadt.*

0·3584 grm. substance gave 0·8344 grm. carbonic acid and 0·1847 grm. water.

C. *Narcotine prepared from Egyptian opium by* Dr. G. Merck.

0·3172 grm. substance gave 0·7451 grm. carbonic acid and 0·1630 grm. water.

D. *Narcotine prepared from Persian opium by* Dr. G. Merck.

0·3192 grm. substance gave 0·7460 grm. carbonic acid and 0·1660 grm. water.

E. *Narcotine prepared from Egyptian opium by* Mr. Morson.

0·3460 grm. substance gave 0·8094 grm. carbonic acid and 0·1790 grm. water.

F. *Narcotine from Turkish opium obtained from* Messrs. Hopkin *and* Williams.—(The whole of this sample of narcotine was divided into three portions by fractional crystallization from alcohol: the following analyses were made with crystals of the first and last crops respectively, each of which was considerably smaller in quantity than the intermediate crop.)

I. 0·4046 grm. substance gave 0·9503 grm. carbonic acid and 0·2085 grm. water.
II. ·3366 grm. substance gave ·7843 grm. carbonic acid and ·1730 grm. water.

These numbers correspond to the following percentages:—

	A.				B.	C.	D.	E.	F.		Mean.
	I.&V.	II.&VI.	III.&VII.	IV.&VIII.					I.	II.	
Carbon .	63·91	63·83	64·00	63·42	63·49	64·01	63·74	63·80	64·05	63·55	63·78
Hydrogen	6·05	5·77	5·69	5·74	5·73	5·71	5·77	5·75	5·72	5·71	5·76
Nitrogen .	3·26	3·31	3·32	3·40	——	——	——	——	——	——	3·32
Oxygen .	——	——	——	——	——	——	——	——	——	——	27·14

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[*] This was the material that served for the greater number of the experiments which follow.

The percentages required by the formula $C_{46}H_{25}NO_{14}$ are—

Carbon	**64·63**
Hydrogen	5·85
Nitrogen	3·28
Oxygen	26·24

from which the above results differ very considerably: we are therefore led to regard the formula $C_{44}H_{23}NO_{14}$, or preferably $\text{Ꞓ}_{22}H_{23}N\text{Ɵ}_7$, which accords much better with our analyses, as more exact: it corresponds to the following proportions in 100 parts:—

Carbon	63·92
Hydrogen	5·57
Nitrogen	3·39
Oxygen	27·12

And since all the different samples of narcotine which we were able to procure gave on analysis identical results *, we conclude that there is no sufficient evidence of the existence of more than one kind of narcotine, especially as we believe that the observations which formerly gave rise to the contrary opinion are explained by an experiment to be hereinafter described.

If the formula $\text{Ꞓ}_{22}H_{23}N\text{Ɵ}_7$ be adopted for narcotine, it is impossible to account in

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* The difference between the highest quantity of carbon (F. I.) and the lowest (A. IV.) given by the whole series of analyses amounts to 0·63 per cent.; but this extreme difference among the analyses of seven distinct specimens is scarcely perceptibly greater than the difference (0·58 per cent.) between the highest and lowest (A. III. & IV.) numbers resulting from the analysis of the same specimen. Although these differences are rather high, we believe that they are due entirely to accidental errors of experiment, and chiefly to the difficulty of ensuring the complete combustion of narcotine. No analysis of that substance ever gave us more than 64·05 per cent. of carbon, but several gave smaller quantities than the lowest quoted above (for instance, 62·7, 62·8, 63·1 per cent.); but in these cases traces of unoxidized carbon could always be detected in the combustion-tube after the experiment. The combustions were made with oxide of copper, and, like all those given in this paper, were finished in a stream of oxygen.

The difference between the percentage of nitrogen required by the old formula of narcotine (Blyth's) and by that which we adopt being no greater than the probable errors of analysis, it was not thought worth while to determine that element in more than one sample.

For the sake of comparison, we append here the results obtained by other chemists who have analysed narcotine or its salts.

Analyses of Narcotine.

	Dumas and Pelletier (1823).	Pelletier (1832).	Liebig (1832).	Regnault (1838).			Hofmann (1844).	
Carbon . .	68·88	65·16	64·09	64·01	64·51	64·09	64·53	——
Hydrogen .	5·91	5·45	5·50	5·96	5·99	5·73	6·21	——
Nitrogen . .	7·21	4·31	2·51	3·46	3·52	——	3·30	2·92

	Mulder (1838).				Varrentrapp and Will (1841).	
Nitrogen	3·03	2·98	2·73	2·44	3·77	3·72

Note.—The analyses of Dumas and Pelletier are calculated with the old equivalent of carbon (6·11); Liebig's analysis is given as recalculated by Blyth (Mem. Chem. Soc. vol. ii. p. 167) with the corrected equivalent; Regnault's analyses are similarly recalculated from the original weighings. For all the foregoing, any simple manner for the well-known decomposition of that substance, under the influence of oxidizing agents, into opianic acid and cotarnine, unless the received formula of either one or other of these bodies be also modified.

It seemed impossible to doubt the accuracy of the formula $C_{10}\,H_{10}\,O_5$ for opianic acid, but nevertheless, for the sake of greater certainty, we have repeated the analysis of this acid, though with no other result than a fresh confirmation of the admitted formula.

I. 0·2494 grm. acid gave 0·5198 grm. carbonic acid and 0·1066 grm. water.
II. ·1921 grm. acid gave ·4030 grm. carbonic acid and ·0855 grm. water.
III. ·2492 grm. acid gave ·5260 grm. carbonic acid and ·1117 grm. water.

	Calculated.		Found.			
			I.	II.	III.	Mean.
C_{10}	120	57·14	56·84	57·21	57·57	57·21
H_{10}	10	4·76	4·75	4·95	4·98	4·89
O_5	80	38·10	——	——	——	37·90
$C_{10}\,H_{10}\,O_5$	210	100·00				100·00

On the other hand, our analyses of cotarnine lead us to adopt the formula $C_{12}\,H_{13}\,NO_3$

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see references †, ‡, §, ‖, page 1; for HOFMANN's analysis see BLYTH's memoir (*loc. cit.*); MULDER's results are quoted from VARRENTRAPP and WILL (Ann. Chem. Pharm. vol. xxxix. p. 282).

Analyses of CHLOROPLATINATE OF NARCOTINE.

	Liebig (1838).		Regnault (1839).		Hofmann (1844).		Blyth (1844).		How (1854).
Platinum . .	14·51	14·64	15·81	15·97	15·85	15·80	15·65	15·73	15·88

	Blyth (1844).		Wertheim (1856).				Calculated.	
							C_{22}.	C_{23}.
Carbon . .	43·72	43·56	42·92	42·27	42·44	43·17	42·62	43·55
Hydrogen .	4·17	4·30	3·94	4·12	4·14	4·15	3·87	4·14
Platinum .	16·00	15·95	15·95	——	——	15·72	15·98	15·62

Note.—LIEBIG, Ann. Pharm. vol. xxvi. p. 52; REGNAULT, Ibid. vol. xxix. p. 60; HOFMANN and BLYTH, Mem. Chem. Soc. vol. ii. pp. 166, 167; HOW, Trans. Roy. Soc. Edinb. vol. xxi. p. 31; WERTHEIM, Traité de Chimie Organique, par M. CHARLES GERHARDT, vol. iv. p. 67.

Analyses of HYDROCHLORATE OF NARCOTINE.

	Robiquet (1832).	Regnault (1838).			Regnault (1839).	Calculated.	
						C_{22}.	C_{23}.
Hydrochloric acid . .	8·22	8·14	8·17	7·96	7·43	8·12	7·87

Note.—ROBIQUET, Ann. de Chim. et de Phys. vol. i. p. 231; REGNAULT, Ibid. vol. lxviii. p. 138, and Ann. Pharm. vol. xxix. p. 60.

Analysis of CHLOROMERCURATE OF NARCOTINE.

	Hinterberger (1852). Ann. Chem. Pharm. lxxxii. 312.	Calculated.	
		$C_{22}\,H_{23}\,NO_7,\ H\,Cl,\ Hg\,Cl.$	$C_{23}\,H_{25}\,NO_7,\ H\,Cl,\ Hg\,Cl.$
Carbon	43·64	45·13	46·08
Hydrogen	3·90	4·10	4·34
Mercury	18·02	17·09	16·69

in preference to the usual formula, $Ȼ_{13}H_{13}NƟ_3$, which represents that substance as containing one atom of carbon more. The following analyses were made with crystallized cotarnine prepared by the action of dilute nitric acid on narcotine, the method recommended by ANDERSON for the preparation of pure cotarnine: it had a slight yellow or buff colour which it was found impossible to remove.

I. 0·3473 grm. substance gave 0·7746 grm. carbonic acid and 0·2043 grm. water.
II. ·2261 grm. substance gave ·5020 grm. carbonic acid and ·1324 grm. water.
III. ·5633 grm. substance gave ·2320 grm. platinum.
IV. ·5057 grm. substance gave ·2113 grm. platinum.

	Calculated.		Found.					Calculated.
			I.	II.	III.	IV.	Mean.	$Ȼ_{13}H_{15}NƟ_4$.
C_{12}	144	60·76	60·83	60·55	——	——	60·70	62·65
H_{15}	15	6·33	6·54	6·50	——	——	6·52	6·02
N	14	5·90	——	——	5·82	5·91	5·87	5·62
$Ɵ_4$	64	27·01	——	——	——	——	26·86	25·71
$Ȼ_{12}H_{13}NO_3, HƟ_2$	237	100·00					100·00	100·00

V. 0·6619 grm. substance lost 0·0493 grm. water at 110° C.
VI. ·5450 grm. substance lost ·0406 grm. water at 110° C.

	Calculated.		Found.		Calculated.
			V.	VI.	$Ȼ_{13}$.
$C_{12}H_{13}NO_3$	219	92·41	——	——	92·77
H_2O	18	7·59	7·45	7·45	7·23
$Ȼ_{12}H_{13}NO_3, H_2O$	237	100·00			100·00

These results are confirmed by the following determinations of the proportion of platinum in chloroplatinate of cotarnine, dried *in vacuo* over sulphuric acid[*]:

I. 0·4312 grm. salt gave 0·1005 grm. platinum.
II. ·3161 grm. salt gave ·0732 grm. platinum.
III. ·4000 grm. salt gave ·0924 grm. platinum.

	Calculated.		Found.		
	$Ȼ_{12}$.	$Ȼ_{13}$.	I.	II.	III.
Platinum per cent	23·3	22·6	23·3	23·2	23·1

The adoption of the formula $Ȼ_{12}H_{13}NƟ_3$[†] for cotarnine enables us to represent its formation by the action of oxidizing agents on narcotine, by means of a very simple equation,—

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[*] Five platinum-determinations, made with chloroplatinate of cotarnine that had been dried in the water-bath, gave quantities of platinum varying from 23·5 to 23·9 per cent., numbers which differ even more than those quoted in the text from that required by the hitherto admitted formula. The substances used for analyses II. and III. were prepared from two distinct specimens of cotarnine: the salt II. was, moreover, precipitated in presence of excess of bichloride of platinum, salt III. in presence of excess of hydrochlorate of cotarnine.

[†] The formula originally proposed for cotarnine by WÖHLER, but which he himself only regarded as provisional, was $C_{26}H_{13}NO_5$; BLYTH's formula was $C_{25}H_{13}NO_6$; the formula $Ȼ_{13}H_{13}NƟ_3$ was proposed by GERHARDT (Précis de Chimie Organique, 1845, vol. ii. p. 298).

$$\underset{\text{Narcotine.}}{\text{Ȼ}_{22}\mathrm{H}_{23}\mathrm{NO}_7} + \Theta = \underset{\text{Cotarnine.}}{\text{Ȼ}_{12}\mathrm{H}_{13}\mathrm{N}\Theta_3} + \underset{\text{Opianic Acid.}}{\text{Ȼ}_{10}\mathrm{H}_{10}\Theta_5},$$

and, as will be shown hereafter, it is further supported by the manner in which cotarnine is decomposed by nitric and hydrochloric acids.

Assuming the accuracy of the formulæ here proposed as sufficiently established, we next endeavoured to ascertain the chemical constitution of narcotine by studying the action of various reagents on cotarnine and on its other principal derivative, opianic acid. Hitherto we have made but few experiments on the action of reagents on narcotine itself, from the conviction that their results would hardly be intelligible without a previous knowledge of the transformations of the bodies into which it splits up with so much ease.

It appears, as the result of a good many trials, that the following is perhaps the most advantageous method of transforming narcotine into opianic acid and cotarnine. 100 grms. of narcotine are dissolved in a considerable excess of dilute sulphuric acid (150 grms. acid and 1500 grms. water), the solution is heated to boiling, and 150 grms. finely powdered peroxide of manganese* is then added as quickly as possible, care being taken that it does not cause the liquid to froth over; when the whole quantity of peroxide of manganese has been added, the mixture is quickly filtered through a funnel surrounded by boiling water. The filtrate, on cooling, becomes half-solid, from the separation of crystals of opianic acid, and by twice recrystallizing this product from boiling water it is obtained sufficiently pure for most purposes, though still retaining a slight brownish-yellow colour, which can be got rid of, when needful, by boiling the acid with a dilute solution of hypochlorite of sodium, in the manner indicated by WÖHLER.

The cotarnine contained as sulphate, together with a large quantity of sulphate of manganese in the original mother-liquor of the opianic acid, may be conveniently extracted therefrom as follows. The mother-liquor is mixed with a quantity of milk of lime sufficient to neutralize the free sulphuric acid and to precipitate part of the manganese, then, without filtering, an excess of carbonate of sodium is added and the whole is heated to boiling for a few minutes, in order to precipitate completely the manganese and lime (or the neutralization and precipitation may be effected entirely by means of carbonate of sodium, without using lime). The mixture is then filtered, first through calico and afterwards, if needful, through paper; the filtrate is neutralized with dilute sulphuric acid, evaporated rapidly to a small bulk, allowed to cool completely, poured off from any sulphate of sodium that may have crystallized out, and finally mixed with an excess of strong potash- or soda-lye, whereby the cotarnine is precipitated.

It is a matter of some importance for the successful preparation of cotarnine in this manner, that its extraction from the original mother-liquors containing it should be proceeded with without much delay, otherwise the quantity obtained is small and its colour dark. Under all circumstances the odour of ammonia, or methylamine, is perceptible on adding potash for the final precipitation of the cotarnine; and this odour is strongest when a long time has elapsed during the previous parts of the process. The decomposition which is thus indicated appears to lessen the amount of product obtained in two ways: not only is the quantity of cotarnine existing in the solution diminished by the amount that has suffered decomposition, but the resulting volatile alkali seems to prevent the precipitation of the unaltered base.

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* 100 parts of our oxide of manganese corresponded to 60 parts of pure peroxide.

Cotarnine that is very dark-coloured may be decolorized, to a considerable extent, by dissolving it in hydrochloric acid, digesting the solution upon bone-charcoal, and re-precipitating with potash; but we have never, by any process, succeeded in obtaining cotarnine quite without colour.

§ II.—DECOMPOSITIONS AND DERIVATIVES OF OPIANIC ACID.

1. *Action of Hydrochloric Acid.*—When opianic acid is heated with three or four times its weight of ordinary strong hydrochloric acid, either to 100° C. in a sealed tube, or to the boiling-point of the acid in an open vessel, it is decomposed with evolution of chloride of methyl and carbonic acid, and on evaporating the remaining solution, first on the water-bath, then at the common temperature over lime and sulphuric acid, a residue is obtained which appears to consist of three distinct crystallizable substances; but the ease with which at least one of them undergoes alteration by heat or exposure to air, and the difficulty of completely separating them from each other, have hitherto prevented us from making a thorough examination of these products, and we therefore postpone to a future communication any further description of them*.

Opianic acid seems to undergo a similar decomposition when boiled with fuming *hydriodic acid*; in this case iodide of methyl is given off, without separation of iodine, but we have not found it possible to remove the excess of hydriodic acid from the residue without destroying the organic constituents.

2. *Action of Potash.*—When opianic acid is mixed with a large excess of potash-lye, and the liquid is evaporated nearly to dryness, the acid splits up into meconin and hemipinic acid. No blackening occurs unless too small a quantity of potash is employed; but when a certain degree of concentration is reached, the decomposition appears to take place almost instantaneously: the mixture, which up to that point is a thick fluid,

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* According to WÖHLER, opianate of ethyl is not formed when a solution of opianic acid in alcohol is saturated with hydrochloric acid. The experiments described in the text made it seem probable that the non-formation of opianic ether under these circumstances might be owing to the decomposition of the opianic acid; but, on sealing up opianic acid with an alcoholic solution of hydrochloric acid and heating it in the water-bath for an hour, it was found that opianate of ethyl was formed in abundance. The ether was precipitated by pouring the contents of the tube into water, and purified by crystallization from alcohol. Thus obtained, it formed hemispherical masses of brilliantly white radiating needles, insoluble in water or dilute alkalis, melting to a colourless oil under hot water, and easily soluble in alcohol and ether. In the dry state it melts at 88° C.

suddenly becomes nearly solid, and as soon as this change has occurred, no more opianic acid can be detected in the mass. The decomposition may even be effected, without the application of external heat, by simply mixing opianic acid with a very strong and warm solution of potash.

In an experiment made in order to ascertain the proportions in which meconin and hemipinic acid are formed in this reaction, 42 grms. of opianic acid yielded

	13·5 grms.	pure meconin,
	18·5 grms.	pure hemipinic acid (weighed as hemipinate of ammonium),
	4·7 grms.	meconin and hemipinic acid mixed,
	5·0 grms.	uncrystallizable residue.
Total . .	41·7 grms.	

In another experiment, a small quantity of the mixture of opianic acid and potash was boiled down in a small flask, fitted with a glass tube about a yard long, bent vertically downwards just outside the cork, and dipping by its lower end into mercury. When the reaction had taken place, the flask was allowed to cool, and the mercury then rose in the vertical tube to within about an inch of the height of the barometer at the time; thus proving that no permanent gas had been evolved. Hence the following equation—according to which 42 grammes opianic acid should yield 19·4 grammes meconin and 22·6 grammes hemipinic acid—may be taken to represent the decomposition:—

$$\underset{\text{Opianic acid.}}{2C_{10}H_{10}O_5} = \underset{\text{Meconin.}}{C_{10}H_{10}O_4} + \underset{\text{Hemipinic acid.}}{C_{10}H_{10}O_6}.$$

The following process was adopted for the purification of the meconin and hemipinic acid thus formed. The alkaline mass obtained by heating opianic acid with potash, was dissolved in a moderate quantity of warm water and mixed with an excess of hydrochloric acid: in this way the meconin was caused to separate out as an oil and to carry down most of the hemipinic acid. After the acidified liquid had been allowed to cool completely, it was poured off from the solid cake of meconin and hemipinic acid which had formed at the bottom, and evaporated to a small bulk, so as to cause the separation of the greater part of the chloride of potassium; this was washed with alcohol; the alcoholic washings were mixed with the concentrated mother-liquor, and the fresh quantity of chloride of potassium which was thus precipitated was removed by filtration or decantation, and the clear liquid was evaporated on the water-bath nearly to dryness; the residue thus obtained was again treated with alcohol, in order to separate the last portions of chloride of potassium, and the alcoholic solution filtered and evaporated. The product of these operations, together with the original precipitate of meconin and hemipinic acid, was next dissolved in boiling water and the solution made slightly alkaline with ammonia. Nearly the whole of the meconin then crystallized out as the solution cooled, and was obtained quite pure by recrystallization from water; the hemipinic acid, on the other hand, remained in solution as hemipinate of ammonium, together with a small quantity of meconin; for although this substance dissolves only very slightly in cold water, it is perceptibly more soluble in a solution of hemipinate of ammonium. In order therefore to complete the purification of the hemipinic acid, the solution of its ammonium-salt was precipitated with acetate of lead, and the hemipinate of lead, after being thoroughly washed, was decomposed under water by hydrosulphuric acid.

Meconin and hemipinic acid thus prepared were found to have all the properties ascribed to them by previous observers. Their identity was further established by analysis.

Analyses of Meconin.

I. 0·2640 grm. substance, dried at 100°, gave 0·5938 grm. carbonic acid and 0·1264 grm. water.

II. 0·3078 grm. substance, dried *in vacuo*, gave 0·6928 grm. carbonic acid and 0·1464 grm. water.

	Calculated.		Found.	
			I.	II.
C_{10}	120	61·85	61·34	61·39
H_{10}	10	5·15	5·32	5·28
Θ_4	64	33·00	——	——
$C_{10}H_{10}\Theta_4$ . .	194	100·00		

Chloromeconin was prepared from this product by treating it in aqueous solution with hypochlorite of sodium and hydrochloric acid.

I. 0·1940 grm. chloromeconin gave 0·1213 grm. chloride of silver.

II. ·2920 grm. chloromeconin gave ·1861 grm. chloride of silver.

	Calculated.	Found.	
	$C_{10}H_9Cl\Theta_4$.	I.	II.
Chlorine per cent.	15·54	15·46	15·76

Nitromeconin was also prepared, but not analysed.

Analyses of Hemipinic Acid.

I. 0·3234 grm. acid, dried at 100°, gave 0·6286 grm. carbonic acid and 0·1361 grm. water.

II. 0·3980 grm. acid gave 0·7748 grm. carbonic acid and 0·1654 grm. water.

	Calculated.		Found.	
			I.	II.
C_{10}	120	53·10	53·01	53·09
H_{10}	10	4·42	4·68	4·62
Θ_6	96	42·48	——	——
$C_{10}H_{10}\Theta_6$ . .	226	100·00		

1·3447 grm. air-dry acid lost 0·1862 grm. at 100°.

	Calculated. $C_{10}H_{10}\Theta_6 2H_2\Theta$.	Found.
Water per cent.	13·50	13·85

The *silver-salt* of this acid was found to have the composition and properties of hemipinate of silver.

I. 0·3855 grm. salt gave 0·1885 grm. silver.

II. ·4225 grm. salt gave ·2060 grm. silver.

	Calculated. $C_{10}H_8Ag_2\Theta_6$.	Found. I.	 II.
Silver per cent. . . .	49·09	48·90	48·76

3. *Action of Nascent Hydrogen.*—When an aqueous solution of opianic acid is warmed for several hours with sodium-amalgam, the subsequent addition of hydrochloric acid causes a precipitate of meconin. The formation of meconin is not due, in this case, to the decomposition of the opianic acid by the soda formed from the sodium-amalgam; for it takes place in a dilute solution, and at a temperature very much below that at which opianic acid is decomposed under the influence of alkali; the quantity of meconin formed from a given weight of acid is also considerably greater than that produced under the latter circumstances; for instance, 5 grms. opianic acid gave 3·65 grms. pure meconin, whereas 5 grms. opianic acid decomposed by alkali would yield 2·3 grms. meconin and 2·7 grms. hemipinic acid. The barely possible supposition that the additional quantity of meconin is owing to the reduction of hemipinic acid formed in a previous stage of the reaction is excluded by the fact that hemipinic acid is not acted on by sodium-amalgam in presence of water. Moreover opianic acid is similarly converted into meconin by the action of zinc and dilute sulphuric acid. Hence the transformation consists in a direct reduction or deoxidation of the acid under the influence of nascent hydrogen.

$$\underset{\text{Opianic acid.}}{C_{10}H_{10}\Theta_5} + H_2 = \underset{\text{Meconin.}}{C_{10}H_{10}\Theta_4} + H_2\Theta.$$

According to this equation, 5 parts of opianic acid correspond to 4·6 parts of meconin.

Of the two following analyses, the first was made with the product of the action of sodium-amalgam, the second with that of the action of zinc and dilute sulphuric acid on opianic acid; this latter product retained a slight colour, even after being several times recrystallized, and was obviously not quite pure.

I. 0·3906 grm. substance gave 0·8845 grm. carbonic acid and 0·1834 grm. water.

II. ·3760 grm. substance gave ·8348 grm. carbonic acid and ·1694 grm. water.

	Calculated.		Found. I.	 II.
C_{10}	120	61·85	61·76	60·55
H_{10}	10	5·15	5·22	5·01
Θ_4	64	33·00	——	——
$C_{10}H_{10}\Theta_4$ . .	194	100·00		

Chemists have long been aware of the simple relation subsisting between the formulæ of

	Meconin	$C_{10}H_{10}\Theta_4$,
	Opianic acid	$C_{10}H_{10}\Theta_5$,
and	Hemipinic acid	$C_{10}H_{10}\Theta_6$;

the foregoing experiments prove that this relation is not confined to the formulæ, but that the bodies themselves are intimately connected. Hence it was evident that, in order to arrive at a knowledge of the constitution of opianic acid, the chemical nature of the other two members of the group must also be examined. The experiments which we have made in this direction are still very far from complete; they relate principally to the action of acids on meconin and hemipinic acid, and the following are the most important results yet obtained.

Action of Hydriodic and Hydrochloric Acids on Meconin.—When meconin is boiled with concentrated hydriodic acid, a considerable quantity of iodide of methyl is formed, but the other products of the reaction are so unstable that it has not been found possible to remove the excess of hydriodic acid without completely destroying them. It was not till after many attempts to purify the products thus formed that it was found that meconin undergoes a similar decomposition when heated in a sealed tube, to a little above 100°, with three times its weight of strong hydrochloric acid. The chloride of methyl, formed under these circumstances, quickly volatilizes when the tube is opened, and by careful evaporation, over lime and sulphuric acid, a crystalline residue may be obtained, whence it is possible to remove completely the excess of hydrochloric acid; but it has not yet been further investigated.

Action of Hydriodic Acid on Hemipinic Acid.—Hemipinic acid is decomposed, when boiled with concentrated hydriodic acid, into carbonic acid, iodide of methyl (boiling-point 42°·8 C., vapour-density 5·05; calculated vapour-density 4·92), and an acid of the formula $C_7H_6\Theta_4$. It will be seen that the composition of this acid is intermediate between that of salicylic acid, $C_7H_6\Theta_3$, and that of gallic acid, $C_7H_6\Theta_5$, and, as will be seen by the description which follows, it is analogous to these acids in some of its properties. Accordingly, in order to recall the fact of its containing one atom of oxygen less than gallic acid, we propose to name it provisionally *hypogallic acid*, reserving to ourselves to suggest, if possible, a more appropriate name when its chemical relations shall have been more thoroughly investigated. The reaction, by which these products are formed, takes place according to the equation

$$\underset{\text{Hemipinic acid.}}{C_{10}H_{10}\Theta_6} + 2HI = C\Theta_2 + \underset{\text{Iodide of methyl.}}{2CH_3I} + \underset{\text{Hypogallic acid.}}{C_7H_6\Theta_4}.$$

Hypogallic acid, when pure, is only slightly soluble in cold water, but dissolves easily in hot water, alcohol, and ether; its solution reacts strongly acid with litmus-paper. It separates from hot water in small prismatic crystals, united into stellate groups, and containing 1½ atom water of crystallization, which they lose at 100°. The acid melts at about 180°, but, as it begins to decompose even at a lower temperature, its melting-point could not be accurately ascertained. Dried at 100° C., it gave the following results on analysis:—

I. 0·3465 grm. substance gave 0·6904 grm. carbonic acid and 0·1238 grm. water.
II. ·4670 grm. substance gave ·9326 grm. carbonic acid and ·1662 grm. water.
III. ·4968 grm. substance gave ·9900 grm. carbonic acid and ·1710 grm. water.

	Calculated.		Found.		
			I.	II.	III.
C_7	84	54·55	54·34	54·46	54·35
H_6	6	3·89	3·97	3·95	3·82
O_4	64	41·56	—	—	—
$C_7H_6O_4$ . .	154	100·00			

I. 1·486 grm. crystallized acid lost 0·2200 grm. at 100° C.
II. 2·132 grms. crystallized acid lost ·3120 grm. at 100°.
III. 1·138 grm. crystallized acid lost ·1716 grm. at 100°.

	Calculated.		Found.		
			I.	II.	III.
$C_7H_6O_4$	154	85·08	—	—	—
$H_3O_{1\frac{1}{2}}$	27	14·92	14·80	14·63	15·08
$C_7H_6O_4, 1\frac{1}{2}H_2O$ . .	181	100·00			

Hypogallic acid gradually turns brown when heated in the air to a little above 100° C.; the same change occurs more quickly when a solution of it, especially if neutral or alkaline, is evaporated. Added to solution of *nitrate* or *ammonio-nitrate of silver*, it causes an immediate precipitation of metallic silver, even in the cold; with *sulphate of copper* and a slight excess of potash it gives a yellowish-green solution from which an orange-yellow precipitate is thrown down on warming; in a mixture of *sesquichloride of iron and red prussiate of potash*, it immediately produces a blue precipitate; when boiled with solution of *corrosive sublimate*, it reduces it to calomel. With *sesquichloride of iron*, it gives an intense indigo-blue coloration, which is changed to violet by a very small quantity of ammonia, and to blood-red by excess of ammonia, no precipitate being produced, unless too much chloride of iron has been used; the colour is destroyed by strong acids, but restored by neutralization with alkali, and partially by addition of water. A solution of the acid immediately becomes brown on addition of *alkali*, the colour quickly becoming darker by exposure to the air. With *ammonia and chloride of barium* or *calcium*, it gives a dirty brown flocculent precipitate; with *acetate of lead*, a pale yellow precipitate.

Hypogallic acid is decomposed by heat into carbonic acid and a substance which solidifies in the neck of the retort to a colourless crystalline mass. The decomposition begins at about 170° C., and goes on rapidly at 200°. The crystalline product melts, in the crude state, at about 90° C.; it dissolves easily in water and crystallizes in needles when the solution is evaporated. It is rapidly attacked by nitric acid, even when diluted, giving a red-brown solution. With sesquichloride of iron it gives a bluish-black amorphous precipitate; with acetate of lead it gives a white or yellowish-white precipitate, soluble in an excess of acetic acid. It slowly assumes a darker colour by exposure to air in contact with alkali. This substance has not yet been prepared in sufficient quantity for complete investigation.

In addition to hypogallic acid, no less than three other acids are known, having the same composition, and resembling it to a remarkable degree in some of their most characteristic properties. They are carbohydrochinonic acid[*] (obtained by OTTO HESSE by the action of bromine in presence of water on chinic acid), protocatechuic acid[†] (obtained by STRECKER as a product of the action of fused potash on piperic acid), and oxysalicylic acid[‡] (obtained by LAUTEMANN by decomposing iodosalicylic acid with potash). All these acids are described as having about the same solubility in water, alcohol, and ether as hypogallic acid; like it, they all give a dark coloration with the smallest trace of sesquichloride of iron, they all reduce nitrate of silver, they all become dark brown when mixed with alkali and exposed to the air, all give a yellowish-white precipitate with acetate of lead, and at a high temperature they are all decomposed into carbonic acid and oxyphenic acid or hydrochinone[§]. Nevertheless no two of these acids appear to have quite the same properties. The following are the most important points in which differences have been observed. Hypogallic acid crystallizes with 1½ molecule of water (14·9 per cent.), carbohydrochinonic and protocatechuic acids with 1 molecule (10·4 per cent.), and oxysalicylic acid without water. Hypogallic and oxysalicylic acids give a dark blue colour with sesquichloride of iron, the other two acids a dark green colour. Hypogallic acid reduces nitrate of silver immediately in the cold; carbohydrochinonic acid reduces it slowly in the cold, rapidly when boiled; oxysalicylic acid has no action on nitrate of silver in the cold, but reduces it completely when boiled. Carbohydrochinonic acid reduces cuprous oxide from a mixture of cupric acetate, tartaric acid, and excess of potash; protocatechuic acid causes no reduction of the same solution. Hypogallic acid causes a precipitate in a mixture of chloride of barium and ammonia; protocatechuic acid only on addition of alcohol.

Whether some of these differences may not be due to accidental causes, depending on the different sources and modes of preparation of the several acids, is a question that readily suggests itself, but it can be answered only by further investigation.

[*] Annalen der Chemie und Pharmacie, vol. cxii. p. 52 (1859); vol. cxiv. p. 292; vol. cxxii. p. 221.

[†] Ibid. vol. cxviii. p. 280.

[‡] Ibid. vol. cxviii. p. 372; more fully vol. cxx. p. 311.

[§] The product obtained by the action of heat on hypogallic acid does not fully agree in its reactions with either of these bodies, so far as yet examined.

Action of Hydrochloric Acid on Hemipinic Acid.—Hemipinic acid is rapidly decomposed when heated with two or three times its weight of strong hydrochloric acid, either in a sealed tube to about 110° or in an open vessel connected with a condenser so arranged that the condensed vapour flows back into the mixture, and with an apparatus for evolving gaseous hydrochloric acid, whereby the liquid can be kept constantly saturated with that acid. The products of the reaction are chloride of methyl, carbonic acid, and an acid crystallizing in beautiful long transparent prisms.

This acid is almost insoluble in cold water, and not much more soluble in boiling water; alcohol and ether dissolve it more easily. When heated it begins to sublime, without decomposition, at about 200° C., and supports a temperature of more than 245° without any further change, though at a still higher temperature it melts, and solidifies again on cooling to a crystalline mass. It dissolves in strong sulphuric acid, and is precipitated unchanged on addition of water. It gives no coloration with sesquichloride of iron; with nitrate of silver it gives a white precipitate, which blackens on boiling. It gave, on analysis, numbers agreeing nearly with the formula $C_8 H_8 O_4$.

0·2747 grm. substance gave 0·5702 grm. carbonic acid and 0·1277 grm. water.

	Calculated.		Found.
C_8	96	57·14	56·64
H_8	8	4·76	5·17
O_4	64	38·10	——
$C_8 H_8 O_4$ . .	168	100·00	

The formation of this body therefore probably takes place in accordance with the equation

$$\underbrace{C_{10} H_{10} O_6}_{\text{Hemipinic acid.}} + HCl = CO_2 + CH_3Cl + C_8 H_8 O_4.$$

We hope to give a more complete description of it in a future communication.

By the prolonged action of hydrochloric acid on the mother-liquor from which this acid has crystallized, hypogallic acid appears to be formed. The product thus obtained has not yet been analysed, but it is identical in all its qualitative reactions with that formed by the action of hydriodic acid on hemipinic acid.

Nascent Hydrogen, resulting from the action of sodium-amalgam on an aqueous solution of hemipinic acid, leaves that acid unacted upon, as has been already stated. A portion of hemipinic acid that had been subjected for a long time to the action of sodium-amalgam was converted into silver-salt, the salt being precipitated in two fractions. Both portions, as shown by the following analyses, consisted of pure hemipinate of silver, and no other organic substance than hemipinic acid could be detected in the solution.

I. 0·7050 grm. salt (first precipitate) gave 0·3466 grm. silver.
II. ·2655 grm. salt (second precipitate) gave ·1302 grm. silver.

	Calculated.	Found.	
	$C_{10}H_8Ag_2O_6$.	I.	II.
Silver per cent. . . .	49·10	49·16	49·04

We may here mention a peculiar property of hemipinate of barium which does not seem to have been before observed, but which has often been of service to us as affording the means of recognizing hemipinic acid when present only in small quantity.

When a solution of hemipinic acid is neutralized with baryta-water, or when solutions of hemipinate of ammonium and chloride of barium are mixed together, the liquid remains clear for a long while if left to itself; but if it is boiled for a short time, small, shining, crystalline plates of hemipinate of barium are precipitated and soon fill the whole liquid, if the solutions used were not too dilute. On allowing the liquid to cool, it redissolves this precipitate, and becomes almost or quite clear; but after standing for a few hours, or for a day or two, according as more or less of the salt is contained in it, it again deposits hemipinate of barium, but this time in the form of feathery tufts of very small silky needles; if the liquid be now again heated, these feathery crystals dissolve, and the crystalline plates already mentioned make their appearance once more.

§ III.—DECOMPOSITIONS AND DERIVATIVES OF COTARNINE.

1. *Action of Nitric Acid.*—By gently heating cotarnine with dilute nitric acid, we have obtained nitrate of methylamine and a new acid, *cotarnic acid*, but have not hitherto found out the conditions necessary for the certain production of the latter substance, many attempts to obtain it having been completely unsuccessful.

Cotarnic acid dissolves easily in water, giving a solution which reacts strongly acid with litmus-paper: it dissolves to a less extent in alcohol, and is precipitated from its alcoholic solution by ether. It yields no trace of cyanide when heated with metallic sodium in excess, and is thus proved to contain no nitrogen. It gives a white precipitate with acetate of lead, and is not affected by sesquichloride of iron. Nitrate of silver throws down a very stable silver-salt, which may be crystallized from boiling water, in which it is slightly soluble, without alteration. This salt contains $C_{11}H_{10}Ag_2O_5$.

I. 0·2513 grm. salt gave 0·2693 grm. carbonic acid, 0·049 grm. water, and 0·1235 grm. silver.

II. ·2065 grm. salt gave 0·2279 grm. carbonic acid, 0·0403 grm. water, and 0·1019 grm. silver.

	Calculated.		Found.	
			I.	II.
C_{11}	132	30·14	29·22	30·10
H_{10}	10	2·27	2·16	2·17
Ag_2	216	49·32	49·14	49·35
O_5	80	18·27	——	——
$C_{11}H_{10}Ag_2O_5$ . .	438	100·00		

Hence cotarnic acid must contain $C_{11}H_{12}\Theta_5$, and its formation from cotarnine must take place according to the equation

$$\underset{\text{Cotarnine.}}{C_{12}H_{13}N\Theta_2} + 2H_2\Theta = \underset{\text{Cotarnic acid.}}{C_{11}H_{12}\Theta_5} + \underset{\text{Methylamine.}}{CH_5N.}$$

The formation of methylamine in this reaction was proved by the analysis of its chloroplatinate.

0·3408 grm. salt gave 0·1407 grm. platinum.

	Calculated. $CH_5N, HCl, PtCl_2$.	Found.
Platinum per cent. . . .	41·7	41·3

Cotarnine heated with undiluted nitric acid was found to yield oxalic and apophyllic acids, in accordance with the statements of WÖHLER and ANDERSON.

2. *Action of Hydrochloric Acid.*—Cotarnine heated with three times its weight of strong hydrochloric acid to about 140° C., in a sealed tube, is decomposed into chloride of methyl—identified, *inter alia*, by its formation of the solid hydrate described by BÄYER *—and a substance which crystallizes in very small, pale yellow, silky needles. This body we designate, provisionally, *hydrochlorate of cotarnamic acid*, assigning to it the formula $C_{11}H_{13}N\Theta_4$, HCl, which, though not agreeing perfectly with the analyses hitherto obtained, expresses their results more closely than any other formula that seems equally probable. The substance was purified for analysis by several crystallizations from water slightly acidulated with hydrochloric acid, and was dried *in vacuo* over lime and sulphuric acid.

I. 0·3295 grm. substance gave 0·6020 grm. carbonic acid and 0·1729 grm. water.
II. ·4046 grm. substance gave ·7429 grm. carbonic acid and ·2017 grm. water.
III. ·3378 grm. substance gave ·6182 grm. carbonic acid and ·1745 grm. water.
IV. ·3649 grm. substance gave ·1480 grm. platinum.
V. ·4826 grm. substance gave ·2760 grm. chloride of silver.
VI. ·5674 grm. substance gave ·3220 grm. chloride of silver.
VII. ·4590 grm. substance gave ·2626 grm. chloride of silver.
VIII. ·4654 grm. substance gave ·2593 grm. chloride of silver.
IX. ·3712 grm. substance gave ·2128 grm. chloride of silver.
X. ·5200 grm. substance gave ·2905 grm. chloride of silver.

	Calculated.		Found.						Mean.
			I.,IV.&V.	II.&VI.	III.&VII.	VIII.	IX.	X.	
C_{11}	132	50·87	49·83	50·08	49·91	——	——	——	49·94
H_{14}	14	5·40	5·83	5·54	5·74	——	——	——	5·70
N	14	5·40	5·73	——	——	——	——	——	5·73
Θ_4	64	24·65	——	——	——	——	——	——	24·61
Cl	35·5	13·68	14·15	14·04	14·15	13·78	14·18	13·82	14·02
$C_{11}H_{13}N\Theta_4$, HCl .	259·5	100·00							100·00

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* Annalen der Chemie und Pharmacie, vol. ciii. p. 183.

Assuming for the present the accuracy of the proposed formula, the action of hydrochloric acid on cotarnine will be represented by the following equation:—

$$\underbrace{C_{12}H_{13}NO_3}_{\text{Cotarnine.}}+H_2O+2HCl=CH_3Cl+\underbrace{C_{11}H_{13}NO_4, HCl}_{\text{Hydrochlorate of cotarnamic acid.}}$$

Hydrochlorate of cotarnamic acid is only slightly soluble in cold water, but is very soluble in hot water; it is less soluble in alcohol than in water, and almost insoluble in ether. When dissolved in hot water, it always undergoes partial decomposition, as is indicated by the separation of an orange-coloured granular precipitate if the quantity of water used is only moderate; when more water is employed, this precipitate remains dissolved, the smallest trace imparting a bright orange-colour to the solution. A similar precipitate (cotarnamic acid, $C_{11}H_{13}NO_4$?) is thrown down on cautiously adding an alkali, or alkaline carbonate or sulphite, to an aqueous or slightly acid solution of hydrochlorate of cotarnamic acid; this precipitate regenerates the original compound when treated with hydrochloric acid; it dissolves with orange-colour in excess of alkali, giving a solution that rapidly becomes brown in the air; by washing with water, out of contact with air, it may be obtained quite free from chlorine. The hydrochlorate dissolves without alteration in water containing a small quantity of free hydrochloric acid; the solution, which is but slightly coloured at first, gradually acquires a beautiful dark-green colour by exposure to the air. If nitric acid is slowly added to a solution of the compound in boiling water, the portions of the liquid with which the acid comes in contact assume a fine opaque crimson colour when seen by reflected light, but appear of a transparent orange tint when seen by transmitted light. After a few minutes a slight effervescence takes place, and this effect disappears. Hydrochlorate of cotarnamic acid mixed with a slight excess of dilute sulphuric acid, and evaporated nearly to dryness on a water-bath, acquires a fine crimson colour rivalling that of acetate of rosaniline; this colour is destroyed by addition of water, but it appears again when the water is evaporated. Nitrate of silver added in excess to a hot solution of the hydrochlorate is rapidly reduced.

Hydriodic and *sulphuric acids* appear to act upon cotarnine in the same way as hydrochloric acid. With hydriodic acid iodide of methyl is formed in unmistakable quantity: the formation of a methyl-compound by the action of sulphuric acid was not proved, but in both cases appropriate treatment of the resulting solutions yielded hydrochlorate of cotarnamic acid with all its characteristic properties above described. Analyses of specimens prepared in this way are included among those already given (page 360).

We have not yet obtained cotarnamic acid itself, nor any of its compounds except the hydrochlorate, in a state that invited analysis; but we hope to be able to do so on continuing our experiments, and thus to remove any uncertainty that may at present exist as to the true formula of this substance.

3. *Action of Potash.*—Cotarnine distilled with caustic potash yields ammonia and

methylamine, but apparently no di- or tri-methylamine or other similar compounds. The ammonia and methylamine were separated by treating their hydrochlorates with absolute alcohol in the usual way. The chloroplatinate of the latter was analysed.

0·3423 grm. salt gave 0·1413 grm. platinum.

	Calculated. $CH_5N, HCl, PtCl_2$.	Found.
Platinum per cent. . .	41·7	41·3

§ IV.—CONCLUSION.

The conclusions which the foregoing experiments enable us to draw, relatively to the constitution of narcotine and its derivatives, are far from being sufficiently comprehensive and precise to admit of expression by a series of rational formulæ; nevertheless it may be allowable to recapitulate briefly the chief points which we think have been established, and to offer some suggestions towards the interpretation of the results that have been obtained.

1. Our analyses indicate the existence of only one kind of narcotine—that, namely, which contains $C_{22}H_{23}NO_7$. It has already been stated (page 345) that this formula was assigned by WERTHEIM to "methyl-narcotine," which he believed to be only one of several varieties of the base; in addition he recognized, on the authority of BLYTH'S investigation, the existence of "ethyl-narcotine," $C_{23}H_{25}NO_7$; and in consequence of having obtained a volatile base, of the composition of propylamine, C_3H_9N, by distilling narcotine with potash, he also admitted the existence of "propyl-narcotine," $C_{24}H_{27}NO_7$.

In a note near the beginning of this paper, the results of all the published analyses we have been able to find are put together. These results do not appear to us to afford any strong evidence that other chemists have operated upon a kind of narcotine different from that represented by the formula $C_{22}H_{23}NO_7$; and we think that the following experiment shows that such a supposition is not required to explain the observation which caused the existence of propyl-narcotine to be admitted.

By distilling 20 grammes of narcotine with an excess of concentrated hydriodic acid, we obtained 19 grammes of pure iodide of methyl, a quantity which is nearly in the proportion of three molecules of iodide of methyl to one molecule of narcotine

$$(C_{22}H_{23}NO_7 : 3CH_3I :: 413 : 436 :: 20 : 21{\cdot}1).$$

Narcotine therefore contains three atoms of methyl so combined as to be easily separated, and hence we think it likely that WERTHEIM'S supposed propylamine was the isomeric trimethylamine. This experiment makes it also seem probable that the distillation of narcotine with potash would yield trimethylamine, dimethylamine, methylamine, and ammonia in proportions varying with the conditions of the experiment, in which case the nature of the product could afford no trustworthy evidence as to the composition of the material employed.

On the other hand, the existence of narcotine containing three atoms of methyl may, at first sight, seem to render probable the existence of other varieties of the base in which ethyl or a similar radicle takes the place of the whole or part of the methyl. Analogy, however, does not support such an inference. Many natural products are known which contain, in some form or other, methyl as one of their constituents, although corresponding compounds containing ethyl are either entirely unknown or have been obtained only by artificial means. We may mention as examples methyl-salicylic acid, methyl-coniine (VON PLANTA and KEKULÉ), brucine (STRECKER), morphine (WERTHEIM), codeine (ANDERSON), caffeine (WURTZ, ROCHLEDER, STRECKER), theobromine (ROCHLEDER and HLASIWETZ, STRECKER), creatine (STRECKER), sarcosine (STRECKER, VOLHARD).

2. According to our analyses, narcotine contains the elements of meconin and cotarnine:—

$$\underset{\text{Narcotine.}}{C_{22}H_{23}NO_7}=\underset{\text{Meconin.}}{C_{10}H_{10}O_4}+\underset{\text{Cotarnine.}}{C_{12}H_{13}NO_3}.$$

After having discussed the probable nature of these two bodies, we shall return to the consideration of their possible functions as constituents of narcotine.

3. We have been unable to find a second reaction altogether analogous to the transformation of opianic acid into meconin and hemipinic acid, regarded as a mere transference of an atom of oxygen from one molecule of opianic acid to another, according to the equation

$$\underset{\text{Opianic acid.}}{C_{10}H_{10}O_5}+\underset{\text{Opianic acid.}}{C_{10}H_{10}O_5}=\underset{\text{Meconin.}}{C_{10}H_{10}O_4}+\underset{\text{Hemipinic acid.}}{C_{10}H_{10}O_6}.$$

Possibly, however, this equation, though correctly expressing the final result, does not represent the actual decomposition which occurs in the first instance. Of course, instead of hemipinic acid, hemipinate of potassium is produced at first, but perhaps also, instead of meconin, a compound of that body with potash, $C_{10}H_{10}O_4$, $KHO(=C_{10}H_{11}KO_5)$, may be the correlative product. The reaction would then be (substituting for potash, in the equation, its equivalent of water)

$$2C_{10}H_{10}O_5+H_2O=C_{10}H_{12}O_5+C_{10}H_{10}O_6,$$

strictly analogous to the transformation of oil of bitter almonds into benzylic alcohol and benzoic acid:

$$\underset{\text{Oil of bitter almonds.}}{2C_7H_6O}+H_2O=\underset{\text{Benzylic alcohol.}}{C_7H_8O}+\underset{\text{Benzoic acid.}}{C_7H_6O_2}.$$

According to this supposition, meconin would be an anhydride of a less stable compound $C_{10}H_{12}O_5$, probably a body possessing more or less the characters of a polyatomic alcohol. If, with BERTHELOT[*], we compare meconin to ethylene, opianic acid to aldehyde, and hemipinic acid to acetic acid, the compound $C_{10}H_{12}O_5$ might be compared to alcohol:

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[*] Annales de Chimie et de Physique, 3rd series, vol. lvi. p. 77.

Meconin	$C_{10}H_{12}O_4$	C_2H_4 . . .	Ethylene.
Hypothetical hydrate of meconin . . .	$C_{10}H_{10}O_5$	C_2H_6O . .	Alcohol.
Opianic acid	$C_{10}H_{10}O_5$	C_2H_4O . .	Aldehyde.
Hemipinic acid	$C_{10}H_{10}O_6$	$C_2H_4O_2$ . .	Acetic acid.

The supposed existence of a hydrate of meconin derives some slight support from the fact that, when the product obtained by heating opianic acid with potash is dissolved in water and the solution made acid with hydrochloric acid, the meconin remains for a long time dissolved in the acid liquid unless a great excess of hydrochloric acid is added—apparently as though it did not exist as such in the solution, but required a certain time, or else the aid of a great excess of acid, to enable it to separate from a previous state of combination*.

4. The conversion of opianic acid into meconin by means of nascent hydrogen is a transformation as anomalous as that last considered, if viewed as a direct deoxidation of the acid; but if it be supposed that a compound $C_{10}H_{12}O_5$ is first formed and is converted by subsequent dehydration into meconin, the reaction appears (not as a removal of oxygen, but) as a fixation of hydrogen, and takes its place among a large number of similar transformations which have been observed within the last few years to occur under like conditions:—

$$\underset{\text{Opianic acid.}}{C_{10}H_{10}O_5} + H_2 = \underset{\text{Hydrate of meconin.}}{C_{10}H_{12}O_5}.$$

Among the many analogous reactions we may mention especially the conversion of glucose into mannite, lately effected by Linnemann†,

$$\underset{\text{Glucose.}}{C_6H_{12}O_6} + H_2 = \underset{\text{Mannite.}}{C_6H_{14}O_6},$$

—a reaction which perhaps justifies the following comparison of formulæ:—

Meconin	$C_{10}H_{10}O_4$	$C_6H_{12}O_5$ . .	Mannitan.
Hydrate of Meconin . . .	$C_{10}H_{12}O_5$	$C_6H_{14}O_6$ . .	Mannite.
Opianic acid	$C_{10}H_{10}O_5$	$C_6H_{12}O_6$ . .	Glucose.
Hemipinic acid	$C_{10}H_{10}O_6$	$C_6H_{12}O_7$ . .	Mannitic acid.

If this view of the action of nascent hydrogen on opianic acid be adopted, and if the decomposition of that acid by potash be interpreted in the manner suggested in the last paragraph, both reactions may be regarded as giving additional weight to the views of Berthelot, who has already pointed out that opianic acid possesses many properties usually considered characteristic of the aldehydes. In any case, however, the fact that nascent hydrogen converts opianic acid into meconin, but has no action on hemipinic acid, excludes the possibility of regarding the first of these bodies as a hemipinate of meconin.

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* Among the products which Anderson obtained by the action of nitric acid on narcotine, was one which he supposed to be a hydrate of meconin. Anderson did not obtain a sufficient quantity of this substance to enable him to ascertain its chemical relations; and we have already shown elsewhere (Proc. Roy. Soc. vol. xi. p. 60) that his analyses agree precisely with the formula of cotarnic acid, $C_{11}H_{12}O_5$: they also agree, though not quite as well, with the composition of a hydrate of meconin $=2(C_{10}H_{10}O_4)H_2O$, but not at all with that of the hydrate whose existence is supposed in the text.

† Annalen der Chemie und Pharmacie, vol. cxxiii. p. 136.

5. The action of hydrochloric and hydriodic acids on opianic acid and its congeners proves that methyl is a constituent of each of them; and since one molecule of hemipinic acid yields two molecules of iodide of methyl when distilled with hydriodic acid, we must assume the existence of two atoms of methyl in each molecule of meconin, opianic acid, or hemipinic acid. It is useless to discuss the function of the methyl in the first two of these substances before possessing more definite knowledge of their chemical nature and of the bodies formed from them when the methyl is eliminated. In the case of hemipinic acid, our knowledge on these points is already sufficient to enable us to form a tolerably clear idea of the state of combination in which its two atoms of methyl may exist.

Hemipinic acid was shown by Anderson to be a well-characterized bibasic acid, and to contain, in one molecule, six atoms ($\Theta=16$) of oxygen. Among the better-known acids, one which resembles it in these particulars is tartaric acid, $\mathrm{C}_4\,\mathrm{H}_6\,\Theta_6$. Now the experiments of Perkin and Duppa, and of Kekulé, as well as those of Schmitt and Dessaignes, clearly show that, although tartaric acid is only bibasic in the strict sense, yet it is tetratomic, or that, in the language of the modern theory of types, it contains four atoms of hydrogen outside the radicle. If we assume the existence of an acid $\mathrm{C}_8\,\mathrm{H}_6\,\mathrm{O}_6$ strictly comparable to tartaric acid, that is, containing outside the radicle four atoms of hydrogen whereof two only are replaceable by ordinary processes of saline double decomposition, and if we further suppose the other two atoms to be replaced by methyl, the resulting compound would have the composition of hemipinic acid

$$(\mathrm{C}_8\,\mathrm{H}_4\,(\mathrm{CH}_3)_2\,\Theta_6 = \mathrm{C}_{10}\,\mathrm{H}_{10}\,\Theta_6),$$

and might be expected to resemble it in being a bibasic acid, yielding two molecules of iodide of methyl when boiled with hydriodic acid, but no methylic alcohol when boiled with potash. The two atoms of methyl in such a compound would be combined in the same manner as the one atom of ethyl in ethyl-lactic acid, which, as proved by the experiments of Wurtz and Butlerow, cannot be eliminated by the action of alkalis, but, as found by the latter chemist*, is easily eliminated by the action of hydriodic acid.

If hemipinic acid be allowed to have the constitution here suggested, the decomposition which gives rise to hypogallic acid must be supposed to consist of two stages:—first, the replacement of $(\mathrm{CH}_3)_2$ by H_2, resulting in the formation of a hypothetical normal hemipinic acid, $\mathrm{C}_8\,\mathrm{H}_6\,\Theta_6$,

$$\mathrm{C}_8\,\mathrm{H}_4\,(\mathrm{CH}_3)_2\,\Theta_6 + 2\mathrm{HI} = \mathrm{C}_8\,\mathrm{H}_6\,\Theta_6 + 2\mathrm{CH}_3\,\mathrm{I};$$

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* On the dissymmetrical constitution of the radicle of lactic acid as the probable cause of the dissimilar functions of its two replaceable hydrogen-atoms, see Kekulé, Lehrbuch der Organischen Chemie (Erlangen, 1861), vol. i. pp. 174 and 730. The same considerations apply, *mutatis mutandis,* to the case of a tetratomic acid such as tartaric acid.

secondly, the decomposition of this product, at the temperature of boiling hydriodic acid (125° C.), into carbonic and hypogallic acids:—

$$C_8H_6O_6 = CO_2 + C_7H_6O_4.$$

6. The action of acids on cotarnine proves that one molecule of that substance contains one atom of methyl. When the methyl is eliminated, as by the action of hydrochloric acid, the first phase of the reaction probably consists in the simple replacement of methyl by hydrogen, giving rise to a non-methylized or normal cotarnine,—

$$\underset{\text{Cotarnine.}}{C_{11}H_{10}(CH_3)NO_3} + HCl = \underset{\text{Normal cotarnine.}}{C_{11}H_{11}NO_3} + CH_3Cl;$$

but this compound, in presence of aqueous hydrochloric acid, fixes the elements of water and is converted into cotarnamic acid:—

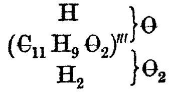
$$\underset{\text{Normal cotarnine.}}{C_{11}H_{11}NO_3} + H_2O = \underset{\text{Cotarnamic acid.}}{C_{11}H_{13}NO_4}.$$

Cotarnic acid, $C_{11}H_{12}O_5$, which may be regarded as the central member of the cotarnine group, is a bibasic acid containing five atoms of oxygen, and is therefore analogous to malic acid, $C_4H_6O_5$. Accordingly, we find, among the derivatives of malic acid, compounds which correspond to all the derivatives of cotarnic acid.

Cotarnic acid	$C_{11}H_{12}O_5$	$C_4H_6O_5$	Malic acid.
Cotarnamic acid . . .	$C_{11}H_{13}NO_4$	$C_4H_7NO_4$	Aspartic acid.
Hydrochlorate of cotarnamic acid	$C_{11}H_{13}NO_4$, HCl	$C_4H_7NO_4$, HCl	Hydrochlorate of aspartic acid.
Cotarnimide (hypothetical normal cotarnine) . .	$C_{11}H_{11}NO_3$	$C_4H_5NO_3$	Malimide.
Methyl-cotarnimide (cotarnine)	$C_{11}H_{10}(CH_3)NO_3$	$C_4H_4(C_6H_5)NO_3$	Phenyl-malimide.

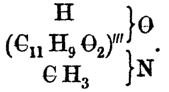
The substance here called *malimide* is produced by the dry distillation of acid malate of ammonium (Pasteur), or of a mixture of equivalent quantities of aspartate of barium and ethyl-sulphate of potassium (Dessaignes): aspartic acid is produced when this compound is boiled with hydrochloric acid (Dessaignes), exactly as we have supposed cotarnamic acid to be formed from cotarnimide.

The close analogy existing between the derivatives of cotarnic acid and those of malic acid indicates that cotarnic acid is probably triatomic; hence the typical formulæ by which we represented this acid and cotarnine in our preliminary communication*, require to be somewhat modified, and would probably be more correctly written thus:—



$$\left.\begin{array}{c} H \\ (C_{11}H_9O_2)''' \end{array}\right\}O \quad \left.\begin{array}{c} \\ H_2 \end{array}\right\}O_2$$

Cotarnic acid.



$$\left.\begin{array}{c} H \\ (C_{11}H_9O_2)''' \end{array}\right\}O \quad \left.\begin{array}{c} \\ CH_3 \end{array}\right\}N.$$

Cotarnine.

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* Proceedings of the Royal Society, vol. xi. p. 60.

This formula for cotarnine, which represents it as derived from the double type $\left.\begin{matrix}H_2\Theta\\H_3N\end{matrix}\right\}$, is still in accordance with the experiments of How[*] (who found it impossible to replace hydrogen in cotarnine by means of iodide of methyl), for it will be seen that the whole of the hydrogen of the ammonia is represented as already replaced.

7. Meconin being regarded as a polyatomic anhydride, and cotarnine as an imide, it is allowable to suppose that the constitution of narcotine may be similar in some degree to that of the oxygenated alkaloids obtained by Wurtz by the union of oxide of ethylene with ammonia.

Meconin digested with aqueous ammonia in a sealed tube forms a solution from which it is not deposited on cooling, nor even when the excess of ammonia is expelled by cautious evaporation. Possibly the investigation of the compound thus formed may throw further light on the constitution of narcotine.

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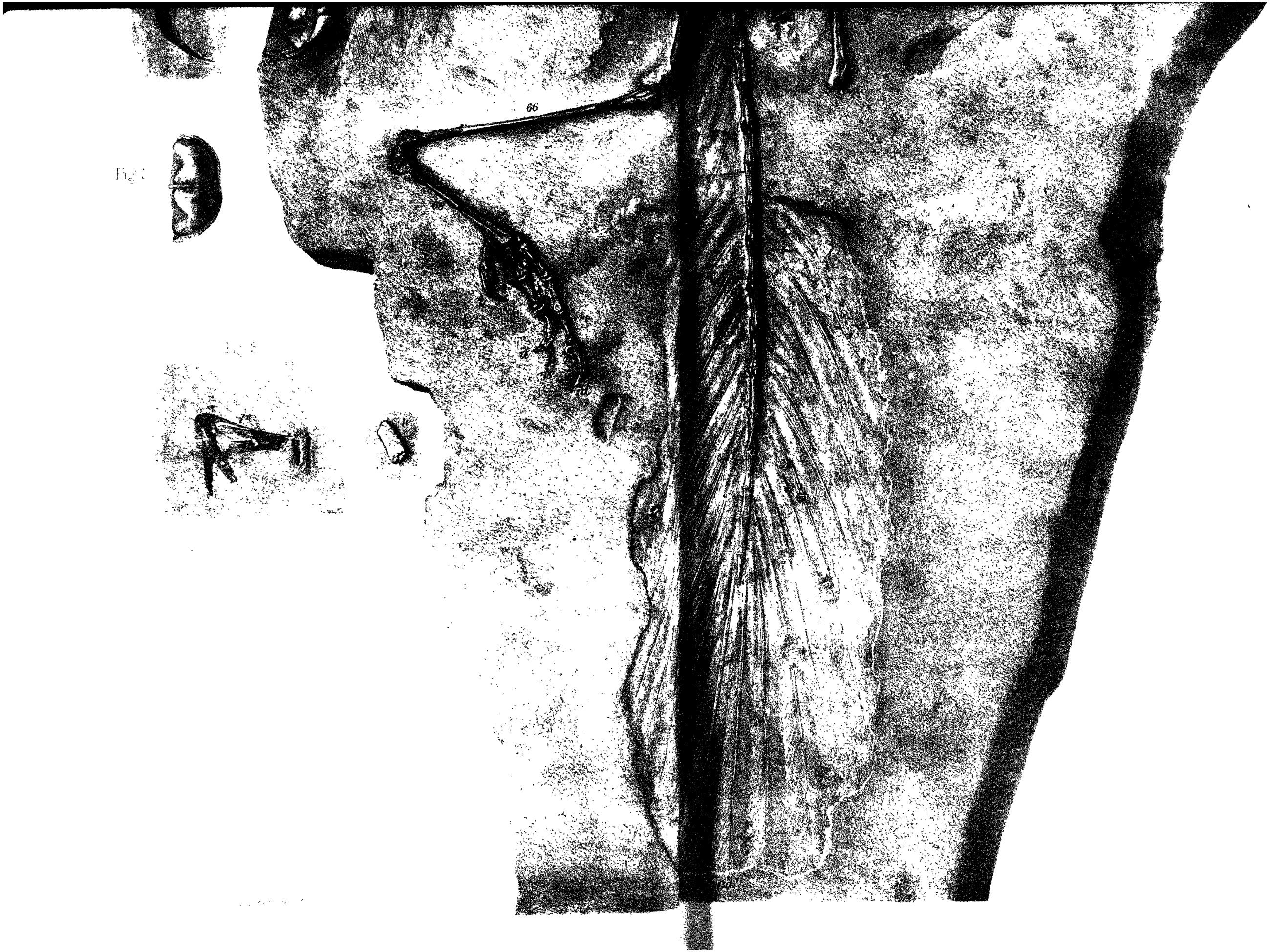
[*] Transactions of the Royal Society of Edinburgh, vol. xxi. p. 31.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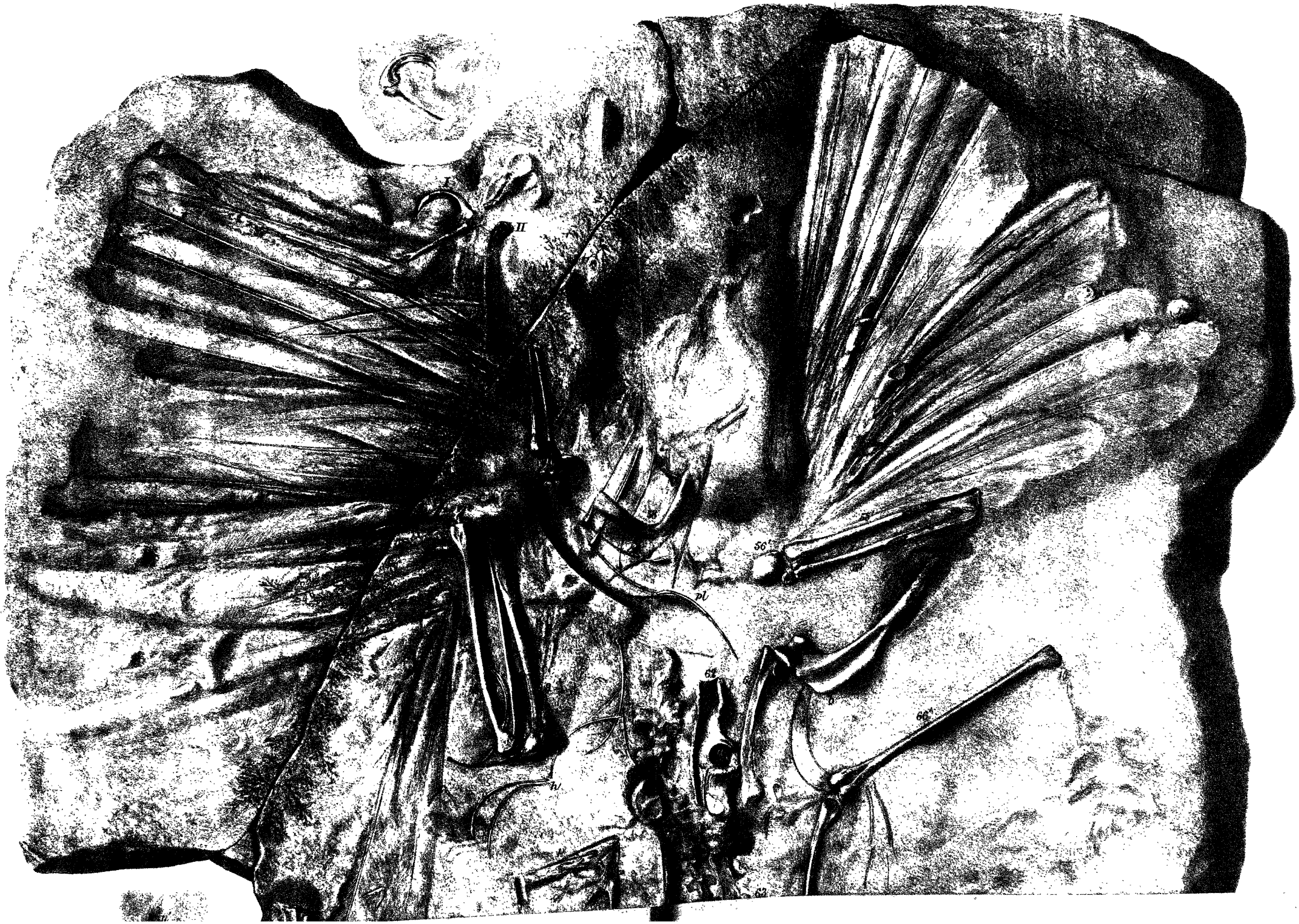
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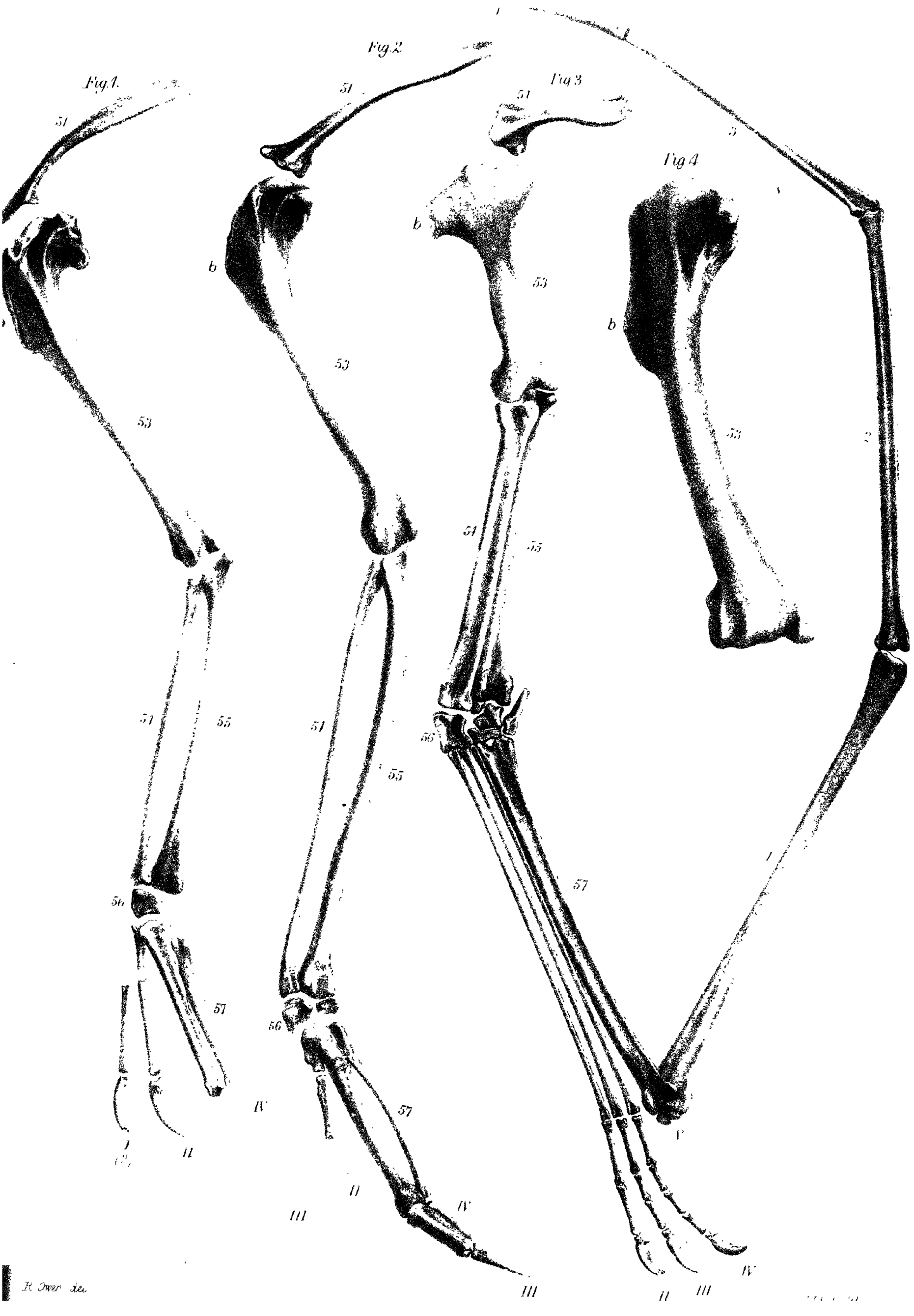
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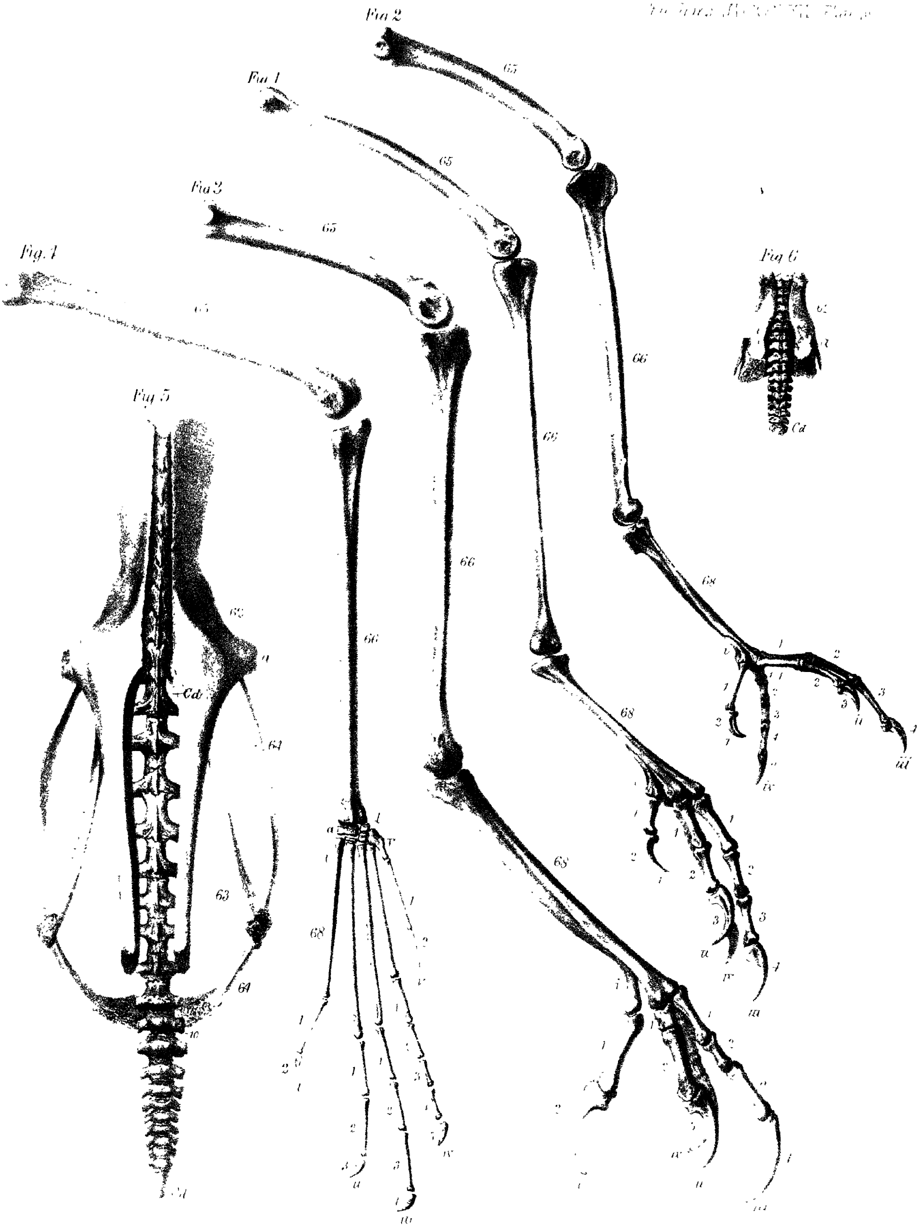
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Fig 1

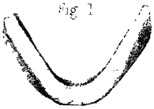


Fig 7

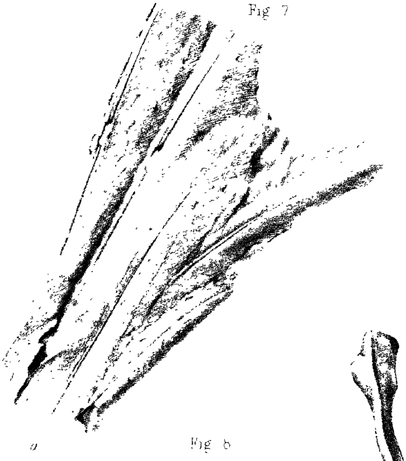
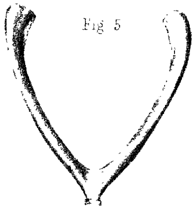


Fig 5



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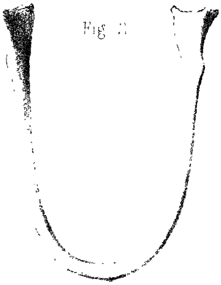


Fig 4

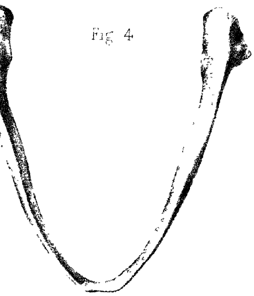


Fig 8

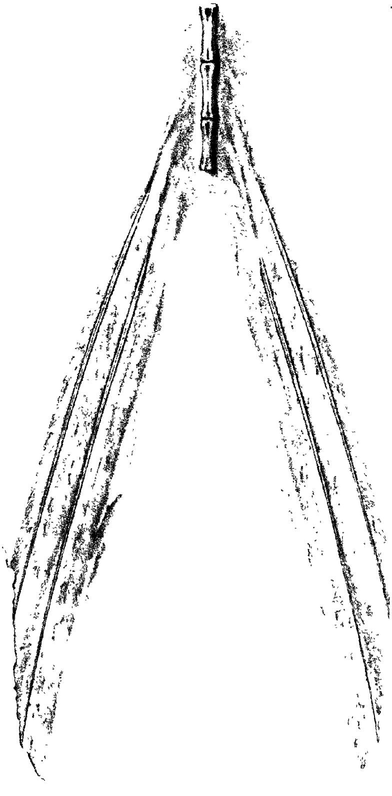


Fig 6

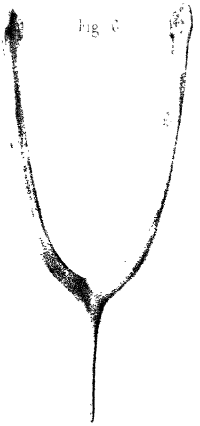


Fig 3

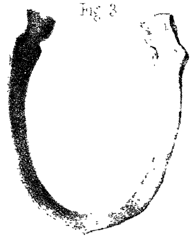


Fig. 9

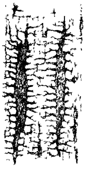


Fig 10

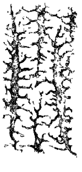


Figure 1

Theory of Strains article 6

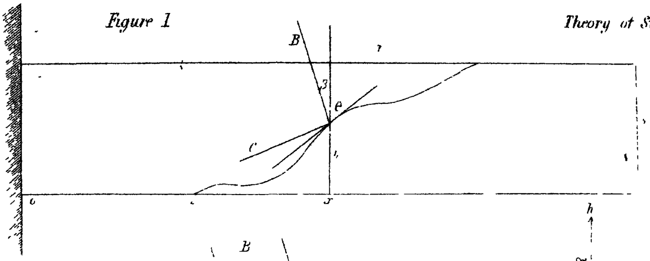


Figure 2
article 7

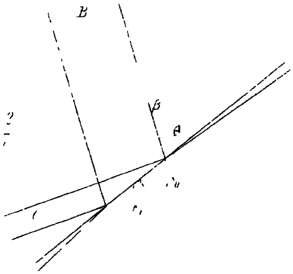


Figure 3 article 10

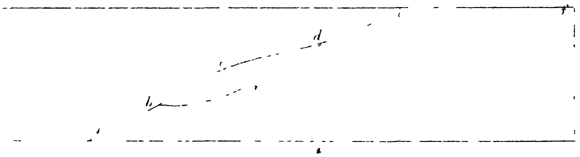
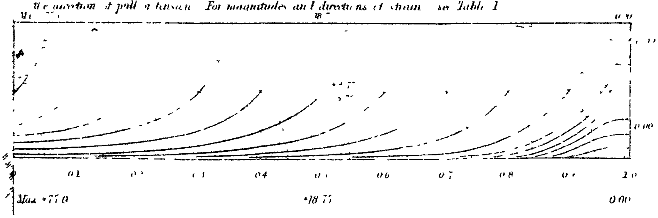


Figure 4

Strains in the interior of a beam which projects from a wall

The continuous curves indicate the direction of thrust or compression the interrupted curves or chain lines indicate the direction of pull or tension For magnitudes and directions of strain see Table I



Theory of Strains

Figure 5

Strains in the interior of a beam whose ends rest upon piers, and which supports no other weight

(The continuous curves indicate the direction of thrust or compression and the interrupted curves or chain-lines indicate the direction of pull or tension)

For magnitudes and directions of strains see Table II

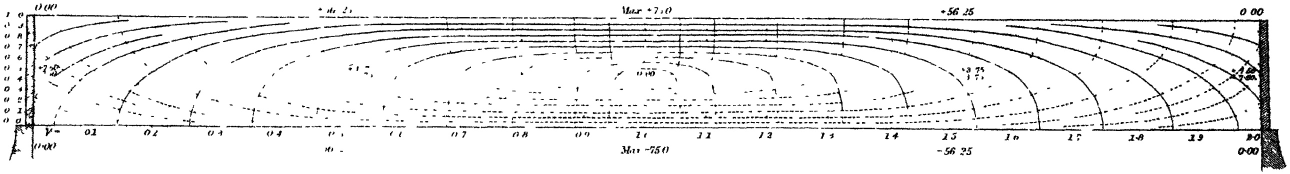


Figure 6

Strains in the interior of a beam whose ends rest upon piers and which supports at the middle of its length a weight equal to half the weight of the beam

For magnitudes and directions of strains see Table III

The curves which unite in the ordinate V=1 0 meet at a small angle

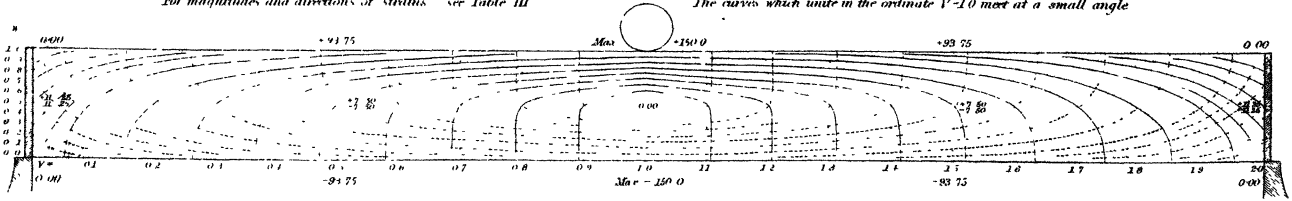
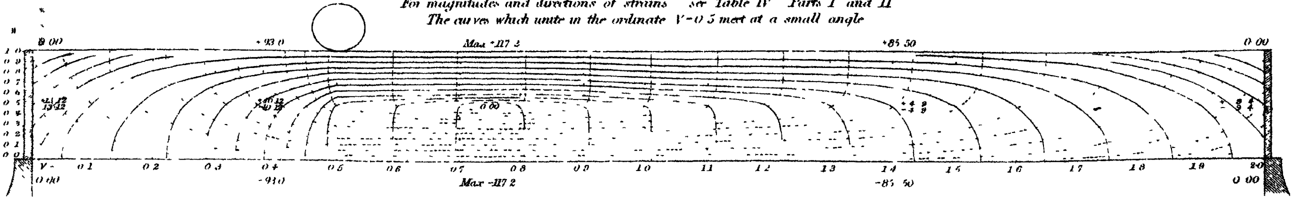


Figure 7

Strains in the interior of a beam whose ends rest upon piers and which supports at the middle of a half-length a weight equal to half the weight of the beam

For magnitudes and directions of strains see Table IV Parts I and II

The curves which unite in the ordinate V=0 5 meet at a small angle



Theory of Strains

Figure 8

Strains in the interior of a beam, whose ends rest upon piers and in which a strain (of the nature of a moment or couple) is impressed on each end, as in the interior tubes of the Britannia Bridge. For magnitudes and directions of strains see Table V

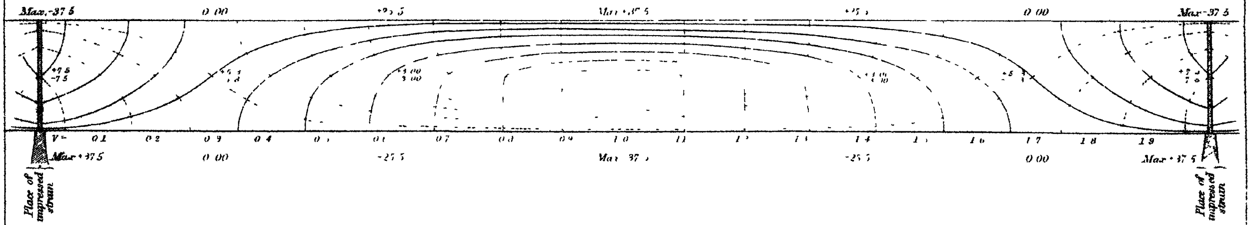
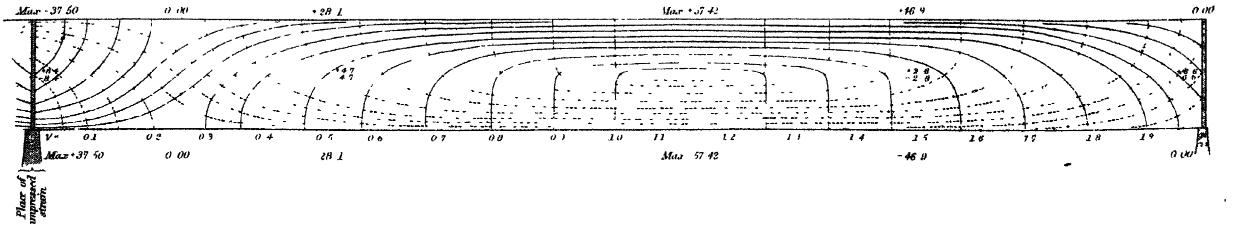


Figure 9

Strains in the interior of a beam whose ends rest upon piers and in which a strain (of the nature of a moment or couple) is impressed on one end, as in the exterior tubes of the Britannia Bridge. For magnitudes and directions of strains, see Table VI, parts I and II



J Basire Sc.

Fig 1

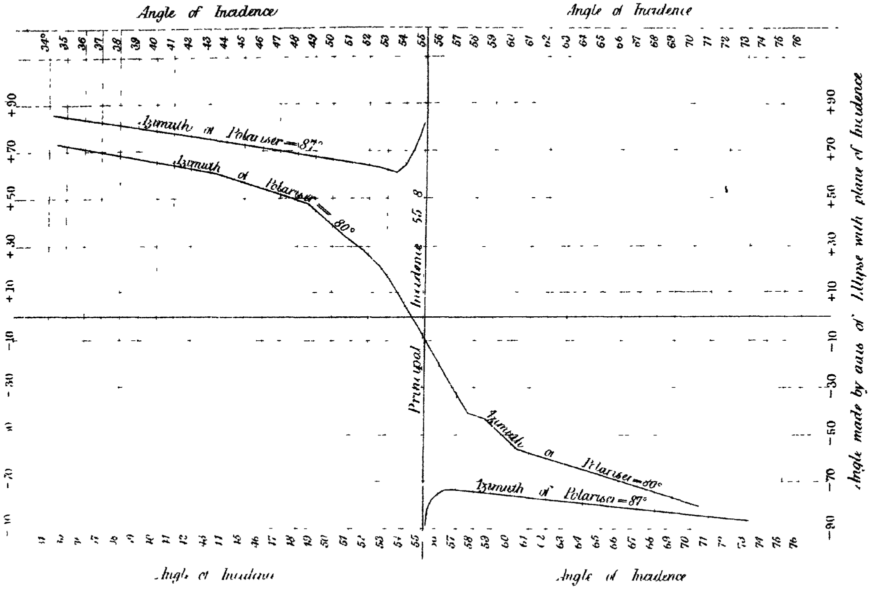
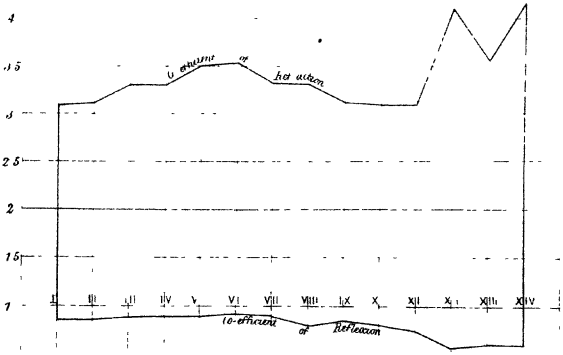
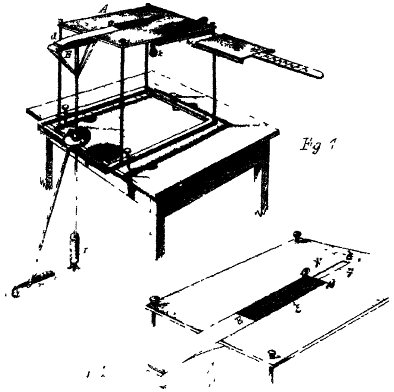

Angle of Incidence
34° 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76
Angle made by axis of Ellipse with plane of Incidence
+90 +70 +50 +30 +10 −10 −30 −50 −70 −90
Azimuth of Polariser = 87°
Azimuth of Polariser = 80°
Principal Incidence 55.8

Fig B


4
3.5
3
2.5
2
1.5
1
Co-efficient of Heat action
Co-efficient of Reflexion
I II III IV V VI VII VIII IX X XI XII XIII XIV



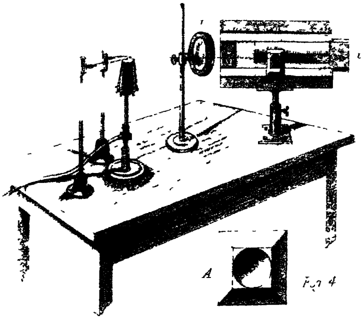
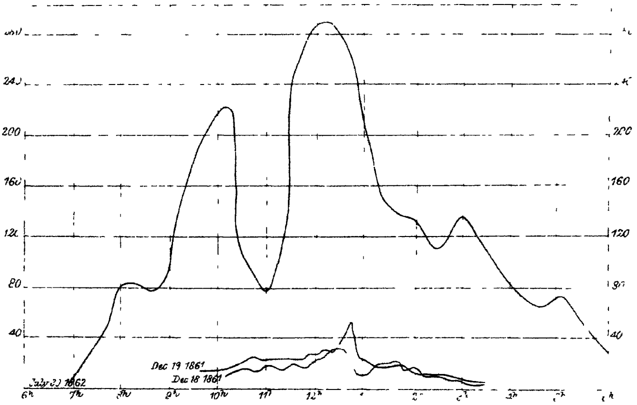
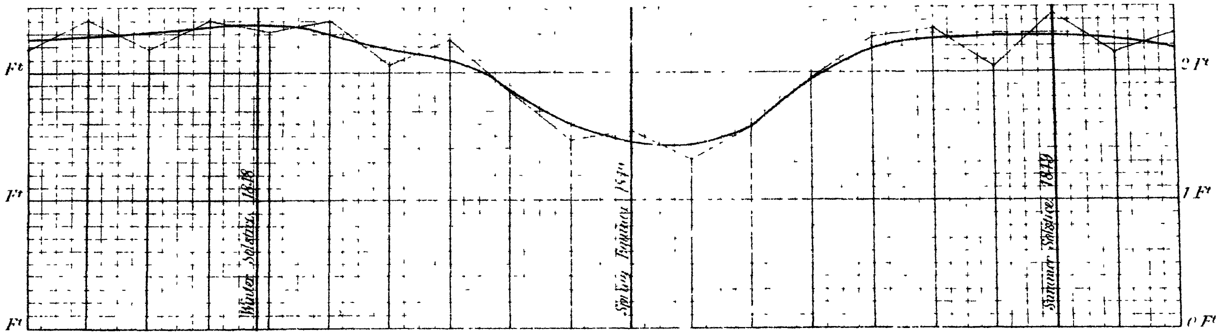


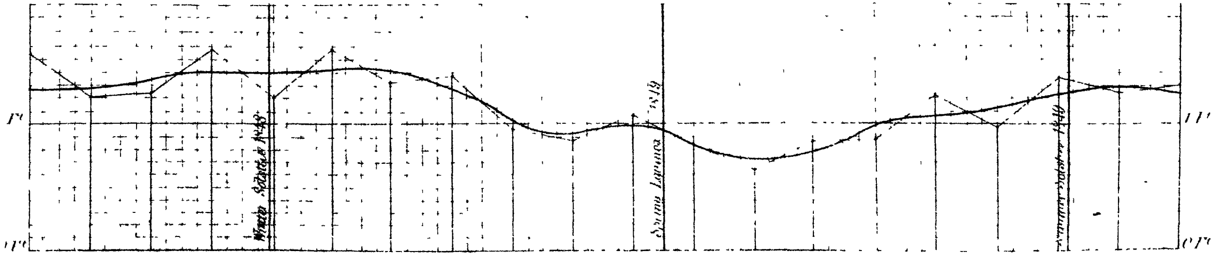
Fig 5



Basire lith

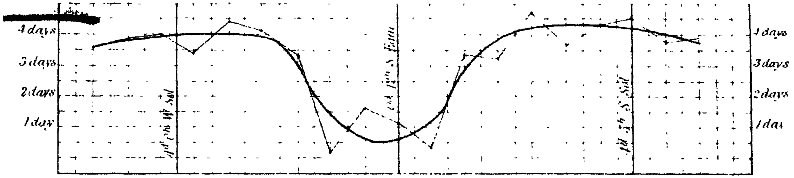


Range of Diurnal Tide, at High Water, at Port Leopold from November 1848 to July 1849



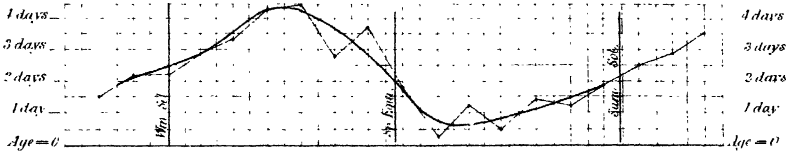
Range of Diurnal Tide, at Low Water, at Port Leopold from November 1848 to July 1849

Basire sc.



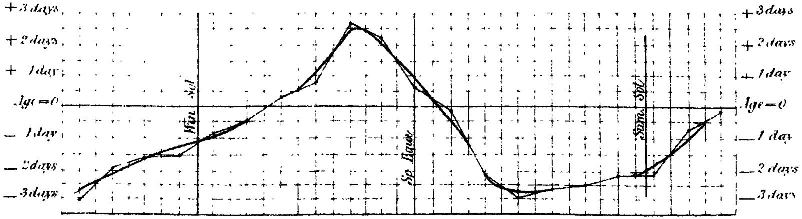
Interval of vanishing of Diurnal at High & Low Water

Fig. IV



Age of Diurnal Tide at time of High Water

Fig. V



Age of Diurnal Tide at time of Low Water.

J. Basire lith.

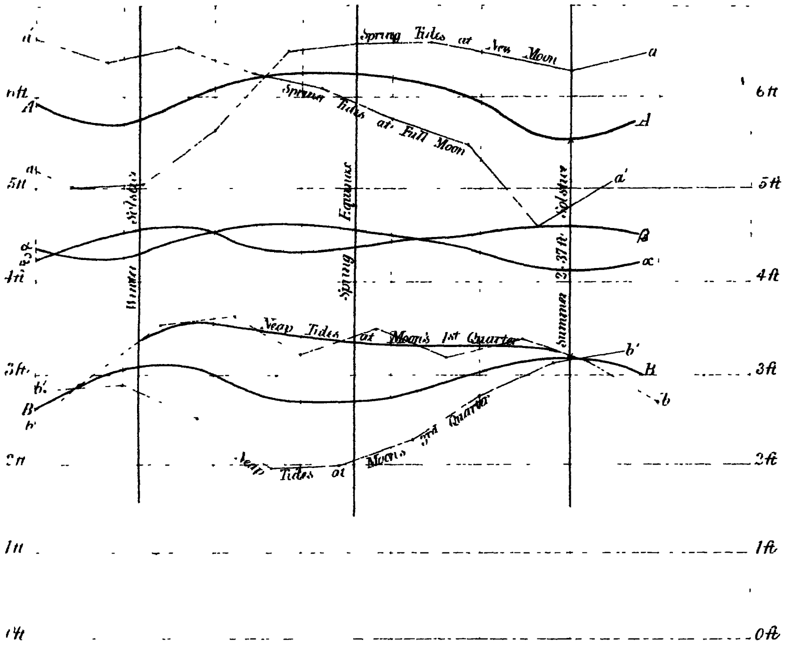
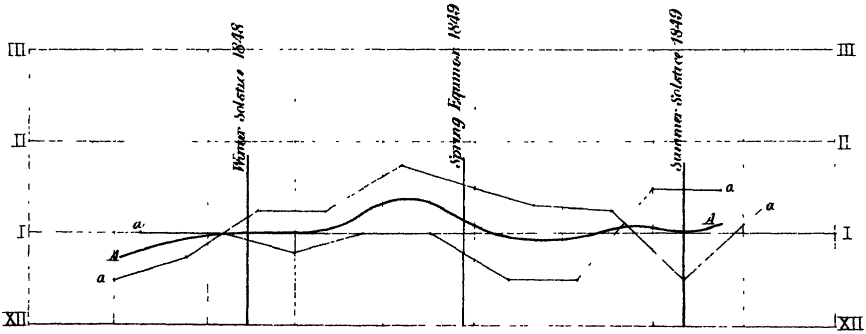
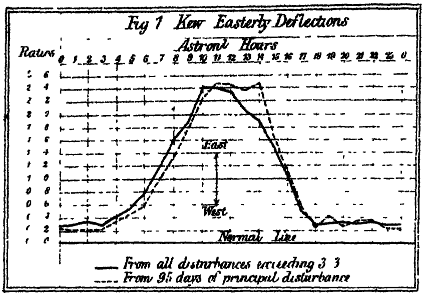


Diagram No 2



J Basire lith.

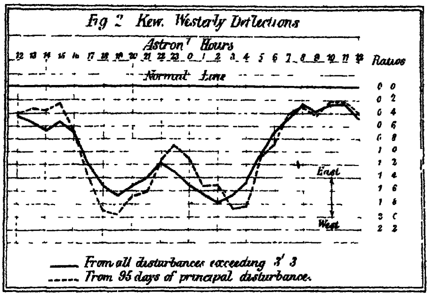
Fig 1 Kew Easterly Deflections



From all disturbances exceeding 3′ 3

From 95 days of principal disturbance

Fig 2 Kew. Westerly Deflections



From all disturbances exceeding 3′ 3

From 95 days of principal disturbance.

Fig 4 Toronto Easterly Deflections

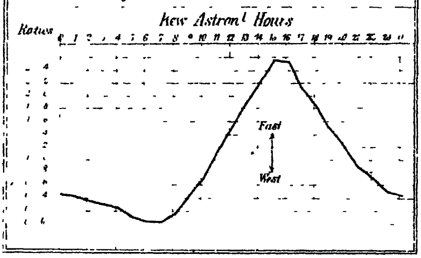


Fig 3 Kew, Easterly or Westerly Excess, or Disturbance – Diurnal Variation

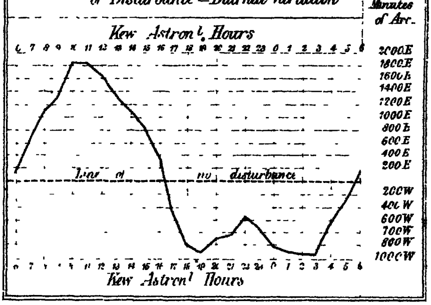


Fig 5 Nertschinsk Westerly Deflections

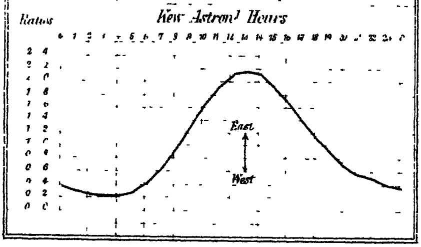


Fig 7 Solar diurnal Variation

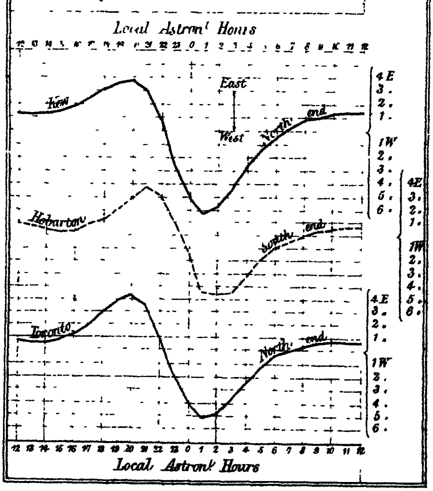
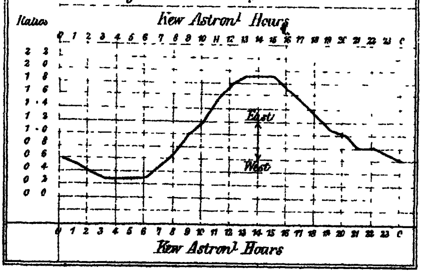
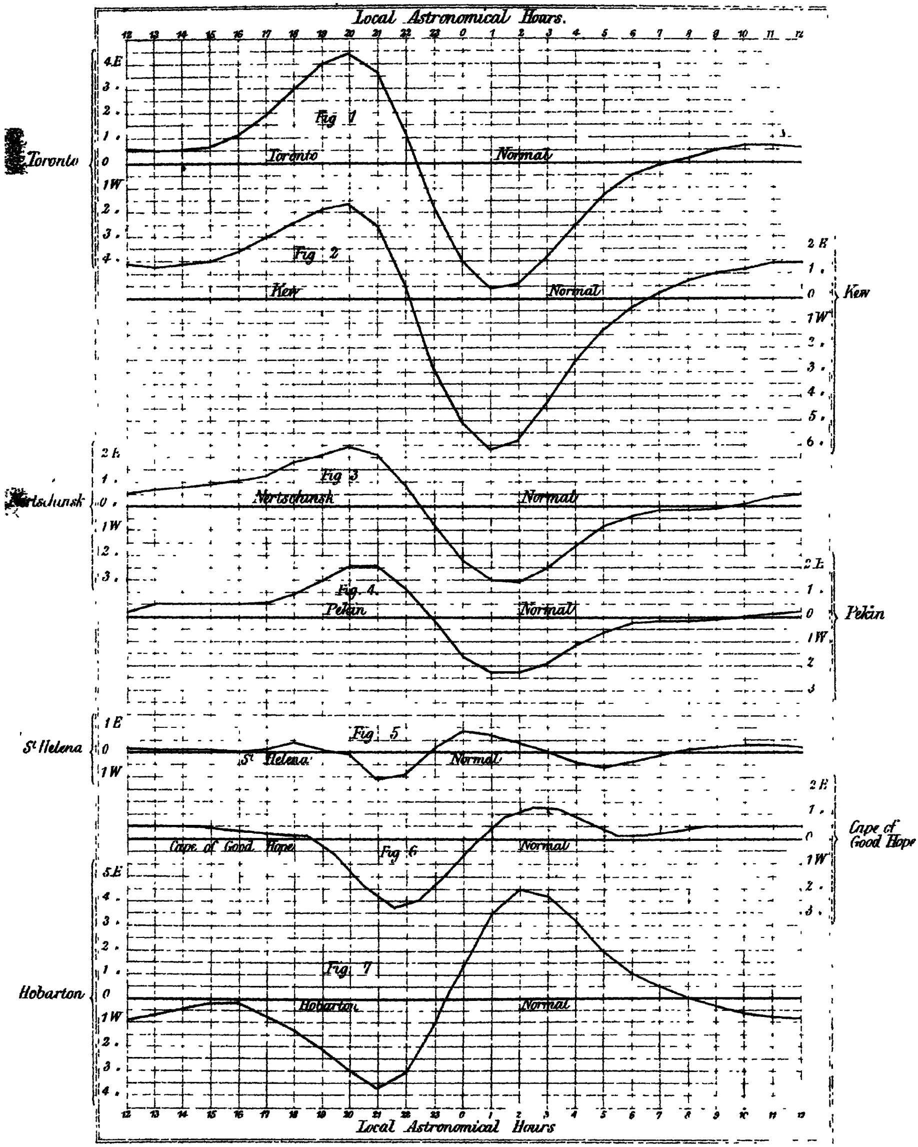


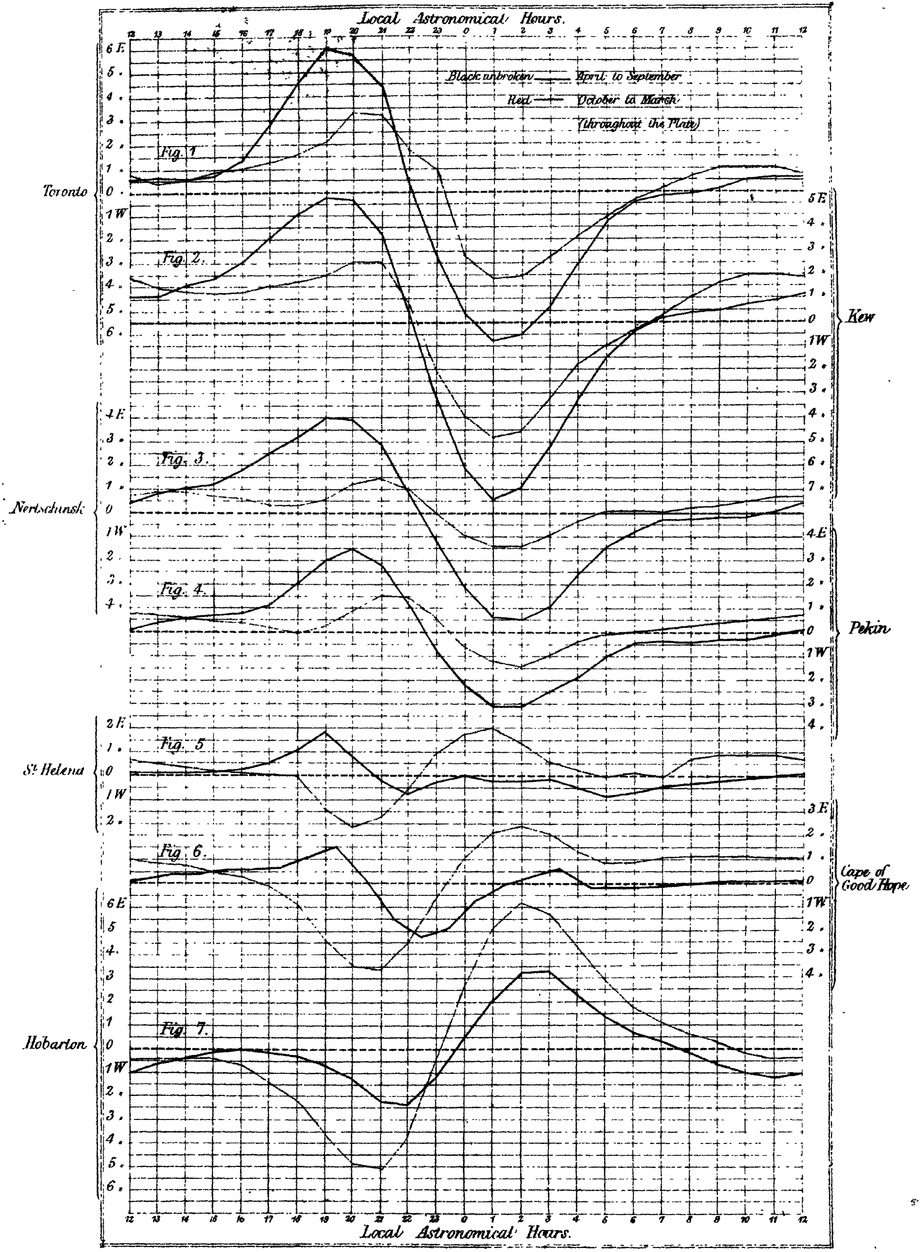
Fig 6 Pekin Westerly Deflections



Solar-diurnal Variation Deflections of the North End of the Magnet



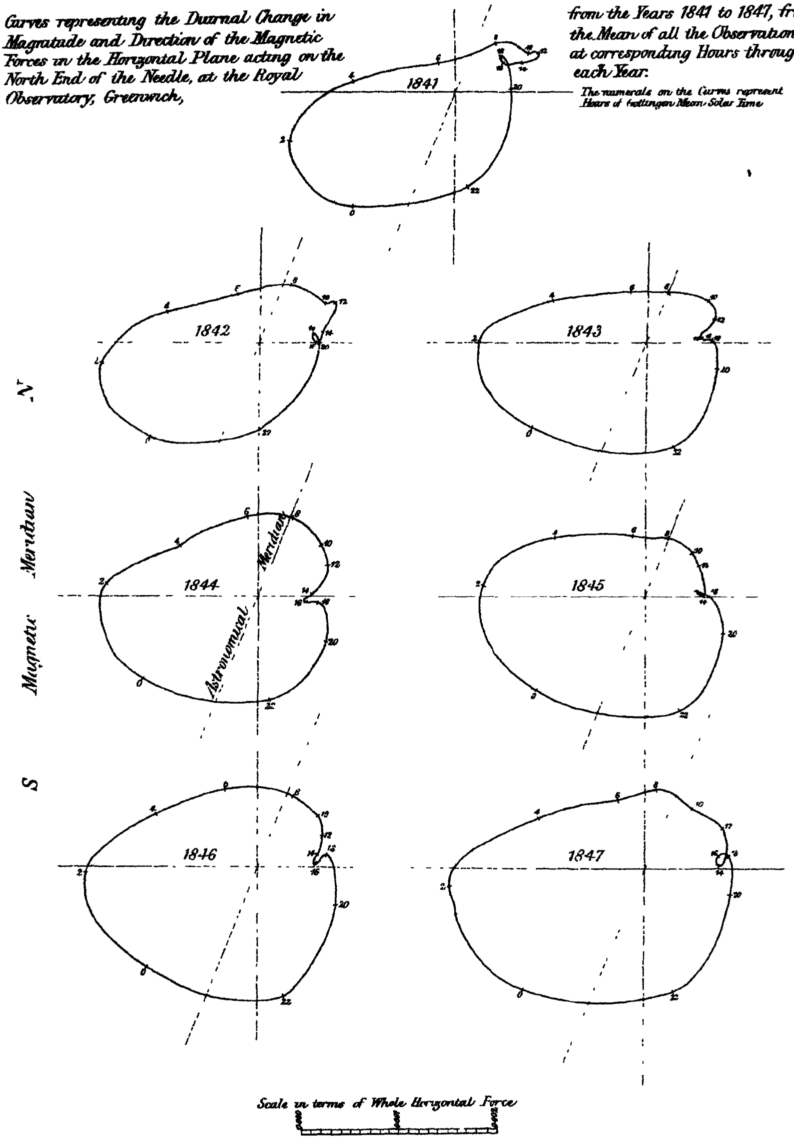
Semi-annual inequality of the Solar diurnal Variation.



J Basire lith.

Curves representing the Diurnal Change in Magnitude and Direction of the Magnetic Forces in the Horizontal Plane acting on the North End of the Needle, at the Royal Observatory, Greenwich, from the Years 1841 to 1847, from the Mean of all the Observations, at corresponding Hours through each Year.

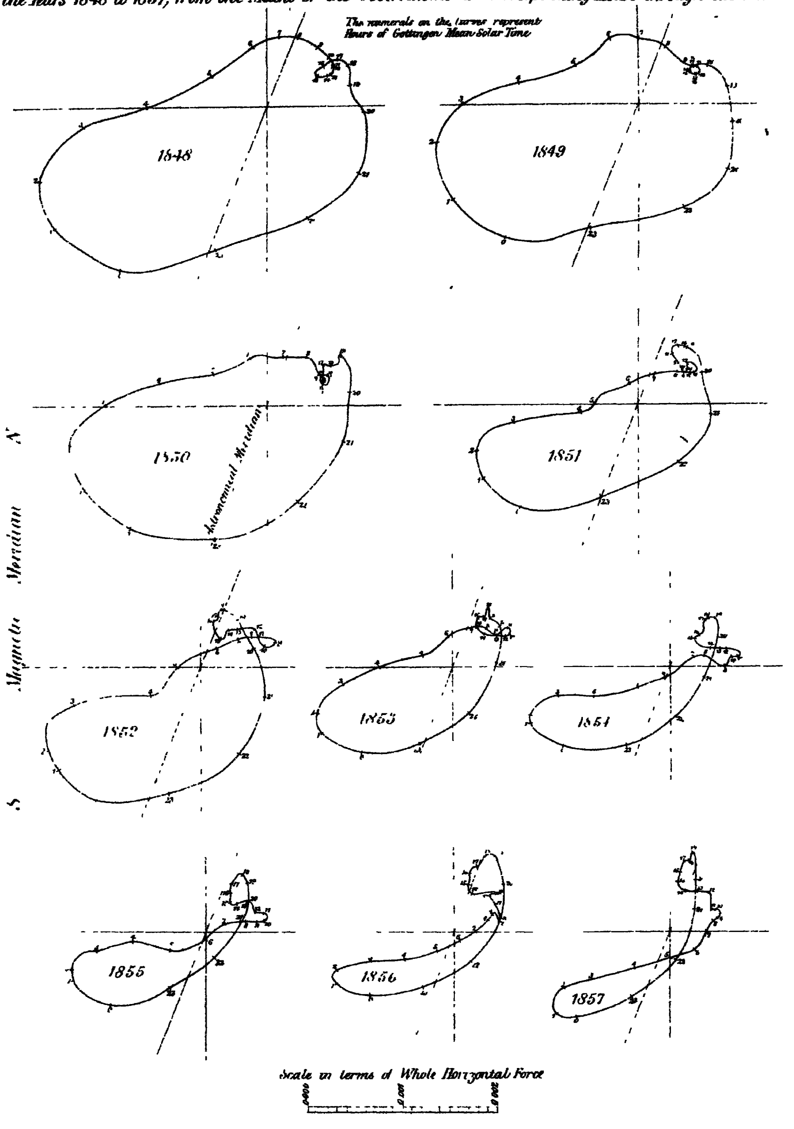
The numerals on the Curves represent Hours of Göttingen Mean Solar Time



T Basire lith

Phil. Trans. MDCCCLIX. Plate XI.

Curves representing the Diurnal Change in Magnitude and Direction of the Magnetic Forces in the Horizontal Plane acting on the North End of the Needle, at the Royal Observatory, Greenwich, for the Years 1848 to 1857, from the Means of the Observations at corresponding Hours through each Year



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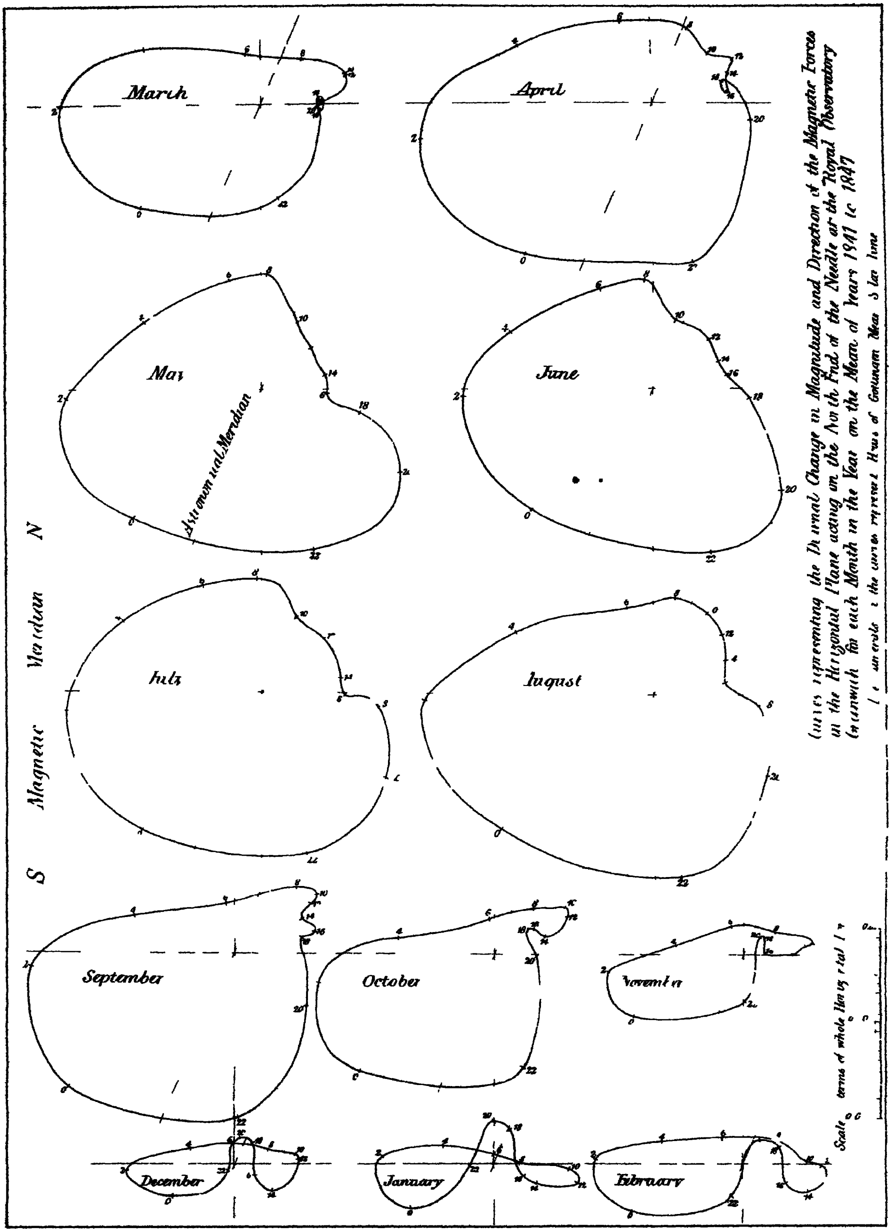
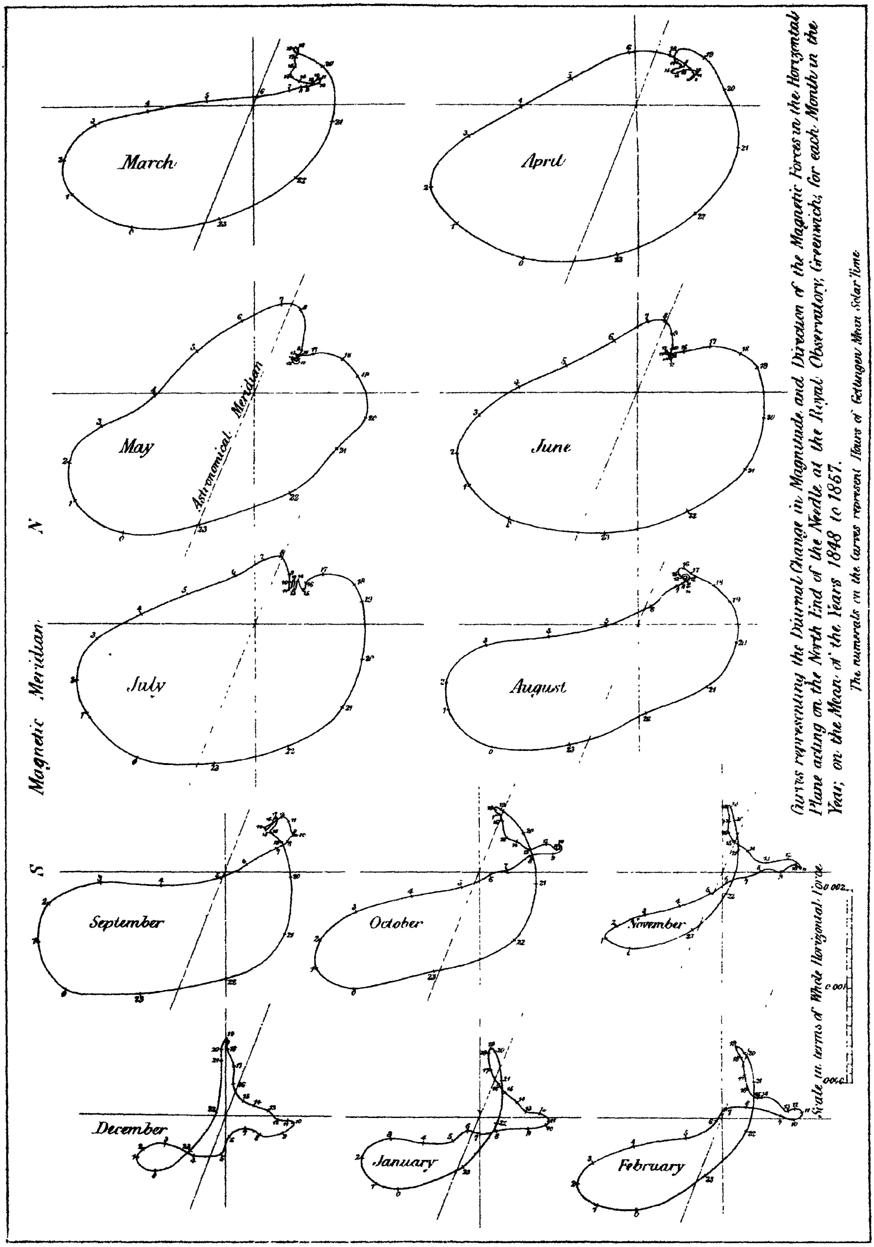

Curves representing the Diurnal Change in Magnitude and Direction of the Magnetic Forces in the Horizontal Plane acting on the North End of the Needle at the Royal Observatory Greenwich for each Month in the Year on the Mean of Years 1841 to 1847

The numerals on the curves represent Hours of Greenwich Mean Solar time

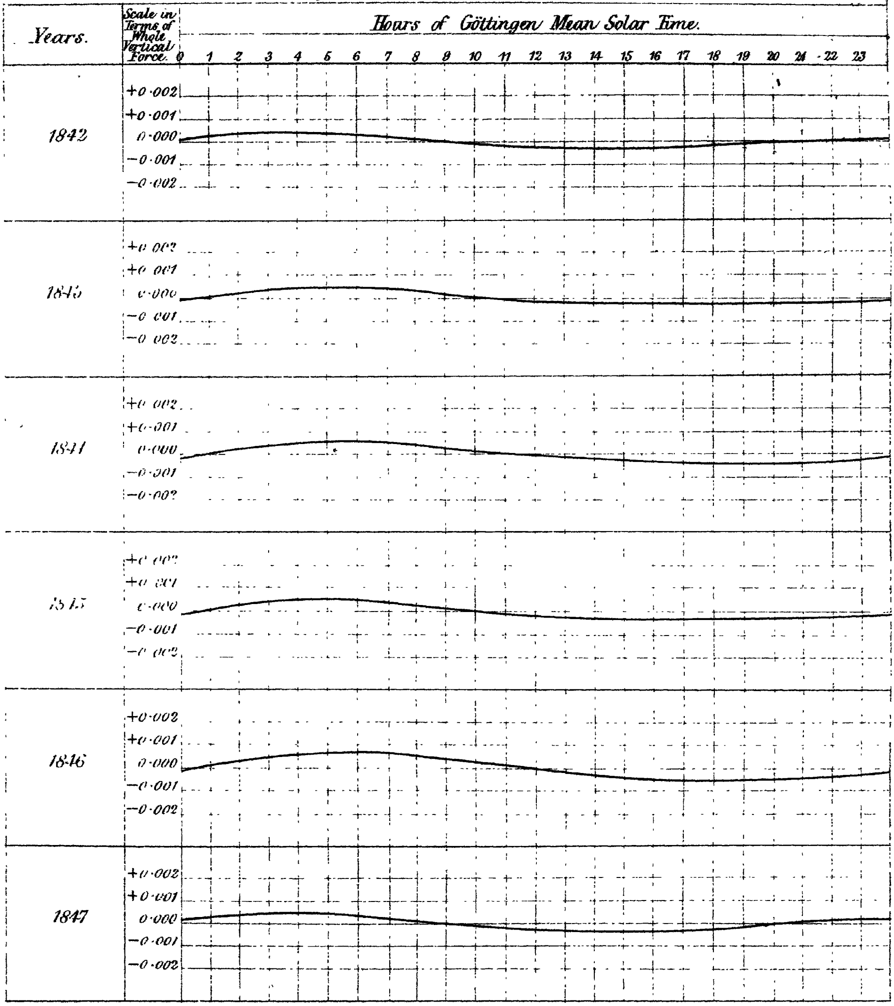
J. Basire lith.



Curves representing the Diurnal Change in Magnitude and Direction of the Magnetic Forces in the Horizontal Plane acting on the North End of the Needle at the Royal Observatory, Greenwich; for each Month in the Year; on the Mean of the Years 1848 to 1857.

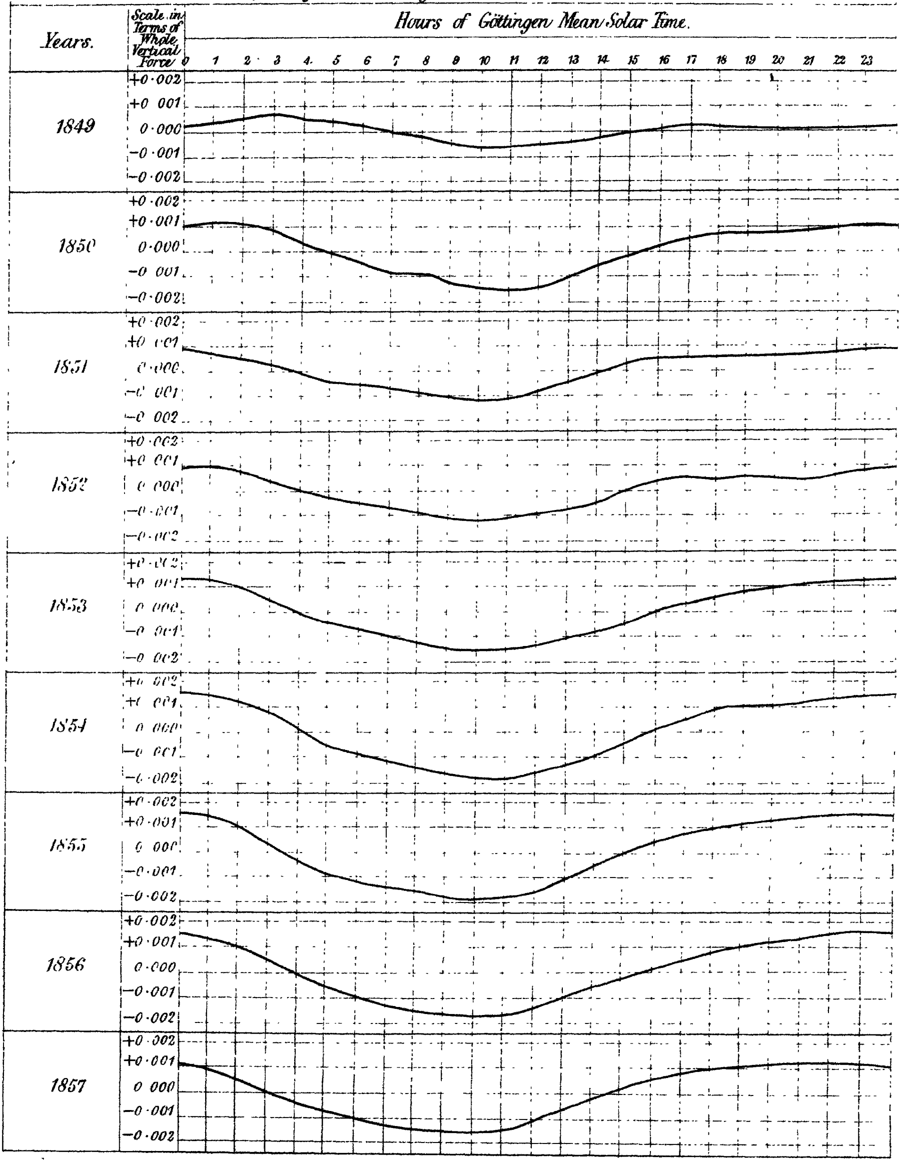
The numerals on the curves represent Hours of Gottingen Mean Solar Time

Curves representing the Diurnal Change in Magnitude and Sign of the Vertical Magnetic Force acting on the North End of the Needle at the Royal Observatory, Greenwich, for the Years 1842 to 1847, from the Mean of all the Observations at corresponding Hours through each Year.



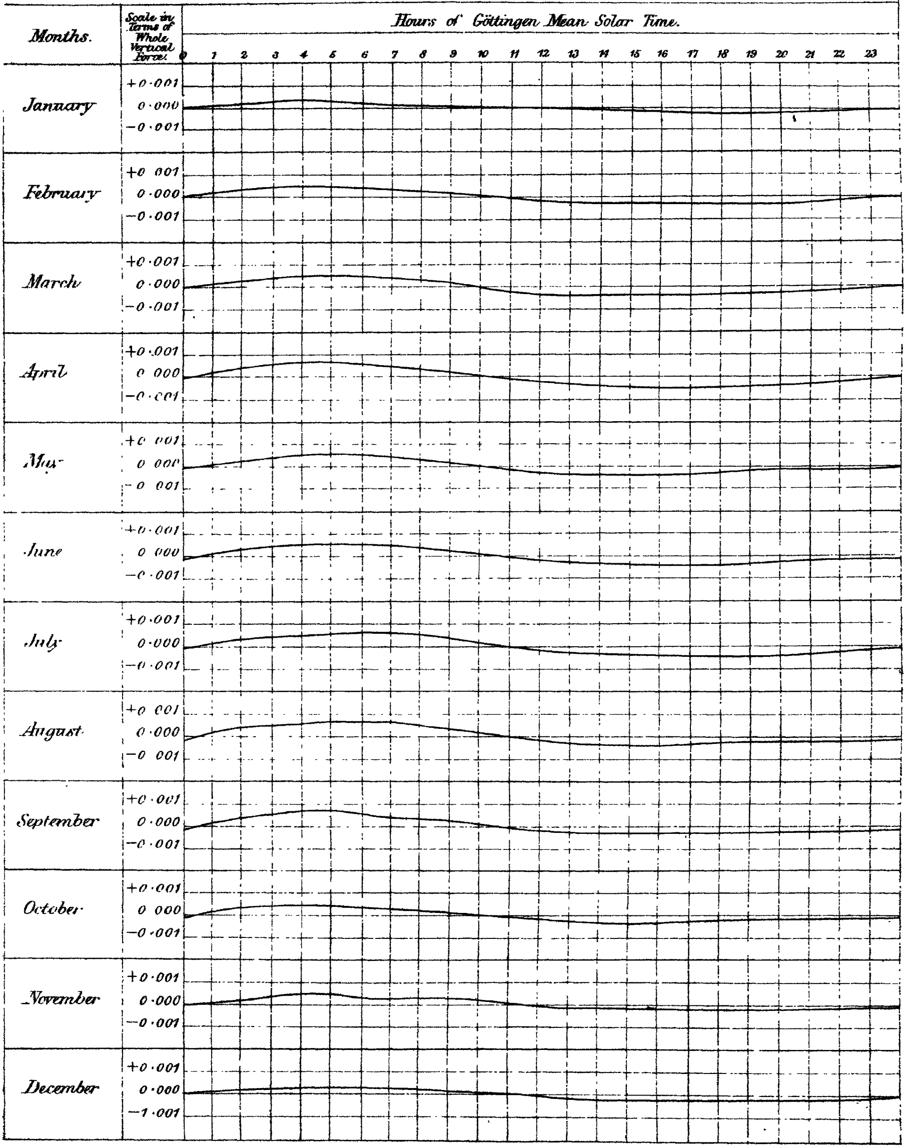
J. Basire lith.

Curves representing the Diurnal Change in Magnitude and Sign of the Vertical Magnetic Force acting on the North End of the Needle at the Royal Observatory, Greenwich, for the Years 1849 to 1857, from the Mean of all the Observations at corresponding Hours through each Year.



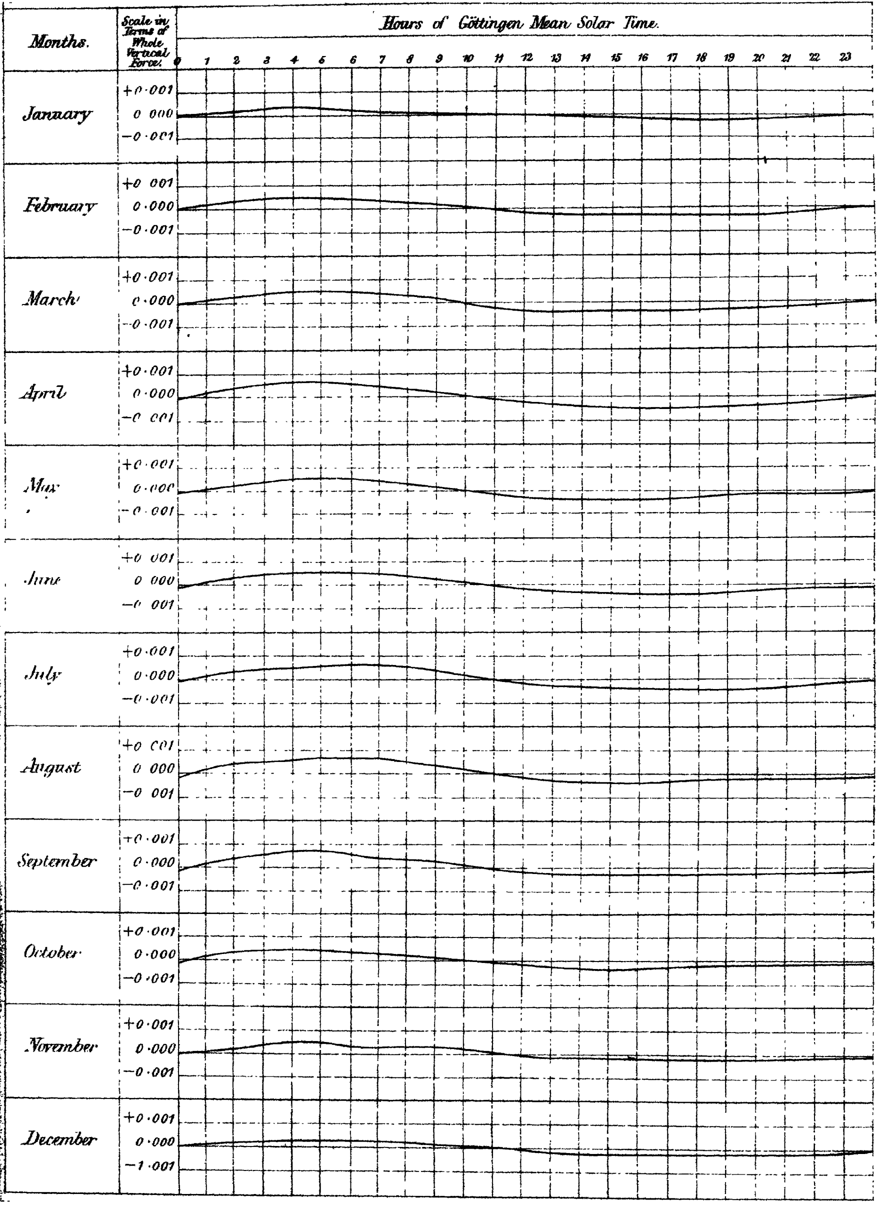
J. Basire lith.

Curves representing the Diurnal Change in Magnitude and Sign of the Magnetic Vertical Force acting on the North End of the Needle at the Royal Observatory, Greenwich; for each Month in the Year, on the Mean of Years 1842–1847.

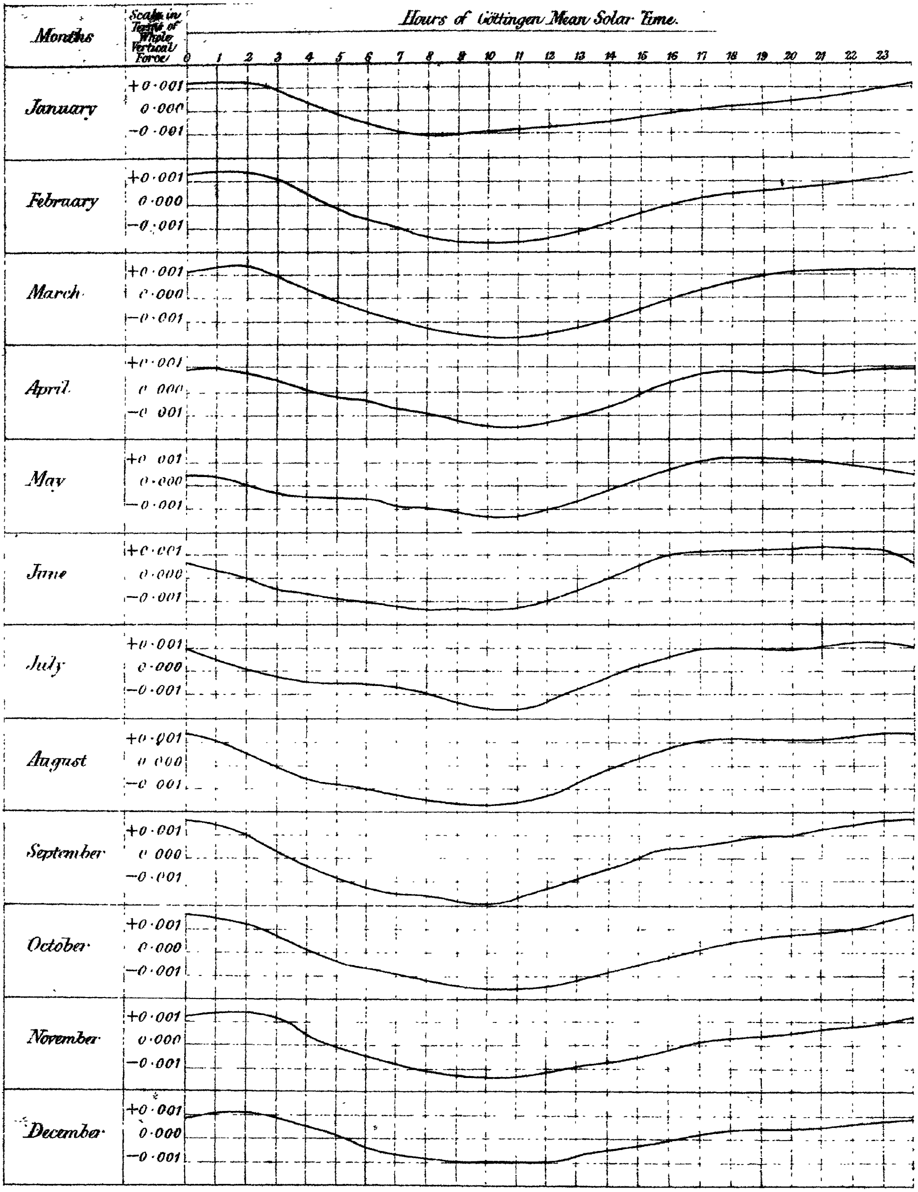


J. Basire, lith.

Curves representing the Diurnal Change in Magnitude and Sign of the Magnetic Vertical Force acting on the North End of the Needle at the Royal Observatory, Greenwich, for each Month in the Year, on the Mean of Years 1842–1847.



Curves representing the Diurnal Change in Magnitude and Sign of the Magnetic Vertical Force acting on the North End of the Needle at the Royal Observatory, Greenwich, for each Month in the Year, on the Mean of Years 1849–1857.



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PHILOSOPHICAL TRANSACTIONS

OF THE

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OF

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FOR THE YEAR MDCCCLXIII.

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MDCCCLXIV.

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XXX. *Results of hourly Observations of the Magnetic Declination made by Sir* Francis Leopold M^cClintock, *and the Officers of the Yacht 'Fox,' at Port Kennedy, in the Arctic Sea, in the Winter of* 1858–59; *and a Comparison of these Results with those obtained by Captain* Rochfort Maguire, *and the Officers of Her Majesty's Ship 'Plover,' in* 1852, 1853, *and* 1854, *at Point Barrow. By Major-General* Edward Sabine, *R.A., President of the Royal Society* 649

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Plates XXXIII. to XL.—Professor Beale on the Structure of the so-called Apolar, Unipolar, and Bipolar Nerve-cells of the Frog.

Plate XLI.—Major-General Sabine on the Results of hourly Observations of the Magnetic Declination at Port Kennedy.

ADJUDICATION of the MEDALS of the ROYAL SOCIETY for the year 1863 by the PRESIDENT and COUNCIL.

The COPLEY MEDAL to the Rev. ADAM SEDGWICK, F.R.S., for his original observations and discoveries in the Geology of the Palæozoic Series of Rocks, and more especially for his determination of the characters of the Devonian System, by observations of the order of Superposition of the Killas Rocks and their Fossils, in Devonshire.

A ROYAL MEDAL to the Rev. MILES J. BERKELEY, for his Researches in Cryptogamic Botany, especially Mycology.

A ROYAL MEDAL to JOHN P. GASSIOT, Esq., F.R.S., for his Researches on the Voltaic Battery and Current, and on the Discharge of Electricity through Attenuated Media.

The BAKERIAN LECTURE was delivered by HENRY CLIFTON SORBY, F.R.S.: it was entitled, "On the Direct Correlation of Mechanical and Chemical Forces."

The CROONIAN LECTURE was delivered by Professor JOSEPH LISTER, F.R.S.: it was entitled, "On the Coagulation of the Blood."

XVI. *On the Influence of Temperature on the Electric Conducting-Power of Thallium and Iron.*

By A. MATTHIESSEN, *F.R.S., Lecturer on Chemistry in St. Mary's Hospital, and* C. VOGT, *Ph.D.*

Received February 12,—Read March 12, 1863.

IT has been shown* that the conducting-power of several of the pure metals decreased between 0° and 100° to the same extent, namely 29·307 per cent. On continuing the research with other metals and alloys, we have found that thallium and iron form an exception to the above; and in the following we will describe the experiments made with these metals.

The thallium was kindly lent to us by Mr. CROOKES, the discoverer of this new metal, who with great readiness placed a small quantity of it at our disposal.

On account of its extreme softness (for it is much softer than pure lead), it was easily cut with a knife to fit one of the small presses described in the 'Philosophical Magazine' (February 1857), and pressed without the application of heat. The wire, as soon as it came out of the small hole of the press, passed into a tube filled with water; for although the metal readily oxidizes in air, yet it may be kept under water, free from air, for some time without oxidation. When, however, the wire is exposed to the air for a short time, it loses its lustre and is soon covered with a coating of oxide, which, as in the cases of lead and zinc, protects in a great measure the metal from further oxidation. The error caused by the slight oxidation during the short time we were obliged to manipulate with the wire in air may therefore be overlooked, more especially as wires of the same piece of metal showed the same conducting-power when pressed at different times. Now, if the slight oxidation had any marked effect, we ought to have found great differences in the conducting-power of different wires, for it can scarcely be supposed that in all cases the same amount of oxidation had taken place. Again, although thallium appears to be attacked by alcohol, yet we found we might varnish the wire with a solution of shell-lac in alcohol; for the small quantity of alcohol contained in the first coating of varnish volatilizes so quickly that it has very little time to act on the wire; in fact the resistance of the wire was found to be the same when determined before being varnished, and after three or four coatings of varnish had dried on it. The reason of varnishing the wire was to protect the metal from the action of the hot oil.

The apparatus and precautions taken whilst determining the conducting-powers at different temperatures have already been fully described†. The normal wires used were made of German silver; these were compared with the gold-silver alloy*, and the values so obtained for their resistances reduced, for the sake of comparison with former observations, to that of a hard-drawn silver wire whose conducting-power at 0° is equal to 100, that of a hard-drawn gold-silver wire being then 15·03 at that temperature. As in the case of wires of most other metals, those of thallium were found to alter in conducting-power after having been kept at 100° for some time. It was therefore necessary to heat them for several days until their conducting-power showed no longer any alteration after cooling to the original temperature.

---

* "On the Influence of Temperature on the Electric Conducting-Power of Metals. By A. MATTHIESSEN and M. VON BOSE," Philosophical Transactions, 1862, p. 1.

† Philosophical Transactions, 1862, p. 1.

The length of the wire experimented with was 158 millims., and its diameter 0·502 millim.

		Reduced to 0°†.
Conducting-power found before heating the wire . . .	8·808 at 13°·2	9·290
,, ,, after being kept at 100° for 1 day .	8·939 ,, 10·8	9·338
,, ,, ,, ,, 2 days .	8·941 ,, 11·8	9·378
,, ,, ,, ,, 3 days .	8·949 ,, 11·6	9·378
,, ,, ,, ,, 4 days .	8·990 ,, 10·2	9·368

The means of the conducting-powers for each of the following temperatures were—

T.	Conducting-power.		Difference.
	Observed.	Calculated.	
10°·40	8·987	8·979	+0·008
25·07	8·460	8·466	−0·006
39·22	7·996	8·006	−0·010
54·43	7·551	7·550	+0·001
69·68	7·138	7·132	+0·006
85·95	6·742	6·729	+0·013
100·13	6·404	6·414	−0·010

The formula deduced from the observations, from which the conducting-powers were calculated, was

$$\lambda = 9{\cdot}364 - 0{\cdot}037936t + 0{\cdot}00008467t^2,$$

or

$$\lambda_1 = 100 - 0{\cdot}40513t + 0{\cdot}0009042t^2,$$

corresponding to a percentage decrement of 31·471 per cent.

To check the above value another wire was made, and its conducting-power determined.

The length of the wire was 187 millims., its diameter 0·421 millim., and the conducting-power found was

8·610 at 15°·6, or, reduced to 0° by the above formula, 9·169.

Now the first observation made with the first wire was 9·29 at 0°; or, as mean, we find the conducting-power of pure thallium at 0° equal to 9·23.

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* Philosophical Magazine, February 1861.

† The manner in which these values were reduced is fully described in the paper already alluded to (Philosophical Transactions, 1862, p. 10).

Although we had already found, as will be shown in the following, that the conducting-power of iron decreases between 0° and 100° more than 29·307 per cent., the mean value deduced from the percentage decrements of eleven metals, yet we thought it would be interesting to check the above results, and we therefore applied to Professor LAMY, of Lille, for a specimen of the metal, who with great kindness lent us two small bars prepared at different times. The results obtained with these specimens fully confirm those obtained with Mr. CROOKES's metal, both in respect to the conducting-power and to the percentage decrement.

The results obtained with these specimens were as follows:—

1st Bar.

Length . . . . 205 millims.
Diameter . . . 0·553 millim.

			Reduced to 0°.
Conducting-power found before heating the wire . . .	8·881	at 11°·8	9·330
„ „ after being kept at 100° for 1 day .	8·986	„ 10·0	9·370
„ „ „ „ 2 days .	8·953	„ 11·9	9·410
„ „ „ „ 3 days .	8·901	„ 13·4	9·413

The means of the conducting-powers for each of the following temperatures were—

T.	Conducting-power.		Difference.
	Observed.	Calculated.	
13°·75	8·903	8·894	+0·009
25·23	8·477	8·483	−0·006
38·80	8·021	8·031	−0·010
54·68	7·544	7·547	−0·003
70·56	7·122	7·111	+0·011
82·71	6·819	6·811	+0·008
99·57	6·433	6·441	−0·008

The formula deduced from the observations, from which the conducting-powers were calculated, was

$$\lambda = 9\cdot419-0\cdot039520t+0\cdot00009656t^2,$$

or

$$\lambda_1=100-0\cdot41958t+0\cdot001025t^2,$$

corresponding to a percentage decrement of 31·706 per cent.

Another wire from the same piece of metal, the length of which was 136 millims. and the diameter 0·532, conducted

8·433 at 17°·8, or, reduced to 0°, 9·082.

A third wire, the length of which was 141 millims., the diameter 0·449 millim., conducted

8·758 at 12°·4, or, reduced to 0°, 9·223.

The first observation made with the first wire reduced to 0° was 9·33; and if we now take the mean of these three values, we find the conducting-power of the first bar to be at 0° equal to

$$9{\cdot}212.$$

2nd Bar.

Length	155 millims.
Diameter . . .	0·502 millim.

			Reduced to 0°.
Conducting-power found before heating the wire . . .	8·377	at 14°·8	8·866
,, ,, after being kept at 100° for 1 day .	8·692	,, 10·0	9·032
,, ,, ,, ,, 2 days .	8·764	,, 8·6	9·058

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
9°·0	8·747
54·5	7·258
100·0	6·240

from which numbers the following formula was deduced,

$$\lambda = 9{\cdot}054 - 0{\cdot}034697t + 0{\cdot}00006554t^2,$$

or

$$\lambda_1 = 100 - 0{\cdot}38322t + 0{\cdot}0007239t^2,$$

corresponding to a percentage decrement of, 31·083 per cent.

A second wire, 135·5 millims. long and having a diameter of 0·542 millim., was found to conduct

$$8{\cdot}507 \text{ at } 21^\circ{\cdot}2, \text{ or, reduced to } 0^\circ,\ 9{\cdot}226.$$

Now, as before, taking the mean of this value and that found for the first determination, we find the conducting-power of the second bar equal to, at 0°,

$$9{\cdot}046.$$

From the foregoing it will be seen that the values obtained for the conducting-powers of the different specimens agree very closely with each other; for that found

	Conducting-power at 0°.	Percentage decrement for the conducting-power between 0° and 100°.
For Mr. Crookes's metal . .	9·230	31·471
For M. Lamy's 1st bar. . .	9·212	31·706
2nd bar . .	9·046	31·083
Mean . . .	9·163	31·420

and calculating, from the mean of the above, the formula for the correction of the conducting-power for temperature of thallium, we find it to be

$$\lambda = 9{\cdot}163 - 0{\cdot}036894t + 0{\cdot}00008104t^2,$$

where λ represents the conducting-power at t°.

The mean of the three formulæ reduced to the same unit at 0° was used for calculating the above, viz.

$$\lambda_1 = 100 - 0{\cdot}40264t + 0{\cdot}0008844t^2.$$

As thallium in many respects resembles lead, it seemed to us of peculiar interest to see whether in its electric behaviour it would resemble that metal, namely, if when alloyed with tin, cadmium, or zinc, the conducting-power of the alloy would be equal to the mean conducting-power of the volumes of the component metals. We are indebted to Mr. CROOKES for an alloy of thallium and tin containing only traces of the latter metal, and on testing its conducting-power we found it lower than that of pure thallium, showing that the addition of a better conductor (the conducting-power of tin being at 0° 12·366) causes a decrement in the conducting-power of the metal. Again, we alloyed a portion of M. LAMY's thallium with traces of cadmium; and here we also found the above observation confirmed (the conducting-power of cadmium being at 0° 23·725). Thallium, therefore, appears to belong to that class of metals* which, when alloyed with lead, tin, cadmium, or zinc, or with one another, do not conduct electricity in the ratio of their relative volumes, but always in a lower degree than the mean of their volumes, and not to that class of metals to which lead belongs, namely, those which when alloyed with one another conduct electricity in the ratio of their relative volumes.

Respecting the conducting-power of the alloys of thallium, we shall discuss them in our paper "On the Influence of Temperature on the Electric Conducting-Power of Alloys," which will shortly be ready for publication.

We are greatly indebted to the kindness of Professor PERCY, who placed at our disposal the specimens of iron used for the following experiments; in fact, with the exception of the two last, they are all from his collection. As several of them have been analysed by Mr. TOOKEY in his laboratory, the results we have obtained will be the more interesting, as they show how traces of foreign matter influence the conducting-power of iron.

We will first give the numerical results, and then make some remarks on them.

1. *Electrotype Iron*, deposited from a solution of pure sulphate of iron. The strips were very thin and porous; we could not, therefore, obtain concordant values for the conducting-power, but were able to determine the percentage decrement in the conducting-power between 0° and 100°. We have therefore taken the first observed conducting-power equal 100.

		Reduced to 0°.
Conducting-power found before heating the strip . .	100·000 at 18°·1	109·698
„ „ after being kept at 100° for 1 day.	100·520 at 16·8	109·539
„ „ „ „ 2 days	100·894 at 15·9	109·443

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
10°·0	103·926
55·0	82·866
100·0	67·528

from which numbers the following formula was deduced,

$$\lambda = 109{\cdot}38 - 0{\cdot}55983t + 0{\cdot}001413t^2,$$

or

$$\lambda_1 = 100 - 0{\cdot}51182t + 0{\cdot}0012915t^2,$$

corresponding to a percentage decrement of 38·262 per cent.

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* Philosophical Transactions, 1860, p. 162.

2. No. 1, annealed and cooled in hydrogen.

		Reduced to 0°.
Conducting-power found before heating the strip . .	100·000 at 20°·8	111·375
,, ,, after being kept at 100° for 1 day .	100·518 at 19·5	111·202
,, ,, ,, ,, 2 days	100·400 at 19·8	111·234
,, ,, ,, ,, 3 days	101·243 at 18·8	111·560

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
12°·0	104·564
56·0	83·622
100·0	68·460

from which numbers the following formula was deduced,

$$\lambda = 111{\cdot}275 - 0{\cdot}57745t + 0{\cdot}0014928t^2,$$

or

$$\lambda_1 = 100 - 0{\cdot}51894t + 0{\cdot}0013415t^2,$$

corresponding to a percentage decrement of 38·479 per cent.

3. Electrotype iron, a strip cut from the same foil as No. 1.

		Reduced to 0°.
Conducting-power found before heating the strip . .	100·000 at 16°·8	108·997
,, ,, after being kept at 100° for 1 day	99·867 at 17·6	109·299

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
11°·0	102·958
55·5	82·324
100·0	67·396

from which numbers the following formula was deduced,

$$\lambda = 108{\cdot}943 - 0{\cdot}55947t + 0{\cdot}0014404t^2,$$

or

$$\lambda_1 = 100 - 0{\cdot}51355t + 0{\cdot}0013221t^2,$$

corresponding to a percentage decrement of 38·134 per cent.

4. No. 3, annealed in air.

		Reduced to 0°.
Conducting-power found before heating the strip . .	100·000 at 21°·3	111·436
,, ,, after being kept at 100° for 1 day .	102·944 at 17·2	112·356
,, ,, ,, ,, 2 days	102·705 at 17·6	112·323

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
10°·0	107·025
55·0	85·427
100·0	69·636

from which numbers the following formula was deduced,

$$\lambda = 112{\cdot}615 - 0{\cdot}57315t + 0{\cdot}0014341t^2,$$

or

$$\lambda_1 = 100 - 0{\cdot}50895t + 0{\cdot}0012735t^2,$$

corresponding to a percentage decrement of 38·16 per cent.

5. This, and likewise Nos. 6, 7, and 8, were cut and drawn into wire from pieces of metal which had been analysed. The analyses were as follows:—

	5. In 100 parts.	6. In 100 parts.	7. In 100 parts.	8. In 100 parts.
Sulphur	0·190	0·121	0·104	0·118
Phosphorus . . .	0·020	0·173	0·106	0·228
Silicon	0·014	0·160	0·122	0·174
Carbon	0·230	0·040	0·020	0·020
Manganese Cobalt Nickel . .	0·110	0·029	0·280	0·250

The length of the wire used from No. 5 was 752 millims. and its diameter 0·658 millim.

		Reduced to 0°.
Conducting-power found before heating the wire . .	17·887 at 11°·4	15·712
,, ,, after being kept at 100° for 1 day .	15·004 at 9·8	15·716

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
10°·0	14·993
55·0	12·162
100·0	10·045

from which numbers the following formula was deduced,

$$\lambda = 15\cdot719 - 0\cdot07437t + 0\cdot0001763t^2,$$

or

$$\lambda_1 = 100 - 0\cdot47312t + 0\cdot0011242t^2,$$

corresponding to a percentage decrement of 36·07 per cent.

6. Length . . . . 1047 millims.
Diameter . . . 0·778 millim.

		Reduced to 0°.
Conducting-power found before heating the wire . .	14·543 at 15°·4	15·640
,, ,, after being kept at 100° for 1 day .	15·002 at 9·4	15·682

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
12°·0	14·809
56·0	12·078
100·0	10·029

from which numbers the following formula was deduced,

$$\lambda = 15\cdot672 - 0\cdot074045t + 0\cdot0001761t^2,$$

or

$$\lambda_1 = 100 - 0\cdot47247t + 0\cdot0011237t^2,$$

corresponding to a percentage decrement of 36·01 per cent.

7.

Length . . . . 589 millims.
Diameter . . . 0·622 millim.

		Reduced to 0°.
Conducting-power found before heating the wire . .	13·351 at 13°·8	14·204
„ „ after being kept at 100° for 1 day .	13·469 at 12·8	14·266
„ „ „ „ 2 days .	13·435 at 13·4	14·268

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
15°·0	13·340
57·5	11·063
100·0	9·312

from which numbers the following formula was deduced,

$$\lambda = 14{\cdot}269 - 0{\cdot}064133t + 0{\cdot}0001456t^2,$$

or

$$\lambda_1 = 100 - 0{\cdot}44946t + 0{\cdot}0010204t^2,$$

corresponding to a percentage decrement of 34·742 per cent.

8. This piece of metal was drawn into wire with great difficulty, owing to its being very brittle. The wire used was somewhat faulty, and this may account for the low conducting-power found.

Length . . . . 160 millims.
Diameter . . . 0·479 millim.

		Reduced to 0°.
Conducting-power found before heating the wire . .	11·242 at 16°·9	12·132
„ „ after being kept at 100° for 1 day .	11·275 at 17·9	12·222
„ „ „ „ 2 days .	11·287 at 18·0	12·241

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
9°·67	11·814
54·77	9·694
99·80	8·137

from which numbers the following formula was deduced,

$$\lambda = 12{\cdot}342 - 0{\cdot}055894t + 0{\cdot}0001379t^2,$$

or

$$\lambda_1 = 100 - 0{\cdot}45291t + 0{\cdot}0011174t^2,$$

corresponding to a percentage decrement of 34·117 per cent.

9. This specimen formed the basis for some experiments made by Dr. PERCY on the absorption of carbon by iron, and was cut from the same piece of foil as Nos. 10, 11, 12. The strips of foil were first annealed in a current of dry hydrogen at a red heat for about two hours. No. 9 was therefore annealed in hydrogen, Nos. 10, 11, 12 treated first with No. 9, and then separately under a layer of sugar charcoal in a current of hydrogen for different lengths of time. They were all hardened. Dr. PERCY's experiments have not yet been published, but he informs us they will be given in his work on Metallurgy.

Length	171 millims.
Diameter . . .	0·402 millim.

		Reduced to 0°.
Conducting-power found before heating the strip . .	14·096 at 9°·4	14·723
„ „ after being kept at 100° for 1 day .	14·123 at 9·0	14·724

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
10°·0	14·009
55·0	11·416
100·0	9·471

from which numbers the following formula was deduced,

$$\lambda = 14{\cdot}673-0{\cdot}067999t+0{\cdot}0001597t^2,$$

or

$$\lambda_1=100-0{\cdot}46343t+0{\cdot}0010884t^2,$$

corresponding to a percentage decrement of 35·459 per cent.

10. Heated for three hours under sugar charcoal in a current of hydrogen; the carbon taken up was 0·99 per cent.

Length	105 millims.
Diameter . . .	0·281 millim.

		Reduced to 0°.
Conducting-power found before heating the strip . .	10·376 at 6°·6	10·666
„ „ after being kept at 100° for 1 day .	10·282 at 9·0	10·676

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
10°·0	10·218
55·0	8·499
100·0	7·177

from which numbers the following formula was deduced,

$$\lambda = 10{\cdot}654-0{\cdot}044560t+0{\cdot}00009789t^2,$$

or

$$\lambda_1=100-0{\cdot}41825t+0{\cdot}0009188t^2,$$

corresponding to a percentage decrement of 32·637 per cent.

11. Heated for four hours under sugar charcoal in a current of hydrogen; the carbon taken up was 0·933 per cent.

Length	177 millims.
Diameter. . . .	0·435 millim.

		Reduced to 0°.
Conducting-power found before heating the strip . .	9·568 at 9°·0	9·921
„ „ after being kept at 100° for 1 day .	9·668 at 6·4	9·921

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
8°·0	9·610
54·0	8·027
100·0	6·832

from which numbers the following formula was deduced,

$$\lambda = 9\cdot925-0\cdot040097t+0\cdot00009168t^2,$$

or

$$\lambda_1=100-0\cdot404t+0\cdot0009237t^2,$$

corresponding to a percentage decrement of 31·163 per cent.

12. Heated for three hours under sugar charcoal in a current of hydrogen; the carbon taken up was 1·06 per cent.

Length . . . . 191 millims.
Diameter . . . 0·436 millim.

		Reduced to 0°.
Conducting-power found before heating the strip . .	9·032 at 11°·4	9·449
,, ,, after being kept at 100° for 1 day .	9·157 at 8·8	9·482
,, ,, ,, ,, 2 days .	9·162 at 8·6	9·480

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
10°·0	9·090
55·0	7·652
100·0	6·564

from which numbers the following formula was deduced,

$$\lambda = 9\cdot457-0\cdot037573t+0\cdot00008642t^2,$$

or

$$\lambda_1=100-0\cdot3973t+0\cdot0009138t^2,$$

corresponding to a percentage decrement of 30·592 per cent.

13. Thin music wire melted with a quarter of its weight of peroxide of iron under a flux of plate glass.

Length . . . . 145·2 millims.
Diameter . . . 0·455 millim.

		Reduced to 0°.
Conducting-power found before heating the wire . .	12·537 at 13°·8	13·293
,, ,, after being kept at 100° for 1 day .	12·727 at 11·2	13·346
,, ,, ,, ,, 2 days .	12·731 at 11·6	13·374
,, ,, ,, ,, 3 days .	12·639 at 13·6	13·389

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
14°·0	12·610
57·0	10·542
100·0	8·929

from which numbers the following formula was deduced,

$$\lambda = 13\cdot381-0\cdot056829t+0\cdot000123t^2,$$

or

$$\lambda_1=100-0\cdot4247t+0\cdot0009192t^2,$$

corresponding to a percentage decrement of 33·278 per cent.

14. A piece of narrow watch-spring.

Length	276 millims.
Diameter . . .	0·615 millim.

		Reduced to 0°.
Conducting-power found before heating the wire . .	8·254 at 11°·0	8·568
„ „ after being kept at 100° for 1 day .	8·297 at 9·4	8·566

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
10°·0	8·279
55·0	7·127
100·0	6·193

from which numbers the following formula was deduced,

$$\lambda = 8{\cdot}565 - 0{\cdot}029099t + 0{\cdot}00005383t^2,$$

or

$$\lambda_1 = 100 - 0{\cdot}33974t + 0{\cdot}0006285t^2,$$

corresponding to a percentage decrement of 27·689 per cent.

15. Commercial iron wire.

Length	1150 millims.
Diameter . . .	0·971 millim.

		Reduced to 0°.
Conducting-power found before heating the wire . .	13·163 at 10°·6	13·774
„ „ after being kept at 100° for 1 day .	13·157 at 10·8	13·779

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.
12°·0	13·082
56·0	10·859
100·0	9·117

from which numbers the following formula was deduced,

$$\lambda = 13{\cdot}772 - 0{\cdot}05897t + 0{\cdot}0001242t^2,$$

or

$$\lambda_1 = 100 - 0{\cdot}43514t + 0{\cdot}0009018t^2,$$

corresponding to a percentage decrement of 33·801 per cent.

We may here mention the reason of our having taken only observations at three intervals between 0° and 100°. It was found that almost the same formula may be deduced from three observations as from seven or more, if the temperature of the second observation is exactly the mean of the other two. Now as at each interval we always made three observations, it was easy with a little practice to regulate the temperature so as to obtain the wished-for temperature as mean. Of course sometimes we had to make four, five, or more observations in order to bring out the desired temperature. By only taking observations at three intervals, the labour of the research, especially of the calculations, was materially diminished.

In the following Table we have placed together the results obtained with the different sorts of iron:—

	Conducting-power at 0°.	Percentage decrement in the conducting-power between 0° and 100°.
Electrotype iron	—	38·262
Electrotype iron (annealed in hydrogen)	—	38·479
Electrotype iron	—	38·134
Electrotype iron (annealed in air)	—	38·160
No. 5	15·712	36·070
No. 6	15·640	36·010
No. 7	14·204	34·742
No. 8	12·132	34·117
No. 9	14·723	35·459
No. 10	10·666	32·637
No. 11	9·921	31·163
No. 12	9·449	30·592
No. 13	13·293	33·278
No. 14	8·568	27·689
No. 15	13·774	33·801

If we look at the above Table, the following important fact will be obvious, namely, the higher the conducting-power the higher the percentage decrement in the conducting-power between 0° and 100°; in fact we have always found this to be the case; and from experiments made with about 100 alloys in this direction we have not found a single case where the percentage decrement in the conducting-power between 0° and 100° is greater than that of the pure metals; further, we have found that we may deduce the conducting-power of the pure metal from that of the impure one when the impurity in it does not reduce it more than, say, 10 to 20 per cent.; for we have proved experimentally that within those limits the percentage decrement between 0° and 100° in the conducting-power of an impure metal varies in the same ratio as the conducting-power of the impure metal at 100°, compared with that of the pure metal at 100°. Thus the percentage decrement in the conducting-power of pure iron between 0° and 100° is 38·261 per cent., that of No. 5, 36·07 per cent. Now, if the above statement be correct, by multiplying the conducting-power of No. 5 at 100° by $\frac{38{\cdot}261}{36{\cdot}070}$, we shall obtain the conducting-power of the pure metal at 100°, and the conducting-power at 0° by dividing that number by 0·61734.

In the following Table we give the results of such a calculation with those specimens of iron the conducting-powers of which do not vary more than 20 per cent. from that deduced for the pure metal.

	Observed conducting-power at 0°.	Calculated conducting-power of pure iron at 0°.
No. 5 . . .	15·712	17·257
No. 6 . . .	15·640	17·223
No. 7 . . .	14·204	16·533
No. 9 . . .	14·723	16·606
No. 13 . . .	13·293	16·516
No. 15 . . .	13·774	16·717
	Mean	16·809

The reason for making the above deduction will be fully explained in our paper "On the Influence of Temperature on the Electric Conducting-power of Alloys." One glance, however, will show that the deduced values agree as well together as can be expected, considering that the results may be modified by the presence of carbon, sulphur, &c. In cases where we may assume that we have only solutions of one metal in another, the concordance in the deduced values is very great; in fact, when we experiment with metals whose conducting-power in a pure state is known, and when the impurity is only dissolved in it, then the deduced conducting-power agrees almost exactly with that found experimentally.

For resistance-thermometers, as described by Siemens*, the use of an iron wire would give much greater differences for the same increment of temperature than copper; for the resistance of pure copper wire increases for each degree about 0·4 per cent., whereas that of pure iron increases about 0·6 per cent. for each degree.

When we found that iron decreased in conducting-power between 0° and 100° more than the pure metals (and here again we will call attention to the fact that we have as yet found no alloy to decrease in conducting-power between 0° and 100° to a greater extent than that which the pure metals composing it would do), we thought that its being a magnetic body might possibly be the reason of it; but after having tested thallium, and found that the conducting-power of that metal also decreases more than that of the pure metals, we knew that this could not well be the case, for thallium is strongly diamagnetic†. Hearing, however, that Professor Wöhler possessed specimens of pure cobalt and nickel wire prepared by M. Deville, we wrote and asked him to lend them to us; this he immediately did, and on testing them we obtained the following results:—

Cobalt wire.

Length . . . . 270 millims.

Diameter . . . 0·468 millim.

		Reduced to 0°.
Conducting-power found before heating the wire . .	12·495 at 11°·6	12·899
„ „ after being kept at 100° for 1 day .	12·466 at 12·6	12·905
„ „ „ „ 2 days .	12·428 at 13·4	12·894

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* Report of Government Submarine Cable Committee, p. 454.

† Lamy, Compt. Rend. 1862, vol. lv. p. 836. Mr. Crookes informs us he has also found thallium strongly diamagnetic.

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.		Difference.
	Observed.	Calculated.	
8°·65	12·623	12·626	−0·003
24·97	12·080	12·073	+0·007
39·95	11·586	11·589	−0·003
54·88	11·127	11·128	−0·001
70·44	10·671	10·670	+0·001
84·00	10·289	10·291	−0·002
99·78	9·873	9·872	+0·001

The formula deduced from the observations, from which the conducting-powers were calculated, was

$$\lambda = 12{\cdot}930 - 0{\cdot}035521t + 0{\cdot}00004887t^2,$$

or

$$\lambda_1 = 100 - 0{\cdot}27472t + 0{\cdot}000378t^2,$$

corresponding to a percentage decrement of 23·692 per cent.

Nickel wire.

Length . . . . 240 millims.
Diameter . . . 0·408 millim.

		Reduced to 0°.
Conducting-power found before heating the wire . .	11·594 at 11°·2	12·035
„ „ after being kept at 100° for 1 day .	11·630 at 14·0	12·185
„ „ „ „ 2 days .	11·739 at 12·4	12·235
„ „ „ „ 3 days .	11·735 at 12·5	12·235

The means of the conducting-powers observed at the following temperatures were—

T.	Conducting-power.		Difference.
	Observed.	Calculated.	
12°·38	11·735	11·728	+0·007
24·20	11·260	11·276	−0·016
40·06	10·708	10·701	+0·007
53·86	10·239	10·230	+0·009
70·14	9·700	9·710	−0·010
83·93	9·302	9·298	+0·004
100·03	8·850	8·851	−0·001

The formula deduced from the observations, from which the conducting-powers were calculated, was

$$\lambda = 12{\cdot}222 - 0{\cdot}040787t + 0{\cdot}00007088t^2,$$

or

$$\lambda_1 = 100 - 0{\cdot}33372t + 0{\cdot}0005799t^2,$$

corresponding to a percentage decrement of 27·573 per cent.

From our experiments with alloys we should deduce that the cobalt and nickel wires were not pure, and we are justified in making this statement by the fact that we have not yet found any metal in a pure and solid state to decrease in conducting-power between 0° and 100° less than 29·307 per cent., and, further, when we consider that, although these metals may have been pure when in the state of powder, yet very little is known about their behaviour to the crucibles at the high temperatures at which they fuse. It is well known how difficult it is to procure chemically pure iron in a fused state, on account of its decomposing the crucibles in which it is melted and taking up some impurities.

Assuming, therefore, that cobalt and nickel behave like most other pure metals, namely, decrease in conducting-power between 0° and 100°, 29·307 per cent., we may deduce from the above data the conducting-power of the pure metals.

The conducting-power of pure cobalt would then be 17·223 at 0°, and that of pure nickel 13·106 at 0°.

We hope shortly to be able to prepare some pure cobalt and nickel by depositing galvanoplastically those metals in the form of foil from solutions of their pure salts, and so to check the above deduced values for the conducting-power of the pure metals.

In conclusion, we give, in the following Table, the conducting-power of some of the pure metals, in order to show the places which the metals treated of in this paper take.

	Conducting-power at 0°.
Silver (hard drawn)	100·00
Copper „	99·95
Gold „	77·96
Zinc	29·02
Cadmium	23·72
Cobalt*	17·22
Iron*	16·81
Nickel*	13·11
Tin	12·36
Thallium	9·16
Lead	8·32
Arsenic	4·76
Antimony	4·62
Bismuth	1·245

* Probable value for the pure metal deduced from the observations with the impure one.

XVII. *On the Molecular Mobility of Gases.*

By THOMAS GRAHAM, *F.R.S., Master of the Mint.*

Received May 7,—Read June 18, 1863.

THE molecular mobility of gases will be considered at present chiefly in reference to the passage of gases, under pressure, through a thin porous plate or septum, and to the partial separation of mixed gases which can be effected, as will be shown, by such means. The investigation arose out of a renewed and somewhat protracted inquiry regarding the diffusion of gases (which depends upon the same molecular mobility), and has afforded certain new results which may prove to be of interest in a theoretical as well as in a practical point of view.

In the Diffusiometer, as first constructed, a plain cylindrical glass tube, about 10 inches in length and rather less than an inch in diameter, was simply closed at one end by a porous plate of plaster of Paris, about one-third of an inch in thickness, and was thus converted into a gas-receiver[*]. A superior material for the porous plate has since been found in the artificially compressed graphite of Mr. BROCKEDON, of the quality used for making writing-pencils. This material is sold in London in small cubic masses about 2 inches square. A cube may easily be cut into slices of a millimetre or two in thickness by means of a saw of steel spring. By rubbing the surface of the slice without wetting it upon a flat sand-stone, the thickness may be further reduced to about one-half of a millimetre. A circular disc of this graphite, which is like a wafer in thickness but possesses considerable tenacity, is attached by resinous cement to one end of the glass tube above described, so as to close it and form a diffusiometer (fig. 1). The tube is filled with hydrogen gas over a mercurial trough, the porosity of the graphite plate being counteracted for the time by covering it tightly with a thin sheet of gutta percha (fig. 2). On afterwards removing the latter, gaseous diffusion immediately takes place through the pores of the graphite. The whole hydrogen will leave the tube in forty minutes or an hour, and is replaced by a much smaller proportion of atmospheric air (about one-fourth), as is to be expected from the law of the diffusion of gases. During the process, the mercury will rise in the tube, if

Fig. 1.

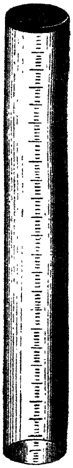
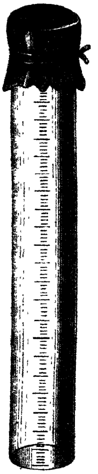


Fig. 2.



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[*] "On the Law of the Diffusion of Gases," Transactions of the Royal Society of Edinburgh, vol. xii. p. 222; or Philosophical Magazine, 1834, vol. ii. pp. 175, 269, 351.

allowed, forming a column of several inches in height—a fact which illustrates strikingly the intensity of the force with which the interpenetration of different gases is effected. Native graphite is of a lamellar structure, and appears to have little or no porosity. It cannot be substituted for the artificial graphite as a diffusion-septum. Unglazed earthenware comes next in value to graphite for that purpose.

The pores of artificial graphite appear to be really so minute, that a gas *in mass* cannot penetrate the plate at all. It seems that molecules only can pass; and they may be supposed to pass wholly unimpeded by friction, for the smallest pores than can be imagined to exist in the graphite must be tunnels in magnitude to the ultimate atoms of a gaseous body. The sole motive agency appears to be that intestine movement of molecules which is now generally recognized as an essential property of the gaseous condition of matter.

According to the physical hypothesis now generally received*, a gas is represented as consisting of solid and perfectly elastic spherical particles or atoms, which move in all directions, and are animated with different degrees of velocity in different gases. Confined in a vessel the moving particles are constantly impinging against its sides and occasionally against each other, and this contact takes place without any loss of motion, owing to the perfect elasticity of the particles. If the containing vessel be porous, like a diffusiometer, then gas is projected through the open channels, by the atomic motion described, and escapes. Simultaneously the external air is carried inwards in the same manner, and takes the place of the gas which leaves the vessel. To this atomic or molecular movement is due the elastic force, with the power to resist compression, possessed by gases. The molecular movement is accelerated by heat and retarded by cold, the tension of the gas being increased in the first instance and diminished in the second. Even when the same gas is present both within and without the vessel, or is in contact with both sides of our porous plate, the movement is sustained without abatement—molecules continuing to enter and to leave the vessel in equal number, although nothing of the kind is indicated by change of volume or otherwise. If the gases in communication be different but possess sensibly the same specific gravity and molecular velocity, as nitrogen and carbonic oxide do, an interchange of molecules also takes place without any change in volume. With gases opposed of unequal density and molecular velocity, the permeation ceases of course to be equal in both directions.

These observations are preliminary to the consideration of the passage through a graphite plate, in one direction only, of gas under pressure, or under the influence of its own elastic force. We are to suppose a vacuum to be maintained on one side of the porous septum, and air or any other gas, under a constant pressure, to be in contact with the other side. Now a gas may pass into a vacuum in three different modes, or in two other modes besides that immediately before us.

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* D. BERNOULLI, J. HERAPATH, JOULE, KRÖNIG, CLAUSIUS, CLERK MAXWELL, and CAZIN. The merit of reviving this hypothesis in recent times and first applying it to the facts of gaseous diffusion, is fairly due to Mr. HERAPATH. See 'Mathematical Physics,' in two volumes, by JOHN HERAPATH, Esq. (1847).

1. The gas may enter the vacuum by passing through a minute aperture in a thin plate, such as a puncture in platinum foil made by a fine steel point. The rate of passage of different gases is then regulated by their specific gravities, according to a pneumatic law which was deduced by Professor JOHN ROBISON from TORRICELLI'S well-known theorem of the velocity of efflux of fluids. A gas rushes into a vacuum with the velocity which a heavy body would acquire by falling from the height of an atmosphere composed of the gas in question, and supposed to be of uniform density throughout. The height of the uniform atmosphere would be inversely as the density of the gas, the atmosphere of hydrogen, for instance, sixteen times higher than that of oxygen. But as the velocity acquired by a heavy body in falling is not directly as the height, but as the square root of the height, the rate of flow of different gases into a vacuum will be inversely as the square root of their respective densities. The velocity of oxygen being 1, that of hydrogen will be 4, the square root of 16. This law has been experimentally verified*. The relative times of the effusion of gases, as I have spoken of it, are similar to those of molecular diffusion; but it is important to observe that the phenomena of effusion and diffusion are distinct and essentially different in their nature. The effusion movement affects masses of gas, the diffusion movement affects molecules; and a gas is usually carried by the former kind of impulse with a velocity many thousand times as great as is demonstrable by the latter.

2. If the aperture of efflux be in a plate of increased thickness, and so becomes a tube, the effusion-rates are disturbed. The rates of flow of different gases, however, assume again a constant ratio to each other when the capillary tube is considerably elongated, when the length exceeds the diameter by at least 4000 times. These new proportions of efflux are the rates of the "Capillary Transpiration" of gases†. The rates are found to be the same in a capillary tube composed of copper as they are in glass, and appear to be independent of the material of the capillary. A film of gas no doubt adheres to the surface of the tube, and the friction is really that of gas upon gas, and is consequently unaffected by the tube-substance. The rates of transpiration are not governed by specific gravity, and are indeed singularly unlike the rates of effusion.

The transpiration-velocity of oxygen being 1, that of chlorine is 1·5, that of hydrogen 2·26, of ether vapour the same or nearly the same as that of hydrogen, of nitrogen and carbonic oxide half that of hydrogen, of olefiant gas, ammonia, and cyanogen 2 (double or nearly double that of oxygen), of carbonic acid 1·376, and of the gas of marshes 1·815. In the same gas the velocity of transpiration increases with increased density, whether occasioned by cold or pressure.

The transpiration-ratios of gases appear to be in direct relation with no other known property of the same gases, and they form a class of phenomena remarkably isolated from all else at present known of gases.

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* "On the Motion of Gases," Philosophical Transactions, 1846, p. 573.

† Ibidem, p. 591; and Philosophical Transactions, 1849, p. 349.

There is one property of transpiration immediately bearing upon permeation of the graphite plate by gases. The capillary offers to the passage of gas a resistance analogous to that of friction, proportional to the surface, and consequently increasing as the tube or tubes are multiplied in number and diminished in diameter, with the area of discharge preserved constant. The resistance to the passage of liquid through a capillary was observed by POISEUILLE to be nearly as the fourth power of the diameter of the tube. In gases the resistance also rapidly increases; but in what ratio, has not been observed. The consequence, however, is certain, that as the diameter of the capillaries may be diminished beyond any assignable limit, so the flow may be retarded indefinitely, and caused at last to become too small to be sensible. We may then have a mass of capillaries of which the passages form a large aggregate, but are individually too small to allow a sensible flow of gas under pressure. A porous solid mass may possess the same reduced permeability as the congeries of capillary tubes. Indeed the state of porosity described appears to be more or less closely approached by all loosely aggregated mineral masses, such as lime-plaster, stucco, chalk, baked clay, non-crystalline earthy powders like hydrate of lime or magnesia compacted by pressure, and in the highest degree perhaps by artificial graphite.

3. A plate of artificial graphite, although it appears to be practically impermeable to gas by either of the two modes of passage previously described, is readily penetrated by the agency of the molecular or diffusive movement of gases. This appears on comparing the time required for the passage through the plate of equal volumes of different gases under a constant pressure. Of the three gases, oxygen, hydrogen, and carbonic acid, the time required for the passage of an equal volume of each through a capillary glass tube, in similar circumstances as to pressure and temperature, was formerly observed to be as follows:—

	Time of capillary transpiration of equal volumes.
Oxygen	1
Hydrogen	0·44
Carbonic acid	0·72

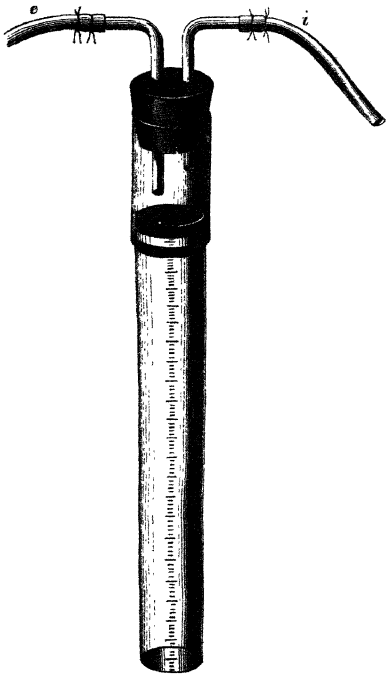
Now through a plate of graphite, half a millimetre in thickness, the same gases were observed to pass, under a constant pressure of a column of mercury of 100 millimetres in height, in times which are as follows:—

	Time of molecular passage.	Square root of density (oxygen 1).
Oxygen	1	1
Hydrogen	0·2472	0·2502
Carbonic acid	1·1886	1·1760

It appears that the times of passage through the graphite plate have no relation to the capillary transpiration-times of the same gases as first quoted. The new times in question, however, show a close relation to the square roots of the densities of the respective gases, as is seen in the last Table; and they so far agree with theoretical *times of diffusion* usually ascribed to the same gases.

These results were obtained by means of the graphite diffusiometer already referred to, which was a plain glass tube about 22 millimetres in diameter, closed at one end by the graphite plate. In order to conduct gas to the upper surface of the graphite plate, a little chamber was formed above the plate, to which the gas was conveyed in a moderate stream by the entrance-tube *e* (fig. 3); while the gas brought in excess was constantly escaping into the air by the open issue-tube *i*. The chamber was formed of a short piece of glass tube, about 2 inches in length, cemented over the upper end of the diffusiometer. The upper opening of this short tube was closed by a cork perforated for the entrance- and exit-tubes. It will be observed that by this arrangement the upper surface of the graphite plate was constantly swept by a stream of gas, which was under no additional pressure beyond that of the atmosphere, a free escape being allowed by the exit-tube. The gas also was always dried before reaching the chamber. The diffusiometer stood over mercury, and was raised or lowered by the lever movement introduced by Professor BUNSEN in his very exact experiments upon gaseous diffusion*. To obtain the pressure of 100 millimetres of mercury, the diffusiometer was first entirely filled with mercury and then raised in the trough. Gas gradually entered till the column of mercury in the tube fell to 100 millimetres. The mercury was then maintained at this height, by gradually raising the tube in proportion as gas continued to enter and the mercury to fall, so as to maintain a constant difference of level of 100 millimetres, as observed by the graduation inscribed upon the tube itself, between the level of the mercury in the tube and trough. The experiment consisted in observing the time in seconds which the mercury took to fall 10 millimetre divisions with each gas. The constant volume of gas which entered was 2·2 cubic centimetres (0·1342 cubic inch). Two experiments were made with each gas.

Fig. 3.



Oxygen entered in 898 and 894 seconds; mean 896 seconds.

Hydrogen in 222 and 221 seconds; mean 221·5 seconds.

Carbonic acid in 1070 and 1060 seconds; mean 1065 seconds.

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* BUNSEN'S 'Gasometry' by ROSCOE.

In such experiments the same gas exists on both sides, and also occupies the pores of the diaphragm. But the molecular movement within the pores in a downward direction is not fully balanced by the molecular movement in an upward direction, owing to the less tension, by 100 millimetres, of the gas below the diaphragm and within the tube than the gas above and without. The influx of gas indicates the difference of molecular movement in opposite directions. Taking the full tension of the gas above the diaphragm at 760 millimetres, that below would be 660 millimetres, and the movement downwards and that upwards are represented by these numbers respectively.

To increase the inequality of tension and favour the passage of gas through the graphite plate, a diffusion-tube was now used, 48 inches in length, or of the dimensions of a barometer-tube, by which a Torricellian vacuum could be commanded. The pneumatic trough in which this gas-tube was suspended consisted of a pipe of gutta percha of equal length, closed at the bottom by a cork, and widening into a funnel-form at the top. In one modification of the instrument it was found convenient to cement a capillary glass tube to the side of the glass diffusiometer, within about 15 millimetres of the upper end of the tube. An opening into the upper part of the glass tube was thus obtained, by means of which the gas contained in the diffusiometer could escape when the latter was depressed in the mercurial trough. A flexible tube with clip was attached to the capillary tube referred to, so that the latter could be closed. From the same opening a specimen of the gas contained in the diffusiometer could be drawn when required for examination.

In another and more serviceable modification of this barometrical diffusiometer a large space was obtained above the mercurial column, by surmounting the long glass tube, unprovided with a graphite plate, by a glass jar about half a litre in capacity. This jar was more correctly a small bell jar (fig. 4) open at top. It was fitted in an inverted position, as in fig. 5, to the open end of the long glass tube *d*, by means of a cork and cement. The large upper opening was closed by a circular plate of gutta percha (fig. 5), about 10 millimetres, or nearly half an inch, in thickness. This disc of gutta percha had two perforations at *f* and *g* (fig. 6), the former of which was fitted above with a wide glass

Fig. 4.

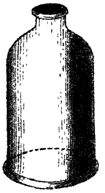


Fig. 5.

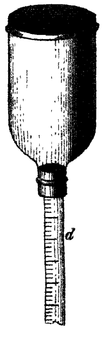
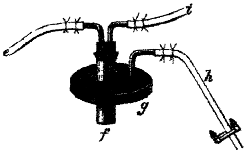
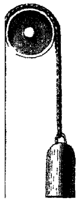


Fig. 6.

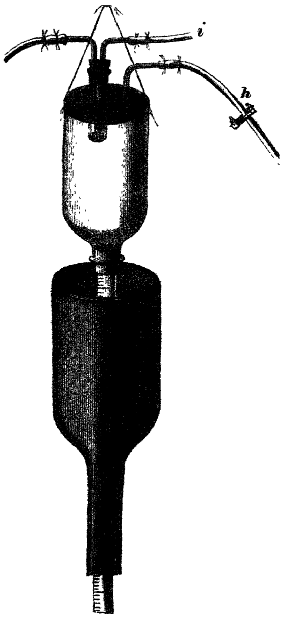


tube. The tube *f* was closed below by the plate of graphite, and above with a perforated cork carrying a quill tube, *e*. This quill tube was the entrance-tube for gas, and was accompanied by the usual issue-tube, *i*. The other aperture in the gutta-percha cover was fitted with a plain quill tube *h*, which did not descend below the level of the gutta percha, and formed a tube of exit. No difficulty was found in making all these junctions air-tight, by applying the heated blade of a knife to fuse the gutta percha in contact with the glass. Gutta percha is indeed of no ordinary value in the construction of pneumatic apparatus. The graphite plate itself required to be not less than 1 millimetre in thickness, in order to support the pressure of a whole atmosphere, to which it is exposed in the present apparatus. This barometrical diffusiometer is supported from above by a cord passing over a pulley, and is duly counterpoised by a hanging weight.

Fig. 7.



In operating, the first point is to expel the air from the barometer-tube and upper chamber. The instrument (fig. 7) is sunk completely in the mercurial trough previously described, till the whole is filled, and mercury enters the quill tube of exit, *h*. The caoutchouc extension of this tube is then closed by a pinch. The diffusiometer is now elevated 30 or 40 inches, when the mercury sinks in the glass tube till it comes to stand at the barometric height for the time, leaving the upper chamber entirely vacuous. The gas to be tried has in the mean time been made to stream over the upper surface of the graphite plate, exactly as in the experiment with the former diffusiometer. The graphite is permeated by the gas, and the mercury in the diffusiometer-tube begins to fall, but it now falls slowly, owing to the considerable vacuous space to be filled. It is allowed to fall about half an inch, and the exact time is then noted, by a watch, when the mercury passes a certain point in the graduation of the tube, and again when the mercury descends to another fixed point an inch or two below the former. The time of permeation of a certain volume of gas is thus ascertained in seconds. The experiment is immediately repeated with two or more gases in succession, in similar circumstances as to pressure, and with great care taken to ensure uniformity of temperature during the whole period.



In a series of four experiments made with hydrogen, the mercury fell from 758 to

685 millims. (29·9 inches to 27 inches) in 252, 256, 254, and 256 seconds; mean 254·5 seconds.

In three experiments with oxygen the mercury fell through the same space in 1019, 1025, and 1024 seconds; mean 1022·7 seconds:

$$\frac{1022{\cdot}7}{254{\cdot}5}=4{\cdot}018.$$

The times of these gases appear therefore to be as 1 to 4·018, while the times calculated as being inversely as the square root of the densities of the same gases are as 1 to 4.

On another day, with a different height of the barometer, four gases were passed through the graphite plate in succession through a somewhat shorter range, namely, from 754 to 685 millims. (29·7 to 27 inches).

The time of permeation of air was 884 and 885 seconds; mean 884·5 seconds.

The time of carbonic acid was 1100 and 1106 seconds; mean 1103 seconds.

The time of oxygen was 936, 924, and 930 seconds; mean 930 seconds.

The time of hydrogen was 229, 235, and 335 seconds; mean 233 seconds.

These times of permeation are in the following proportion:—

	Times of the permeation of equal volumes of gas through graphite.
Oxygen	1
Air	0·9501
Carbonic acid	1·1860
Hydrogen	0·2505

These numbers approach so closely to the square roots of the density, or the theoretical diffusion-times of the same gases, namely, oxygen 1, air 0·9507, carbonic acid 1·176, and hydrogen 0·2502, that they may be held to indicate the prevalence of a common law. They exclude the idea of capillary transpiration, which gives to the same gases entirely different numbers.

The movement of gases through the graphite plate appears to be solely due to their own proper molecular motion, quite unaided by transpiration. It seems to be the simplest possible exhibition of the molecular or diffusive movement of gases. This pure result is to be ascribed to the wonderfully fine (minute) porosity of the graphite. The interstitial spaces appear to be sufficiently small to extinguish capillary transpiration entirely. The graphite plate is a pneumatic sieve which stops all gaseous matter in mass, and permits molecules only to pass.

It is worth observing what result a plate of more open structure, such as stucco, will give in comparison with graphite. For the graphite plate, a cylinder of stucco, 12 millims. in thickness, was accordingly substituted, and gas allowed to percolate at both low and high pressures, as in the former experiments with graphite.

1. Under a constant pressure of 100 millims. of mercury, gas was allowed to enter through 100 millim. divisions of the diffusiometer.

With air, the time in two experiments was 515, and again 515 seconds.
With hydrogen 178 seconds, and again 178 seconds:

$$\frac{515}{178}=2{\cdot}894.$$

2. Under a pressure beginning with 710 millims. (28 inches) and ending with 660 millims. (26 inches), the time with air was 374 and 375 seconds; mean 374·5 seconds. The time with hydrogen was 129 and 130 seconds; mean 129·5 seconds:

$$\frac{374{\cdot}5}{129{\cdot}5}=2{\cdot}891.$$

The stucco cylinder of the preceding experiments had been dried over sulphuric acid, without the application of heat. It was further desiccated at 60° C. for twenty-four hours, in order to find whether the porosity would be altered. The ratio of the time of hydrogen to that of air now became 1 to 2·788 at the lower degree of pressure, and 1 to 2·744 at the higher degree of pressure.

It will be observed that the theoretical diffusion-ratio of hydrogen to air, which is 1 to 3·80, is greatly departed from in these experiments with stucco. The ratio appears to be tending to the proportion of the transpiration-times of the same gases, namely, 1 to 2·04. In an experiment recorded by BUNSEN, the ratio observed between the times of hydrogen and oxygen in passing, under a small degree of pressure, through stucco dried by heat was so low as 1 to 2·73, the stucco being probably less dense than in the experiments before us.

With stucco the permeation of gases *under pressure* appears to be a mixed phenomenon—to some extent molecular diffusion into a vacuum, such as holds with the plate of graphite, but principally capillary transpiration of gas in mass.

The diffusiometer was now closed by a plate of white biscuitware, 2·2 millims. in thickness. The time of fall at the constant pressure of 100 millims., through a range of forty divisions of the diffusiometer, was, for air 1210 seconds, for hydrogen 321 seconds.

$$\begin{array}{l}\text{Air} \;.\;.\;.\;.\;.\;. \\ \text{Hydrogen}\;.\;.\;.\end{array}\;\frac{1210}{321}=3{\cdot}769.$$

The time, again, from 736 to 685 millims. (29 to 27 inches) was, for air 685 and 684 seconds; mean 684·5 seconds; and for hydrogen 183, 183, and 184 seconds; mean 183·3 seconds.

$$\begin{array}{l}\text{Air} \;.\;.\;.\;.\;.\;. \\ \text{Hydrogen}\;.\;.\;.\end{array}\;\frac{684{\cdot}5}{183{\cdot}5}=3{\cdot}754.$$

The stoneware was evidently of a much closer texture than stucco, and the ratio appears again less influenced by capillary transpiration. In fact the molecular ratio of 1 to 3·80 is approached within 1 per cent. Biscuitware therefore appears to be but little inferior to graphite for such experiments, a circumstance which is important, as the latter is not easily procured and cannot be converted into tubes and other convenient forms like plastic clay.

Further, the rate of passage of gas through the plate of graphite appears to be closely proportional to the pressure. The resistance was increased by augmenting the thickness of the plate to 2 millims.; and with air and hydrogen at a pressure maintained constant at 50 and 100 millims., the time was observed that the gas took to enter 10 linear millimetre divisions of the tube.

	Seconds.	Ratio.
Air under pressure of 100 millims. . .	1925	1
Air under pressure of 50 millims. . . .	3880	2·015
Hydrogen under pressure of 100 millims.	497	1
Hydrogen under pressure of 50 millims.	1022	2·056

By halving the pressure the time of passage is doubled, or increased somewhat more. Greater pressures might probably give a rate of passage corresponding more exactly with the pressure.

The ratio between the comparative times of the two gases in the last experiments may also be noticed, the observations having been made in similar circumstances as to pressure and temperature.

Barom. 760 millims.; Therm. 12°·9 C. At pressure of 50 millims.

$$\begin{array}{l} \text{Air} \;.\;.\;.\;.\;.\;.\;. \\ \text{Hydrogen} \;.\;.\;.\;. \end{array} \frac{3880}{1022}=3{\cdot}796.$$

Barom. 760 millims.; Therm. 12°·9 C. At pressure of 100 millims.

$$\begin{array}{l} \text{Air} \;.\;.\;.\;.\;.\;.\;. \\ \text{Hydrogen} \;.\;.\;.\;. \end{array} \frac{1925}{497}=3{\cdot}873.$$

The observation was repeated at the pressure of 100 millims. with barometer at 754 millims. and thermometer at 10° C.

$$\begin{array}{l} \text{Air} \;.\;.\;.\;.\;. \\ \text{Hydrogen} \;.\;.\;. \end{array} \frac{1920}{498}=3{\cdot}855.$$

The velocity of hydrogen appears, as usual, to be nearly 3·8 times that of air; $\frac{1}{\sqrt{0{\cdot}6926}}=3{\cdot}7994$.

An experiment was made at the same time as the former series upon a mixture of 95 hydrogen and 5 air, which gave an unlooked for result that led to a great deal of inquiry. It is known that such a mixture is effused through an aperture in a fine plate in a time which is as the square root of the density of the mixture, and therefore nearly the arithmetical mean of the two gases effused separately. But in transpiration by a capillary, a mixture of 95 hydrogen and 5 air requires a considerably longer time than the gases transpired separately. In fact 5 per cent. of air retards the transpiration of hydrogen nearly as much as 20 per cent. of air would retard the effusion of hydrogen*. Now the mixture in question permeates the graphite plate in 527·5 seconds, while the calculated mean of the times of the two gases is 562·1 seconds.

The mixture has therefore passed neither in the effusion time, nor in a longer time as it would do by capillary transpiration, but, singular to say, in a time considerably shorter than either. The gas that came through was found by analysis to be *altered in composition.* It contained more hydrogen and less air than the original mixture. Hence it passed through with increased rapidity. On consideration it appeared that such a separation of the mixed gases must follow as a consequence of the movement being molecular. Each gas is impelled by its own peculiar molecular force, which, as has been seen, is capable of causing hydrogen to permeate the graphite plate about 3·8 times as rapidly as air.

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* Philosophical Transactions, 1846, p. 628.

Each gas may permeate a graphite plate into a vacuum with the same relative velocity as it diffuses into another gaseous atmosphere, but it remains a question whether the velocities of permeation and diffusion are absolutely as well as relatively the same. To illustrate this point, hydrogen and air were first allowed to permeate into a vacuum, and then to diffuse into each other, through the same graphite plate, which was 1 millim. in thickness. The plate was a circular disc of 22 millims. in diameter.

The mercurial column in the barometrical diffusiometer fell from 762 to 685 millims. (30 inches to 27) with air in 878 seconds, and with hydrogen in 233 seconds.

$$\begin{array}{ll}\text{Air} \;.\;.\;.\;.\;. & 878\\ \text{Hydrogen} \;\;.\;\;. & \overline{233}\end{array}=3{\cdot}768.$$

The volume of gas which produced this effect was found by the calibration of the tube to be 8·85 cub. centims. Hence 1·22 cub. centim. of the hydrogen entered the diffusiometer in 60 seconds, or one minute. But the pressure under which the hydrogen gas entered was the mean of 762 to 685 millims., or 723·5 millims.; while a whole atmosphere (the height of the barometer at the time) was 765 millims. The volume of the gas has therefore to be increased as 723·5 to 765 to give the full action of a vacuum. The volume becomes 1·289 cub. centim. in one minute.

When the diffusiometer was filled with hydrogen and the gas allowed to diffuse into air, the rise of the mercury was pretty uniform for the first five minutes, being 15·5 millim. divisions in the first two minutes, 7 in the third minute, 7·5 in the fourth minute, and 7 in the fifth minute, making 37 divisions in five minutes. But as in diffusion 1 air may be supposed to enter the tube for 3·8 hydrogen which escape, the hydrogen which diffused was more than 37 divisions, by $\frac{1}{3{\cdot}8}$, that is, by about 10 divisions. Hence 47 divisions of hydrogen have diffused into air in five minutes. These divisions measured, by the calibration of the tube, 6·215 cub. centims. One-fifth of this amount, that is, 1·243 cub. centim., diffused in one minute. The result of the whole is that in one minute there passed of hydrogen through the graphite plate,

1·289 cub. centim. by permeation into a vacuum,
1·243 cub. centim. by diffusion into air.

The numbers indicate a close approach to equality in the velocities of permeation into a vacuum and of diffusion into another gas, through the same porous diaphragm. The diffusion appears the slower of the two by a small amount; but this is as it should be, our estimate of the diffusion-velocity being certainly underrated; for the initial diffusion, or even the diffusion in the first minute, must obviously be somewhat greater than the average of the first five minutes, which we have taken to represent it—the hydrogen necessarily diffusing out in a diminishing progression, or more slowly in proportion as air has entered the diffusiometer. It is strictly the initial velocity of diffusion (that of the first second if it could be obtained) that ought to be compared with the percolation into a vacuum.

In fine, there can be little doubt left on the mind that the permeation through the graphite plate into a vacuum and the diffusion into a gaseous atmosphere, through the same plate, are due to the same inherent mobility of the gaseous molecule. They are the exhibition of this movement in different circumstances. In interdiffusion we have two gases moved simultaneously through the passages in opposite directions, each gas under the influence of its own inherent force; while with gas on one side of the plate and a vacuum on the other side, we have a single gas moving in one direction only. The latter case may be assimilated to the former if the vacuum be supposed to represent an infinitely light gas. It will not involve any error, therefore, to speak of both movements as gaseous diffusion,—the diffusion of gas into gas (double diffusion) in one case, and the diffusion of gas into a vacuum (single diffusion) in the other. The inherent molecular mobility may also be justly spoken of as the diffusibility or diffusive force of gases.

The diffusive mobility of the gaseous molecule is a property of matter fundamental in its nature, and the source of many others. The rate of diffusibility of any gas has been said to be regulated by its specific gravity, the velocity of diffusion having been observed to vary inversely as the square root of the density of the gas. This is true, but not in the sense of the diffusibility being determined or *caused* by specific gravity. The physical basis is the molecular mobility. The degree of motion which the molecule possesses regulates the volume which the gas assumes, and is obviously one, if not the only, determining cause of the peculiar specific gravity which the gas enjoys. If it were possible to increase in a permanent manner the molecular motion of a gas, its specific gravity would be altered, and it would become a lighter gas. With the density is also associated the equivalent weight of a gaseous element, according to the doctrine of equal combining volumes.

Diffusion of mixed gases into a vacuum, with partial separation—Atmolysis.

Oxygen and hydrogen.—A diffusiometer of the same construction as that described (fig. 3, p. 389), with a graphite plate of 1 millim. in thickness, was now employed. The upper surface of the plate was swept by a current of the mixed gas proceeding from a gas-holder, the excess of gas being allowed to escape into the atmosphere, as usual, by an open exit-tube. The gas was drawn through the graphite by elevating the diffusiometer containing a column of mercury, from its well, so as to command a partial vacuum in the upper part of the tube. Care is taken that any gas, left in the upper part of the diffusiometer-tube before the experiment begins, should be of the same composition as the gas to be allowed afterwards to enter, so that, on starting, the gas may be uniform in composition on both sides of the graphite plate. The height of the mercurial column, which measures the aspirating force of the diffusiometer, is preserved uniform by gradually raising the tube in the mercurial trough in proportion as gas enters and the mercury falls. The diffusiometer is suspended from the roof of the apartment by a cord passing over a pulley and properly weighted, as in former experiments.

The mixture to be diffused consisted of nearly equal volumes of oxygen and hydrogen. The effect of different degrees of pressure on the amount of separation produced was first observed. It will be seen that as the pressure or aspirating force is increased the amount of separation becomes greater. Barom. 0·759 millim.; therm. 18°·3 C.

Diffusion into a partial vacuum.

	Oxygen.	Hydrogen.
Composition of original mixture in 100 parts	49·3	50·7
Diffused by pressure of 100 millims.	47·0	53
Diffused by pressure of 400 millims.	37·5	62·5
Diffused by pressure of 673 millims. (mean of 635–710) . . .	26·4	73·6
Diffused by pressure of 747 millims. (mean of 736–759) . . .	22·8	77·2

In the last observation, or that with the greatest pressure (747 millims.), the oxygen is reduced to 22·8 per cent. and the hydrogen increased to 77·2 per cent. of the diffused mixture, showing a considerable separation. The mixed gases appear to make their way through the graphite plate independently, each following its own peculiar rate of diffusion.

But it is only under the aspiration of a complete vacuum that the separation can attain its maximum, and reach the full difference that may exist between the special diffusibilities of the two gases. The reason is that while we have the original mixture on both sides of the plate, and of equal tension, the gases are not at rest, but diffusion is proceeding as actively through the plate in opposite directions, as if the gases were different or the tension unequal on the two sides. This is a condition of the molecular mobility of gases (p. 386). The tension therefore being supposed to differ by 100 millims. only, as when the gas above the plate was of 759 millims. tension, and below of 659 millims. (in the first experiment of the last series), then 100 volumes only out of 759 of the mixture are subject to separation. But while these 100 volumes press through they are accompanied by 659 volumes of unchanged mixture. The latter 659 volumes are replaced by an equal bulk of unchanged mixture diffused from below, so that the volumes are not disturbed by this portion of the molecular interchange.

The amount of separation, then, attainable by transmitting a mixed gas through a porous diaphragm by pressure will be in proportion to the pressure—that is, to the inequality of tension on different sides of the diaphragm.

Oxygen and nitrogen.—The separation of the gases of the atmosphere by transmission through the graphite plate has a peculiar interest.

In an experiment resembling those last described, atmospheric air was swept over the upper surface of a graphite plate having a thickness of 2 millims. The gas that penetrated into the vacuum contained, as was to be expected, the lighter and more diffusible constituent in excess. It gave by the pyrogallic acid and potash process of LIEBIG,

Oxygen	20
Nitrogen	80

This was an increase in the nitrogen of quite 1 per cent.; for air, analysed for comparison at the same time and in the same manner, gave oxygen 21·03 and nitrogen 78·97.

It may be legitimately inferred from the last experiment, that if pure hydrogen in a diffusiometer were allowed to diffuse into the atmosphere through a porous plate, the portion of air which then enters the diffusiometer should also have its composition disturbed. A diffusion of hydrogen through a graphite plate was interrupted before completion. The air which had entered was found to consist of

Oxygen	19·77
Nitrogen	80·23
	100·00

The increase of nitrogen is 1·23 per cent.

While the nitrogen is increased and the oxygen diminished in the air which makes its way under pressure through the graphite, the converse effect must be produced on the air left behind. But the latter result of atmolysis cannot be made apparent without a change in the mode of experimenting.

With the view of effecting an increase in the proportion of oxygen, a volume of air, confined in a jar suspended over mercury, was allowed to communicate through a graphite plate of 2 millims. in thickness, with a vacuum sustained by means of an air-pump, the gauge being about 1 inch only below the height of the barometer during the whole time of experimenting.

The jar containing the air to be atmolysed was formed of a plain glass cylinder, open at both ends, and about 400 millims. in height (15·75 inches). The upper end was closed by a thick plate of gutta percha cemented on. This plate was itself penetrated by a wide glass tube, descending about an inch into the jar. The last tube carried the graphite disc, which was 27 millims. (1·04 inch) in diameter, sufficient to close the lower end of the tube upon which it was cemented. The other or upper end of the same tube was fitted with a cork and quill tube, and was put into communication with a large bell jar upon the plate of the air-pump.

The permeation was slow, owing to the unusual thickness of the graphite plate, occupying three hours to drain away one-half of the original volume of air in the jar. The air remaining behind in the jar was examined in a series of experiments, in which the original volume was reduced to one-half, one-fourth, one-eighth, and one-sixteenth.

The residual air, reduced to one-half, gave in two experiments 21·4 and 21·57 per cent. of oxygen, the air of the atmosphere being by the same analytical process 21 per cent.

Reduced to one-fourth of its volume, the residual air gave, in two experiments, 21·95 and 22·01 per cent. of oxygen.

Reduced to one-eighth of its volume, the air gave 22·54 per cent. of oxygen.

Reduced to one-sixteenth of its volume, the air gave 23·02 per cent. of oxygen. The proportion of oxygen had therefore increased about one-tenth in the last experiment, where the effect is greatest.

When the numbers are compared, it appears that by a reduction to half its volume the air gains about one-half per cent. of oxygen; when this last air is reduced to one-half again, another half per cent. of oxygen is gained, and so on—the gain in the proportion of oxygen increasing in an arithmetical ratio, while the volume of air is diminished in a geometrical ratio, or as the powers of the number 2.

Reduction of 1 volume of air	Proportion of oxygen per cent.	Increase of oxygen.
To 1 volume	21	0
To 0·5 volume	21·48	0·48
To 0·25 volume	21·98	0·98
To 0·125 volume	22·54	1·54
To 0·0625 volume	23·02	2·02

The densities of oxygen and nitrogen approach too nearly to admit of any considerable separation being effected by this method. The density of oxygen being taken as 1, that of nitrogen is 0·8785 The square roots of these numbers are 1 and 0·9373, which are inversely as the diffusive velocity of the two gases.

	Diffusive velocity.
Oxygen	1
Nitrogen	1·0669

The velocity of nitrogen therefore exceeds that of oxygen by about 6·7 per cent. Hence by a simple diffusion of a whole volume of air the oxygen could only be increased 6·7 per cent., according to theory. In experiments such as the preceding only one-half of the volume of the air is diffused, and consequently only one-half of the stated amount of concentration of oxygen could possibly be produced at each step. About three-fourths of the theoretical separation is actually obtained, although the apparatus works at an obvious disadvantage from the air within the jar being at rest.

This diffusive method of separation recalls the original observation of Döbereiner on the escape of hydrogen gas from a fissured jar standing over water, which will always hold its place in scientific history as the starting-point of the experimental study of gaseous diffusion. That observation proved to be an instance of double diffusion, air entering the jar by the fissure at the same time that hydrogen escaped by it—although, as Döbereiner looked upon the phenomenon, it was more akin to single diffusion or the passage of gas in one direction only*.

The atmolytic power of other diffusing plates was tested, besides the artificial graphite.

The barometrical diffusiometer already described was closed by a plate of *red unglazed*

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* Annales de Chimie, 1825.

earthenware, 4 millims. in thickness, which was attached to the glass by resinous cement.

Dry air was swept over the upper surface, as in operating with the graphite plate. With a mercurial column of 340 millims. falling to 200 millims., the air which entered was found to contain 79·45 per cent. of nitrogen, instead of 79. With a column of mercury, maintained at 508 millims. in the tube, the air entering contained 79·72 nitrogen, and with a column beginning at 761 millims., the full barometrical height, and falling to 679 millims. in seven minutes, the air entering contained 80·21 nitrogen. This is a full degree of separation, exceeding 1 per cent., while the time was greatly shorter than with graphite. Thermometer 19°·5 C.

With a diffusing plate of gypsum (stucco) 10 millims. in thickness, the proportion of nitrogen was also increased, although less considerably than with biscuitware. The standard proportion of nitrogen observed in atmospheric air being 78·99 per cent., the air drawn into the diffusiometer was as follows:—

	Proportion of nitrogen per cent.
In air entering over column of 330—200 millims. mercury	79·26
In air entering over column 508 millims.	79·32
In air entering over column 761—685 millims.	79·53
In air entering over column 761—685 millims.	79·69

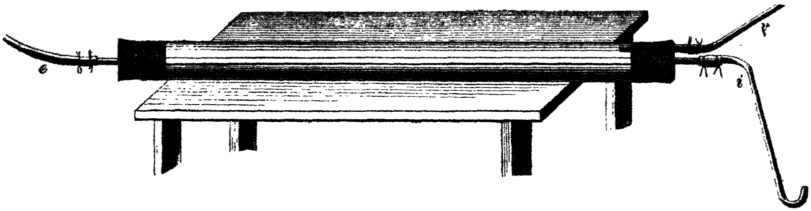
The separation is sufficiently decided, and is certainly remarkable considering the comparatively loose texture of the stucco plate. The gas entered in the two last experiments in about one minute, which appears too rapid a passage, and not to be attended with increased separation, compared with the immediately preceding experiment, in which the pressure was less and the passage of the gas proportionally slower. In all such highly porous plates, we have always to apprehend the passage of a large proportion of the gas in the manner of capillary transpiration, where no separation takes place.

It may be concluded that all porous masses, however loose their texture, will have some effect in separating mixed gases moving through them under pressure. The air entering a room by percolating through a wall of brick or a coat of plaster will thus become richer in nitrogen, in a certain small measure, than the external atmosphere.

The Tube Atmolyser.

In the application of diffusion through a porous septum to separate mixed gases, as a practical analytical method, it is desirable that the process should be more rapid than it can be made with the use of graphite and other diffusing-plates of small size, and also that the process should if possible be a continuous one. Both objects are attained in a considerable degree by adapting a tube of porous earthenware to the purpose. Nothing has been found to answer better than the long stalk of a Dutch tobacco-pipe used as the porous tube. A tube of this description, about 2 feet long and having an internal diameter of 2·5 millims., is fixed by means of perforated corks within a glass or metallic tube, a few inches less in length and about 1½ inch in diameter (*e*, *i* fig. 8), as in the construction of a Liebig condenser. A second quill tube (*v*) is inserted in one of the end corks, and affords the means of communication between the annular space and the vacuum of an air-pump. The external surface of the corks, and of those portions of the pipe-stalk which project beyond the enclosing tube, should be coated with a resinous varnish, to render them impermeable to air. Now, a good vacuum being obtained within the outer tube, and sustained by the action of an air-pump, the mixed gas is made to enter and traverse the clay tube. More or less of gas is drained off through the porous walls and pumped away, while a portion courses on and escapes by the other extremity of the clay tube, where it may be collected. The stream of gas diminishes as it proceeds, like a river flowing over a pervious bed. The lighter and more diffusive constituent of the mixed gases is drawn most largely into the vacuum, leaving the denser constituent, in a more concentrated condition, to escape by the exit end of the clay tube. The more slowly the mixed gas is moved through that tube, the larger the proportion of light gas that is drained off into the vacuum, and the more concentrated does the heavy gas become. The rate of flow of the mixed gas can be commanded by either discharging it from a gas-holder, or drawing it into a gas-receiver, in either case by a regulated pressure.

Fig. 8.



To observe the effect of a more or less rapid passage through the tube atmolyser, the impelling pressure was varied so as to allow a constant volume of half a litre of atmospheric air to pass through and be collected in different periods of time. The clay tube used in these particular experiments was not a tobacco-pipe, but a wide unglazed tube, about 431 millims. (17 inches) long and 19 millims. (0·75 inch) in internal diameter. It was required to place so wide a tube in a vertical position, and to admit the air by the upper and draw it off by the lower extremity of the tube. The proportion of oxygen in the half-litre of air collected was as follows:—

	Oxygen per cent.		
	Experiment 1.	Experiment 2.	Mean.
When collected in 1 minute	21·00	......	......
When collected in 13 minutes	22·33	22·25	22·29
When collected in 75 minutes	22·77	23·02	22·89
When collected in 120 minutes	23·25	23·22	23·23
When collected in 304 minutes	23·54	23·51	23·53

The proportion of oxygen in the air circulated appears thus to increase with the slowness of its passage through the tube atmolyser. The proportion of air drawn into the air-pump vacuum must be very large when the time is protracted; but the additional concentration of oxygen appears small.

The preceding observations being made by means of a porous tube which may be considered wide and of considerable capacity with reference to its internal surface, the experiment was varied by substituting a porous tube about eight times as long, very narrow, and therefore of small internal capacity. This second atmolyser was composed of twelve ordinary tobacco-pipe stems, each about 10 inches in length and of 1·9 millim. internal diameter, connected together by vulcanized caoutchouc adapters so as to form a single tube. Having flexible joints, the tube was folded up and placed within a glass cylinder that could be exhausted. Air was then circulated through this atmolyser by the pressure of several inches of water. The instrument appeared to work with most advantage when the air delivered at the exit-tube amounted to about one-fourth of a litre per hour. A volume of 268 cubic centimetres, which had circulated in one hour, was found to contain 24·37 per cent. of oxygen. The current was then made slower, so that only 108 cub. centims. of gas passed and were collected in one hour, but with little further concentration of the oxygen. The result, however, is interesting as being the highest concentration of oxygen yet obtained by an instrument of this kind. The air collected was composed of

Oxygen	24·52
Nitrogen	75·48
	100·00

The increase of oxygen is 3·5 per cent.; that is, an increase of 16·7 upon 100 oxygen originally present in the air.

With the single pipe-stalk, 24 inches long, first described, the oxygen of atmospheric air was concentrated about 2 per cent. when one litre was transmitted in one hour. Of 450 cub. centims. of air collected in that time, the composition proved to be

Oxygen	23·12
Nitrogen	76·88
	100·00

About 9 litres were drawn into the vacuum at the same time.

The separation of the gases of atmospheric air is a severe trial of the powers of the atmolyser, owing to the small difference in the specific gravities of these gases. But where a great disparity in density exists, the extent of the separation may become very considerable.

Several experiments were made upon a mixture of equal volumes of oxygen and hydrogen carried through the single tube atmolyser, 24 inches in length.

1. Of the mixture described, 7·5 litres entered the tube and 0·45 litre was collected in one experiment. The mixture was composed as follows:

	Oxygen.		Hydrogen.
Before traversing the atmolyser . .	50	+	50
After traversing the atmolyser . .	92·78	+	7·22

2. In another similar experiment, 14 litres of the mixed gas entered the tube and 0·45 litre was delivered in a period of two hours. The result was—

	Oxygen.		Hydrogen.
Before traversing the atmolyser .	50	+	50
After traversing the atmolyser .	95	+	5

Here the proportion of hydrogen is reduced from 50 to 5 per cent.

3. Of the explosive mixture, consisting of 1 volume oxygen and 2 volumes hydrogen, 9 litres were transmitted and 0·45 litre collected in one hour. The change effected was found to be as follows:—

	Oxygen.		Hydrogen.
Before traversing the atmolyser .	33·33	+	66·66
After traversing the atmolyser . .	90·7	+	9·3

The result in such experiments is striking, as the gas ceases to be explosive after traversing the porous tube, and a lighted taper burns in it as in pure oxygen. A mixture of oxygen and hydrogen is not explosive till the hydrogen rises to 11 per cent.

To illustrate the analogy of diffusion into a vacuum with diffusion into air, the outer glass tube of the diffuser was now withdrawn, and the porous tube of the instrument was exposed directly to the air of the atmosphere. A mixture of equal volumes of oxygen and hydrogen was again transmitted at the same rate of velocity as in experiment 1.

The gas atmolysed and collected was found to consist of

Oxygen	51·75
Hydrogen	5·47
Nitrogen	42·78
	100·00

And may be represented as containing

Oxygen	40·38
Hydrogen	5·47
Air	54·15
	100·00

A nearly similar concentration of the oxygen of the mixed gas is here observed as appeared in experiment 1; but the gas collected is now diluted with air which has entered by diffusion. The external air manifestly discharges the same function in the latter experiment which the air-pump vacuum discharged in the former experiment.

Interdiffusion of Gases—double diffusion.

The diffusiometer was much improved in construction by Professor BUNSEN, from the application of a lever arrangement to raise and depress the tube in the mercurial trough; but the mass of stucco forming the porous plate in his instrument appears too voluminous, and, from being dried by heat, is liable to detach itself from the walls of the glass tube. The result obtained of 3·4 for hydrogen, which diverges so far from the theoretical number, is, however, no longer insisted upon by that illustrious physicist. It is indeed curious that my old experiments generally rather exceeded than fell short of the theoretical number for hydrogen; $\sqrt{0{\cdot}06926} = 3{\cdot}7994$. With stucco as the material, the cavities existing in the porous plate form about one-fourth of its whole bulk, and affect sensibly the ratio in question according as they are or are not included in the capacity of the instrument. Beginning the diffusion always with these cavities, as well as the tube, filled with hydrogen, the numbers now obtained with a stucco plate of 12 millims. in thickness and dried without heat, were 3·783, 3·8, and 3·739 when the volume of the cavities of the stucco is added to both the air and hydrogen volumes diffused; and 3·931, 3·949, and 3·883 when such addition is not made to these volumes. The graphite plate, on the other hand, being very thin, and the volume of its pores too minute to require to be taken into account, its action is not attended with the same uncertainty. With a graphite plate of 2 millims. in thickness, the number for hydrogen into air was 3·876, instead of 3·8; and for hydrogen into oxygen 4·124, instead of 4. With a graphite plate of 1 millim. in thickness, hydrogen gave 3·993 to air 1. With a plate of the same material 0·5 millim. in thickness, the proportional number for hydrogen to air rose to 3·984, 4·068, and 4·067. An equally considerable departure from the theoretical number was observed when hydrogen was diffused into oxygen or into carbonic acid, instead of air. All these experiments were made with dry gases and over mercury. It appears that the numbers are most in accordance with theory when the graphite plate is thick, and the diffusion slow in consequence. If the diffusion be very rapid, as it is with the thin plates, something like a current is possibly formed within the channels of the graphite, taking the direction of the hydrogen and carrying back in masses a little air, or the slower gas, whatever it may be. I cannot account otherwise for the slight predominance which the lighter and faster gas appears always to acquire in diffusing through the porous septum.

Interdiffusion of Gases without an intervening septum.

The relative velocity with which different gases diffuse is shown by the diffusiometer, but the absolute velocity of the molecular movement cannot be ascertained by the same instrument. For that purpose it appears requisite that a gas should be allowed to diffuse into air through a wide opening.

In certain recent experiments, a heavy gas, such as carbonic acid, was allowed to rise by diffusion into a cylindrical column of air, pretty much as the saline solution is allowed to rise into a column of water in my late experiments upon the diffusion of liquids. This method of gaseous diffusion appears to admit of considerable precision, and deserves to be pursued further. A glass cylinder of 0·57 metre (22·44 inches) in height had the lower tenth part of its volume occupied with carbonic acid, and the upper ninetenths with air, in a succession of experiments: thermometer 16° Cent. After the lapse of a certain number of minutes, the upper tenth part of the volume was drawn off from the top of the jar and examined for carbonic acid. Before the carbonic acid appeared above, it had ascended, that is, it had diffused a distance of 0·513 metre, or rather more than half a metre. After the lapse of 5 minutes, the carbonic acid so found in two experiments amounted to 0·4 and 0·32 per cent. respectively. In 7 minutes, the carbonic acid observed was 1·02 and 0·90 per cent.; mean 0·96 per cent. The effect of diffusion is now quite sensible, and it may be said that about 1 per cent. of carbonic acid has diffused to a distance of half a metre in seven minutes.

A portion of carbonic acid has therefore travelled by diffusion at an average rate of 73 millims. per minute. It may be added that hydrogen was found to diffuse downwards, in air contained in the same cylindrical jar, at the rate of 350 millims. per minute, or about five times as rapidly as the carbonic acid ascended. In these experiments the glass cylinder was loosely packed with cotton wool, to impede the action of currents in the column of air; but this precaution was found to be unnecessary, as similar results were afterwards obtained in the absence of the cotton. To illustrate the regularity of the results, I may complete this statement by exhibiting the proportion of carbonic acid found in the upper stratum already referred to, after the lapse of different periods of time.

	Carbonic acid per cent.		
	Experiment 1.	Experiment 2.	Mean.
After 5 minutes	0·4	0·32	0·36
After 7 minutes	1·02	0·90	0·96
After 10 minutes	1·47	1·56	1·51
After 15 minutes	1·70	1·68	1·69
After 20 minutes	2·41	2·69	2·55
After 40 minutes	5·60	5·15	5·37
After 80 minutes	8·68	8·82	8·75

In eighty minutes the proportion of carbonic acid had risen to 8·75 per cent., 10 per cent. being the proportion which would indicate the completion of the process of diffusion.

The same intestine movement must always prevail in the air of the atmosphere, and with even greater velocity, in the proportion of 1 to 1·176, the relative diffusion-ratios of carbonic acid and air. It is certainly remarkable that in perfectly still air its molecules should spontaneously alter their position, and move to a distance of half a metre, in any direction, in the course of five or six minutes. The molecules of hydrogen gas disperse themselves to the distance of a third of a metre in a single minute. Such a molecular movement may become an agency of considerable power in distributing heat through a volume of gas. It appears to account for the high convective power observed in hydrogen, the most diffusive of gases.

XVIII. *On the Peroxides of the Radicals of the Organic Acids.*

By Sir B. C. Brodie, *Bart., F.R.S., Professor of Chemistry in the University of Oxford.*

Received June 18,—Read June 18, 1863.

In a former communication[*] I announced to the Royal Society the discovery of a new group of chemical substances, the peroxides of the radicals of the organic acids—bodies which, in the systems of the combinations of these radicals, occupy the same relative position as is held by the peroxides of hydrogen, barium, or manganese in the systems of the combinations of those elements.

The investigation of these peroxides is attended with peculiar difficulties. It is by no means easy to prepare in any considerable quantity the anhydrous acids and chlorides themselves, which is only the first step in the preparation of the peroxides. The greater number also of these substances is excessively unstable; they are decomposed in the very reactions by which they are produced, and the quantity actually obtained is very far from corresponding to that which is indicated by theory. There can be little hope of a complete and successful investigation of the decompositions of these bodies, until methods are discovered by which the substances themselves can be more readily procured. I have for these reasons not yet been able to submit their transformations to the profound study which the subject merits, and which will doubtless be some day followed by a rich harvest of discovery.

One exception should be made to the above remarks, the peroxide of benzoyl. This beautiful substance can be procured with comparative facility, and I hope to pursue the investigation of its metamorphoses. It appeared to me, however, of primary importance to establish the perfect generality of the fundamental reaction by which these bodies are prepared. This I have effected by forming several members of the group; and I have also, in one instance at least, succeeded in ascertaining the constitution of the peroxide of a bibasic acid, a member of a new class of chemical substances, fundamentally different (as the chemist would perhaps anticipate) from the peroxides of the monobasic acids, and characterized by well-marked reactions.

The peroxide of barium, in respect to the definite and universal character of its reactions, may be placed by the side of the alkalies themselves. Every anhydrous organic acid with which I have made the experiment, without any exception, has been found to be converted by its agency into an organic peroxide. It is a new instrument of chemical research, admirable for the power and the simplicity of its action, and which will certainly find in the future of chemistry many applications besides those which are here recorded. Its preparation is a matter of importance.

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[*] Proceedings of the Royal Society, vol. ix. p. 361.

Preparation of Peroxide of Barium.

The peroxide of barium as prepared by leading oxygen over heated baryta is useless for the purposes of the following experiments, for the reason that the oxidation of the baryta is never complete, and that the peroxide is mixed with large quantities of the oxide of barium. However, the first step towards the preparation of a pure peroxide is the preparation of a crude material.

When oxygen gas is passed over fragments of baryta heated in a porcelain tube, the absorption of the gas proceeds at first with great rapidity; and if the heat be properly regulated, not a trace of oxygen will pass through the apparatus. It is nevertheless extremely difficult to prepare in this manner a peroxide which shall contain more than about 6 parts of oxygen to 100 of baryta, however long the action of the oxygen be continued,—the theoretical amount of oxygen required for the formation of the peroxide being 10·46 parts of oxygen to 100 of baryta. By far the simplest and most practical process for the oxidation of baryta is that devised by LIEBIG, which consists in exposing to a gentle heat an intimate mixture of powdered baryta and chlorate of potassium. The mixture is thrown by degrees into a crucible heated to low redness; an ignition is perceived when the chlorate of potassium melts. The fused mass is powdered and extracted with water, which leaves an insoluble residue containing large quantities of peroxide of barium.

On determining the amount of oxygen combined with the baryta in this experiment, I found that the peroxide did not contain above half the theoretical amount of oxygen. The following experiments leave little doubt that in this reaction the baryta is oxidized to the condition, not of peroxide, but of sesquioxide of barium. When oxygen is passed over baryta, the action does not appear to be absolutely arrested at this point, but nevertheless only small quantities of peroxide are formed.

Expt. I. 4·726 grms. of baryta were powdered, and mixed with one-third that weight of chlorate of potassium. The mixture was thrown by degrees into a platinum crucible, heated to low redness. After the first ignition, which proceeded without any external increase of temperature, the crucible was heated somewhat more strongly, so as to ensure the decomposition of the chlorate of potassium. The mass was powdered, mixed with water, and dissolved in very dilute hydrochloric acid, to which the hydrate was added. The solution was rendered feebly alkaline by the addition of baryta water, which precipitated the traces of iron and alumina invariably contained in the baryta. The solution was then rapidly filtered, and immediately rendered acid; and by the addition of water, it was brought to the bulk of 500 cub. centims. 10 cub. centims. of this solution, titred with a standard solution of permanganate, of which 1 part was equivalent to 0·000516 grm. of oxygen, required in two experiments 8·6 and 8·7 parts of that solution. Since

8·65 parts of permanganate were equivalent to 0·0004465 grm. of oxygen, 0·22325 grm. of oxygen were contained in the 500 cub. centims. of solution of peroxide of barium; 100 parts of baryta had therefore absorbed 4·718 parts of oxygen.

In three other experiments conducted in a similar manner, in which respectively 5 grms. of baryta were mixed with $\frac{1}{3}$, with $\frac{2}{3}$, and with an equal weight of chlorate of potassium, precisely the same result was obtained.

Expt. II. 5·084 grms. of baryta were mixed with $\frac{1}{4}$ that weight of perchlorate of potassium: the experiment was otherwise conducted precisely as before. 9·8 cub. centims. of the resulting solution of peroxide required 10 parts of permanganate. 1 part of the permanganate employed was equivalent to 0·000548 grm. of oxygen. Hence 100 parts of baryta had combined with 5·375 parts of oxygen. In the case of the formation of the sesquioxide of barium, 100 parts of baryta would combine with 5·228 parts of oxygen.

When lime or strontia were substituted for baryta in the preceding experiment, not a trace of peroxide of hydrogen could be detected in the resulting solution.

To prepare pure peroxide of barium, the crude peroxide, as prepared by either of the above processes, is finely pulverized and rubbed with water in a mortar, so as entirely to convert it into hydrate. It is then mixed gradually with a very dilute solution of hydrochloric acid, care being taken to keep the solution constantly acid. This solution is filtered, and rendered alkaline with a slight excess of baryta-water. The addition of the latter effects the precipitation of the alumina and iron. The alkaline solution, which immediately begins to decompose, is rapidly filtered through linen filters, and to the clear filtrate is added an excess of baryta-water. The hydrated peroxide of barium is precipitated in brilliant plates, which are insoluble in water, and may be washed by decantation. In order to ascertain whether the whole of the peroxide is precipitated, a small portion of the solution may be filtered, rendered acid, and tested with a dilute solution of bichromate of potassium.

The washed precipitate is to be collected on a filter, pressed out between blotting-paper, and dried under the air-pump, by which means the whole of the water of crystallization may be driven off. The dry peroxide appears in the form of a fine white powder, resembling magnesia. I have analysed this substance, and found it to consist of anhydrous peroxide of barium, $Ba_2 O_2$, the only impurity being a trace of carbonate. It is in this condition a perfectly stable substance.

The absolute amount of peroxide of barium contained in the different preparations employed in the following experiments was ascertained either by a direct determination of the oxygen evolved by the action of platinum-black and a dilute acid, or by means of a standard solution of permanganate of potassium, according to the method which I have given in a former paper.

Peroxide of Benzoyl, $C_{14} H_{10} O_4$.

The peroxide of benzoyl is prepared by the action either of the chloride of benzoyl, or of the benzoic anhydride on hydrated peroxide of barium. When the following precautions are taken, the reaction is perfectly definite.

Equivalent quantities are to be weighed out of chloride of benzoyl and pure peroxide of barium. The peroxide of barium is converted into hydrate, and pressed between blotting-paper, to remove any great excess of water; it is then added by degrees to the chloride of benzoyl in a small mortar, and the two substances well mixed by means of a pestle. The mixture is allowed to remain for some hours; and the resulting substance having been mixed with water is thrown on a filter, and washed until the chloride of barium is removed. It is then treated with a weak solution of carbonate of sodium, so as to render the solution decidedly alkaline. After thus removing the benzoic acid, of which a certain portion is always formed in the reaction, the substance is pressed out between blotting-paper, and dried under the air-pump. When perfectly dry, it is to be dissolved in bisulphide of carbon at a temperature not exceeding 35° C., and three or four times crystallized from that fluid.

Of several slightly different methods of preparation this gave by far the most satisfactory results. In one experiment, for example, from 20 grms. of chloride of benzoyl, 15·2 grms. of the crude peroxide, as dried under the air-pump, were obtained, corresponding to 88·26 per cent. of the theoretical yield. If an excess of water be present, the amount of peroxide of benzoyl obtained is greatly reduced. The hydration of the peroxide of barium appears to be essential to the reaction. An equivalent quantity of peroxide of barium, mixed with anhydrous benzoic acid dissolved in ether and heated for five or six hours in a water-bath, was entirely without action. The same appeared to be the case with equivalent quantities of chloride of benzoyl and peroxide of barium under the same circumstances.

If, in the preparation of this substance, the peroxide of barium be taken in excess, that is, more than one equivalent of that peroxide, $Ba_2 O_2$, to two of chloride of benzoyl, $2Bz\,Cl$, the amount of peroxide of benzoyl formed is reduced; and if a great excess of peroxide of barium be employed, as for example one equivalent of that substance, $Ba_2 O_2$, to one equivalent of chloride of benzoyl, $Bz\,Cl$, oxygen gas is evolved, and hardly a trace of peroxide will be formed. This arises from the circumstance that the reaction in which the peroxide of benzoyl is formed is immediately succeeded by a second reaction in which that substance is destroyed, according to the equations

$$2Bz\,Cl+Ba_2 O_2=2Ba\,Cl+Bz_2 O_2,$$
$$Bz_2 O_2+Ba_2 O_2=2Ba\,Bz\,O+O_2.$$

I have ascertained by direct experiment that the peroxide of barium, when mixed in water with the peroxide of benzoyl, is decomposed with evolution of oxygen gas.

This affords a striking example of a class of decompositions which I recently brought before the Society,—in which one equivalent of the peroxide of barium acts as an agent of oxidation, while a second equivalent acts as a reducing agent, destroying the substance formed in the first reaction.

The peroxide of benzoyl thus prepared is in splendid crystals. From large quantities of solution I have occasionally obtained these crystals as much as three-fourths of an inch in diameter. They are referable to the right prismatic system, and their form, as crystallized from ether, has been examined by Professor W. H. MILLER, of Cambridge*. It is difficult to ascertain with absolute accuracy the melting-point of this substance. The point of decomposition lies close upon the melting-point; and it is only in very small quantities that it can be melted without being decomposed. My experiments, however, place the melting-point at 103°·5 C. At 15° C. 100 parts of bisulphide of carbon dissolve 2·53 parts of the peroxide of benzoyl. It is also soluble in ether and benzole.

This substance gave on analysis the following results:—

I. 0·3975 grm. of substance gave

Carbonic acid	1·0103
Water	0·1513

II. 0·4412 grm. of the same substance, twice recrystallized, gave

Carbonic acid	1·1213
Water	0·1661

These numbers give, as the percentage composition,

	I.	II.
Carbon . . .	69·31	69·31
Hydrogen . .	4·23	4·18
Oxygen . . .	26·46	26·51
	100·00	100·00

The numbers required by theory are

C_{14}	=168	69·42
H_{10}	= 10	4·13
O_4	= 64	26·45
	242	100·00

I have repeatedly prepared and analysed this substance with the same results.

When the peroxide of benzoyl is boiled with a solution of hydrate of potassium, oxygen gas is evolved and benzoate of potassium is formed.

$$Bz_2O_2+2KHO=2BzKO+H_2O+O.$$

If the peroxide of benzoyl be heated, it is decomposed with a slight explosion. By mixing the finely powdered peroxide with sand the action may be moderated; under these circumstances carbonic anhydride is evolved. The decomposition commences at about 85° C. I have estimated the loss of weight which the substance undergoes in this decomposition: in two experiments 100 parts of peroxide lost 18·6 and 18·18 parts; in three other experiments somewhat lower numbers were obtained, 17·78, 16·56, and 16·7 per cent. The theoretical loss, if one equivalent of carbonic anhydride, CO_2, were

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[*] See Proceedings of the Royal Society, January 15, 1862.

evolved from one equivalent of the peroxide, $C_{14}H_{10}O_4$, would be 18·18 per cent. The substance formed by the removal of one equivalent of carbonic anhydride from the peroxide of benzoyl, $C_{13}H_{10}O_2$, would be isomeric with the benzoate of phenyl.

I have not, however, yet succeeded in so moderating the action as to form only one substance. During the decomposition a small quantity of benzoic acid sublimes, and on extracting the sand with ether, filtering, and evaporating the ethereal solution, a soft glutinous residue is obtained, of which a portion dissolves on prolonged boiling in water. Benzoic acid passes over with the vapour of the water, and ultimately a hard and perfectly transparent resin remains, which is soluble in potash, and in all respects resembles a natural resin. I hope again to recur to this substance.

If peroxide of benzoyl be treated with a large excess of concentrated nitric acid, it is dissolved by the acid. When this solution is poured into water, a slightly yellow flocculent substance separates, which, dried under the air-pump, is soluble in bisulphide of carbon.

3 grms. of peroxide of benzoyl were thrown into about 3 fluid-ounces of fuming nitric acid, specific gravity 1·505. There was no perceptible increase of temperature or evolution of gas. The peroxide was rapidly dissolved, the mixture became deeper in colour, and after some time the vessel was filled with fumes of hyponitric acid. After standing about twenty-four hours, the solution in nitric acid was mixed with ten times its bulk of water. The precipitate formed was brought on a filter and washed free from acid. It was then dried under the air-pump, and dissolved in bisulphide of carbon. On the cooling of the bisulphide, a slightly yellow flocculent body separated. This was again dried under the air-pump and analysed.

I. 0·4167 grm. of the substance gave

Carbonic acid	0·7735 grm.
Water	0·0969 grm.

II. 0·433 grm. of the same substance, at a temperature of 18° C., and a barometric pressure of 758·5 millims., gave 32·5 cub. centims. of nitrogen gas. This corresponds to a weight of 0·0374 grm. of nitrogen.

We have as the results of these determinations,

Carbon	50·60
Hydrogen	2·58
Nitrogen	8·49
Oxygen	38·33
	100·00

The formula of the substance derived from the peroxide of benzoyl by the substitution of two atoms of peroxide of nitrogen, NO_2, for two atoms of hydrogen, H, is $C_{14}H_8(NO_2)_2O_4=C_{14}H_8N_2O_8$, and requires

$C_{14}=168$	50·60
$H_8=8$	2·41
$N_2=28$	8·43
$O_8=128$	38·56
332	100·00

This body, when heated, decomposes with a slight explosion, leaving a resinous matter similar in appearance to that formed by the decomposition of the peroxide of benzoyl.

Gerhardt did not succeed in procuring in a state of purity the anhydrous nitrobenzoic acid*, on account of the facility with which it decomposes water. The nitrobenzoic peroxide stands to the acid in the same relation as does the peroxide of benzoyl to the anhydrous benzoic acid.

Peroxide of Cumenyl.

The peroxide of barium is decomposed by the chloride of cumenyl precisely as by the chloride of benzoyl. The resulting substance crystallizes from ether in long and beautiful needles; when heated it explodes, leaving a resinous residue.

I have only once prepared this substance, and did not succeed in procuring it in a state of absolute purity.

0·3798 grm. of the substance gave 1·020 grm. of carbonic acid and 0·2392 grm. of water. These numbers give per cent.,

Carbon	73·24
Hydrogen	7·00
Oxygen	19·76
	100·00

The formula $C_{20}H_{22}O_4$ requires

$C_{20}=240$	74·23
$H_{22}=22$	6·75
$O_4=64$	19·02
326	100·00

The peroxide of benzoyl itself is mixed with traces of some substance which it is very difficult to remove by crystallization, and which lowers the percentage of carbon in the analyses; and, notwithstanding the difference of 1 per cent. in the carbon from that required by theory, we may assume the substance to be the peroxide of cumenyl.

Peroxide of Acetyl.

In the preparation of the peroxides of the acetic series, the use of the anhydrous acid has great advantages over the use of the corresponding chloride. By the action of the anhydride of the acid on peroxide of barium, I have succeeded in preparing three of these peroxides, the peroxides of acetyl, butyl, and valeryl.

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* Annales de Chimie, iii. p. 37 & 321.

The peroxide of acetyl is prepared by dissolving anhydrous acetic acid in pure ether, and adding gradually to this solution an equivalent quantity of peroxide of barium.

The decomposition takes place according to the equation

$$2C_4 H_6 O_3 + Ba_2 O_2 = 2C_2 H_3 Ba O_2 + C_4 H_6 O_4.$$

The reaction is attended with an elevation of temperature which causes the ether to boil; the temperature is not to be allowed to reach this point. After standing some time, the solution is filtered from the gelatinous residue, which does not contain a trace of peroxide of barium, and the ether is distilled off at a very low temperature, great care being taken not to allow the temperature to rise towards the end of the operation. The residue, washed first with water and then with a very weak solution of carbonate of sodium, appears as a thick and viscid fluid. I have in this manner experimented on as much as 20 grms. of anhydrous acetic acid, dissolved in about four times its bulk of pure ether. The addition of the equivalent quantity of peroxide of barium occupied two hours. From these 20 grms. of anhydrous acetic acid, only as much peroxide of acetyl was procured as to be sufficient for the two following determinations. The analysis was thus effected.

An undetermined quantity of the peroxide of acetyl was placed in a little water at the bottom of the bulb-apparatus used for the estimation of oxygen in peroxide of barium, which I have described in a former paper. The bulb was filled with baryta-water, a small tube containing platinum-black introduced into the apparatus, and the whole was weighed. The peroxide was now decomposed by allowing the baryta-water to flow into the flask from the bulb. Acetate of barium and peroxide of barium are formed. The peroxide of barium was decomposed by bringing the platinum-black contained in the small tube in contact with it. After the completion of the reaction, the apparatus was again weighed, and thus the loss of oxygen determined.

A current of carbonic acid was now passed through the solution, which was boiled and filtered, and the barium estimated as sulphate. The sulphate of barium thus formed is the measure of the acetate produced by the decomposition of the peroxide of acetyl.

Experiment I. The weight of the apparatus before and after the experiment gave a loss of oxygen of 0·1225 grm.

The solution precipitated by sulphuric acid gave 1·776 grm. of sulphate of barium.

Experiment II. The loss of oxygen estimated as before was 0·137 grm.

The solution precipitated by sulphuric acid gave 1·944 grm. of sulphate.

In Experiment I., 100 parts of sulphate being formed, 6·89 parts of oxygen were evolved.

In Experiment II., 100 parts of sulphate were obtained and 7·04 of oxygen evolved.

Theory requires that for every 100 parts of sulphate of barium formed 6·86 parts of oxygen should be evolved.

When a small drop of the peroxide of acetyl is heated in a watch-glass, it is decomposed with an explosive violence, only to be paralleled by the decomposition of chloride of nitrogen. Hence the greatest care is necessary in its preparation, especially during the distillation of the ether in which it is dissolved. I had frequently effected this operation without accident; but on one occasion my assistant was engaged in distilling off the ether from a rather considerable quantity of the substance, which was contained in a flask placed in warm water on a small copper water-bath; the temperature was probably allowed to rise too high, and towards the close of the operation a violent explosion took place with a report as of a cannon. A large hole was made in the copper water-bath, through which the hand might be passed, the copper being folded back upon the sides of the bath. The explosion, though of excessive violence, was local, and nothing in the laboratory in which the explosion took place was injured.

The peroxide of acetyl is readily decomposed under the influence of sunlight. A measured quantity of the substance was kept unaltered in bulk for above eighteen hours in the dark, but when placed in water in the bright sunlight, the same substance rapidly disappeared.

This peroxide is a most powerful agent of oxidation; like chlorine it rapidly bleaches indigo, it separates iodine from hydriodic acid and from iodide of potassium, it converts a solution of ferrocyanide of potassium into ferricyanide, and immediately oxidizes the hydrated protoxide of manganese.

These properties it has in common with the peroxide of hydrogen; but it is readily distinguished from that substance by not producing the peculiar effects of reduction, by which the peroxide of hydrogen is characterized. It does not reduce an acid solution of chromic or permanganic acids. The addition of baryta-water to the peroxide of acetyl suspended in water, causes an immediate precipitate of crystals of the hydrated peroxide of barium.

We can have no more convincing proof, if such proof were needed, than that furnished by this experiment, that the difference of properties which oxygen manifests in its different combinations is due, not, as has been imagined, to the existence of certain distinct varieties of that element, but to the circumstance that the combining properties of oxygen, as of other elements, vary with the nature of the chemical substances with which it is combined or associated.

Peroxide of Butyl.

The butylic peroxide is readily prepared by mixing hydrated peroxide of barium with anhydrous butyric acid. Experiments made with the view of preparing this substance by the action, in ether, of the dry peroxide of barium on anhydrous acid, were unsuccessful.

The result of this reaction is given in the equation

$$2C_8H_{14}O_3+Ba_2O_2=2C_4H_7BaO_2+C_8H_{14}O_4.$$

The experiment may be advantageously conducted as follows:—

The anhydrous acid is placed in a small mortar, and an equivalent quantity of hydrated peroxide of barium, from which any great excess of water has been removed, is gradually added to it, the whole being well mixed after each addition of the peroxide. An excess of peroxide of barium is to be carefully avoided, as it again decomposes the peroxide of butyl. To this end it is desirable towards the close of the operation, to examine from time to time the contents of the mortar, by placing a drop on a watch-glass, acidifying with hydrochloric acid, and testing with a dilute solution of bichromate of potassium. The appearance of a feeble blue colour indicates that sufficient peroxide has been added. The substance is mixed with a small quantity of water, and the solution agitated repeatedly with ether, which dissolves the peroxide of butyl. This operation is readily effected in a burette, provided with a glass stopcock. The ethereal solution is then repeatedly washed, first with dilute hydrochloric acid, then with a weak solution of carbonate of sodium, until the solution has a strong alkaline reaction, and then again with water until the alkaline reaction disappears. The solution is filtered and allowed to evaporate in a current of air at a low temperature. An oily residue is left, which is to be washed once or twice with a small quantity of water, in which it is only slightly soluble. It is then removed with a pipette, and allowed to stand for some time in contact with a few fragments of chloride of calcium. The substance thus prepared is pure peroxide of butyl.

This peroxide was analysed with oxide of copper in the usual manner.

I. 0·3562 grm. of the substance gave 0·721 grm. of carbonic acid and 0·2668 grm. of water.

II. 0·3144 grm. of the substance gave 0·6355 grm. of carbonic acid and 0·2344 grm. of water.

These analyses give per cent.,

	I.	II.
Carbon . . .	55·21	55·11
Hydrogen . .	8·29	8·28
Oxygen . . .	36·50	36·61
	100·00	100·00

Theory requires

C_8	=96	55·172
H_{14}	=14	8·046
O_4	=64	36·782
	174	100·000

A drop of the peroxide of butyl on a watch-glass decomposes with a slight explosion. Suspended in water, it possesses the oxidizing properties of the acetic peroxide.

Peroxide of Valeryl.

The peroxide of valeryl is prepared by the action of anhydrous valerianic acid on hydrated peroxide of barium, the result of the reaction being expressed by the equation

$$2C_{10}H_{18}O_3+Ba_2O_2=2C_5H_9BaO_2+C_{10}H_{18}O_4.$$

The method of preparation is in all respects the same as that by which the peroxide of butyl is prepared.

The peroxide of valeryl is a dense oily fluid, heavier than water. It gives a slight explosion when heated, and possesses the oxidizing properties of the other analogous peroxides.

The substance, dried by means of chloride of calcium, gave to analysis the following numbers:—

I. 0·3055 grm. of the substance gave 0·6615 grm. of carbonic acid and 0·2523 grm. of water.

II. 0·4005 grm. of the substance gave 0·873 grm. of carbonic acid and 0·3310 grm. of water.

These numbers give as the percentage constitution of the substance,

	I.	II.
Carbon . . .	59·05	59·39
Hydrogen . .	9·17	9·17
Oxygen . . .	31·78	31·44
	100·00	100·00

Theory requires

C_{10}	=120	59·40
H_{18}	= 18	8·91
O_4	= 64	31·69
	202	100·00

Peroxide of Camphoryl.

The action of the anhydrides of the bibasic acids on the alkaline peroxides affords a remarkable illustration of the profound differences by which this group is distinguished from the anhydrides of the monobasic acids. In the latter case we have seen that the monobasic anhydride decomposes with the alkaline peroxide, forming the peroxide of the radical and the baryta salt of the corresponding acid. In the case of the bibasic anhydride, a combination takes place of the anhydride with the peroxide, with the formation of a new and peculiar compound, which we may regard as the baryta salt of the peroxide of the bibasic radical. The compounds thus formed have but little permanence; and although in several cases we have evidence of their formation, in only one example, namely that of camphoric acid, have I been able to effect the analysis of the compound.

If hydrated peroxide of barium be gradually added to anhydrous succinic acid, and carefully mixed with it in a small quantity of water, the mixture becomes fluid; but long before the addition of the equivalent quantity of peroxide of barium, oxygen gas is evolved. If the fluid be filtered when this effervescence commences, it will be found to have the following properties:—

1. The solution is alkaline. It may be assumed therefore to contain but little, if any, succinate of barium, which is insoluble in water.

2. The solution rendered acid gives no blue colour with bichromate of potassium, and does not discolour permanganic acid. It therefore contains no peroxide of hydrogen. If peroxide of barium be mixed with hydrated succinic acid, a solution is obtained containing peroxide of hydrogen with the above characteristic reactions.

3. The solution bleaches indigo, gives a precipitate of peroxide of manganese with a solution of acetate of manganese, oxidizes ferrocyanide of potassium, and boiled with hydrochloric acid evolves chlorine.

4. When the solution is boiled, oxygen gas is evolved and a crystalline precipitate formed of succinate of barium.

Similar results are obtained if hydrated peroxide of barium be mixed with lactide, the lactic anhydride. The peroxide is rapidly dissolved, and a powerfully bleaching solution is obtained, possessing the same oxidizing properties as that procured from the succinic anhydride. This solution is, however, excessively unstable; even when cooled by ice, it is in a constant state of decomposition; and although it doubtless contains the lactic peroxide, I have been unable to effect its analysis.

With the camphoric anhydride I have been somewhat more successful. The anhydrous camphoric acid used in the following experiments was prepared by the oxidation of camphor by means of nitric acid. It is better not to attempt the prior preparation of a pure camphoric acid, which is attended with much difficulty, but after the product of the oxidation of camphor has been once or twice crystallized, to distil the crude acid. After two distillations and two or three crystallizations from alcohol of the distilled product, the camphoric anhydride is obtained quite pure. The substance was analysed with the following results:—

	Calculated.		Found.
C_{10} . . .	120	65·93	65·51
H_{14} . . .	14	7·69	7·87
O_3 . . .	48	26·38	26·62
$C_{10}H_{14}O_3$.	182	100·00	100·00

A portion (about 3 grms.) of anhydrous camphoric acid thus prepared was triturated in a mortar with ice-cold water, and the equivalent quantity of hydrated peroxide of barium was gradually added to the same, fragments of ice being mixed with the solution. No evolution of gas was observed during the experiment. The filtered solution was slightly alkaline, doubtless from the trace of baryta present in the peroxide. The solution, rendered acid, had the following properties. It gave no blue reaction with chromic acid, nor did it discolour permanganic acid. It bleached indigo, oxidized ferrocyanide of potassium, and decomposed hydriodic acid. The residue from which the solution was filtered was small in amount, and contained a little peroxide of barium. The solution when boiled evolves oxygen. Evaporated to dryness, it leaves a residue, which, dissolved in water, gives a precipitate with a solution of acetate of lead. This precipitate was suspended in water, and decomposed by sulphide of hydrogen. The acid thus separated was after one crystallization analysed. It was pure camphoric acid.

	Calculated.		Found.
C_{10} . . .	120	60·00	60·37
H_{16} . . .	16	8·00	8·13
O_4 . . .	64	32·00	31·50
$C_{10}H_{16}O_4$.	200	100·00	100·00

A solution of the camphoric peroxide thus prepared was analysed in the following manner:—

1. A measured quantity of the solution was rendered acid, and titred by means of a standard solution of iodine.

2. To another measured quantity of the same, a solution of carbonic acid in water was added. The liquid was raised to the boiling-point, filtered, and precipitated by sulphuric acid. The addition of the carbonic acid effects the removal of a small quantity of baryta, invariably present through the decomposition of the peroxide.

3. Another portion of the solution, similarly treated, was precipitated by acetate of lead, and the precipitate collected and weighed. The precipitate thus obtained is pure camphorate of lead, as is shown by the following determination:—

0·5901 grm. of the precipitate, ignited in a porcelain crucible, gave 0·3257 grm. of oxide of lead. Hence 100 parts gave 55·19 parts of oxide of lead. 100 parts of neutral camphorate of lead, $C_{10}H_{14}O_4Pb_2$, contain 55 parts of oxide.

Experiment I.—(1) One part of this solution required for its titration in two concordant experiments, 12·98 cub. centims. of a standard solution of iodine, which contained in 1 cub. centim. 0·002531 grm. of iodine. Hence 1000 parts of the solution of camphoric peroxide contained 2·07 grms. of oxygen.

(2) Six parts of the same solution, treated in the manner described, gave 0·1949 grm. of sulphate of barium. Hence 1000 parts of the solution contained 21·511 grms. of baryta, Ba_2O.

(3) Six parts of the same solution, treated as described, gave 0·3354 grm. of camphorate of lead. Hence 1000 parts of the solution contained 25·12 grms. of anhydrous camphoric acid, $C_{10}H_{14}O_3$.

Experiment II.—(1) In a similar experiment made with another solution, one part titred with the same standard solution of iodine was equivalent to 12·26 cub. centims. of the iodine solution. In a second experiment, one part of the same was equivalent to 12·317 cub. centims. of the same iodine solution. Hence, on the mean of the two experiments, 1000 parts of the solution contained 1·97 grm. of oxygen.

(2) Six parts of the solution gave 0·1508 grm. of sulphate of barium. Ten parts of the solution gave 0·2927 grm. of sulphate.

From the first determination, 1000 parts of the fluid contained 16·5 grms. of baryta; from the second, 1000 parts of the fluid contained 19·22 grms. of baryta. The mean of these two determinations (in which doubtless there is some error) gives 17·88 grms. of baryta in 1000 parts of fluid.

(3) Six parts of the solution gave 0·3354 grm. of camphorate of lead. Hence 1000 parts of the solution contained 21·43 grms. of anhydrous camphoric acid.

These numbers agree with the hypothesis that the solution contains the substance $C_{10}H_{14}O_5Ba_2$,—the reaction taking place according to the equation

$$C_{10}H_{14}O_3+Ba_2O_2=C_{10}H_{14}O_5Ba_2.$$

For we should have, assuming the camphoric acid to be correct as determined by precipitation with acetate of lead in 1000 parts of the solution,

Experiment I.

	Atomic weight.	Ratio calculated.	Found.
$C_{10}H_{14}O_3$ . . .	182	25·12	25·12
O . . .	16	2·20	2·07
Ba_2O . . .	153	21·12	21·51

Experiment II.

$C_{10}H_{14}O_3$ . . .	182	21·43	21·43
O . . .	16	1·88	1·96
Ba_2O . . .	153	18·00	17·88*

The oxygen-determinations show that even in this case there is a gradual, although but slight, decomposition of the substance taking place during the time which the determinations occupy. But this peroxide is far more stable than the corresponding succinic and lactic peroxides. I have made several unsuccessful attempts to analyse these substances by methods similar to the above; but, from the excessive instability of the solutions, I have been obliged to abandon the attempt. In the case, for example, of the lactic peroxide, three successive determinations required 60·8, 54·3, and 48·6 cub. centims. of the standard iodine solution, showing so rapid a change as to render hopeless the accurate determination of the oxygen. These substances stand, as it were, upon the very verge of chemical possibility, and have only a momentary and fugitive existence.

That in the above reaction the oxygen is transferred from the peroxide of barium to the anhydrous camphoric acid—in other words, that the compound formed is to be regarded as the barium salt of the peroxide of camphoryl, and not as the camphorate of the peroxide of barium, is shown by the reactions of the solution. The action of acids upon it does not form peroxide of hydrogen, and the action of alkalies does not reproduce the peroxide of barium. These reactions must take place if the solution contained the salt of the peroxide*.

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* Mean of the two determinations 16·5 and 19·22.

The analogy of the bisulphide of carbon to the anhydrous acids induced me to try its action on the alkaline peroxide. When bisulphide of carbon suspended in water is agitated with hydrated peroxide of barium, the peroxide is dissolved with the formation of a yellow solution. The solution when filtered is at first clear; but on standing, and more rapidly on boiling, a precipitate is formed of carbonate of barium. The solution contains a sulphide of barium. If sulphide of hydrogen be led through water in which peroxide of barium is suspended, a clear yellow solution is formed similar in appearance to the preceding.

I have not fully investigated the reaction; but the experiments point to the conclusion that in the first instance we have formed the combination of bisulphide of carbon and peroxide of barium, which subsequently decomposes into carbonate of barium and bisulphide of barium, according to the equations

$$CS_2 + Ba_2O_2 = Ba_2CS_2O_2,$$

$$Ba_2CS_2O_2 + 2BaHO = Ba_2CO_3 + Ba_2S_2 + H_2O.$$

The reaction is undoubtedly complicated by the action of the bisulphide of barium on the peroxide. This solution in presence of an excess of peroxide becomes colourless, hyposulphite of barium being probably formed,

$$Ba_2S_2 + 3Ba_2O_2 + 3H_2O = Ba_2S_2O_3 + 6BaHO.$$

In the case of the action of carbonic acid on peroxide of barium, I could detect no indication of the formation of a higher oxide.

The previous investigation has placed beyond doubt the existence of a new and extensive group of chemical substances, the peroxides of the radicals of the organic acids, a group in all probability as numerous as the anhydrides of the acids, and characterized by singular properties, which have never hitherto been discovered in any combination of carbon, and which greatly enlarge our view of the system of analogies by which the organic and inorganic worlds of chemistry are connected. These bodies are, so to say, the organic representatives of chlorine. Indeed no compound substance, unless perhaps the peroxide of hydrogen, can be compared with them in chemical similarity to that element. The solution of these peroxides in water can hardly be discriminated from a solution of chlorine; the solution bleaches indigo, oxidizes the protosalts of iron and manganese, decomposes the alkaline peroxides, is decomposed by the action of sunlight, and breaks up with water into the hydrated acid and oxygen. We have

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* The question regarded as one of notation may be considered as, whether we are to write the formula of the compound $\left.\begin{matrix}Ba'\\(C_{10}H_{14}O_3)''\\Ba'\end{matrix}\right\}O_2$, or $\left.\begin{matrix}Ba''\\(C_{10}H_{14}O_2)''\\Ba''\end{matrix}\right\}O_3$, or on the dualistic method as $C_{10}H_{14}O_3, Ba_2O$, or $C_{10}H_{14}O_2, Ba_2O_2$. If we are to write the formula of the sesquioxide and its hydrate as $\begin{matrix}Fe_2'''\\Fe_2'''\end{matrix}O_3$, and $2\left.\begin{matrix}H_3\\Fe_2'''\end{matrix}\right\}O_3$, we must in consistency write the formula of the peroxide and its hydratě thus, $\begin{matrix}Na''\\Na''\end{matrix}O_2$ and $2\begin{matrix}H_2\\Na''\end{matrix}O_2$. (Chem. Soc. Q. J. vol. xiv. p. 280.) The normal salt of the peroxide would be the body derived from the peroxide by the substitution of the equivalent quantity of the acid radical for the equivalent of sodium. The compound of one equivalent of camphoric anhydride with one equivalent of the peroxide of barium, would be an oxide intermediate between the protoxide and peroxide, just as the magnetic oxide, and its derivatives, is intermediate between the protoxide and sesquioxide.

$$Cl_2+2HKO+2Mn\,HO=2K\,Cl+H_2\,O+Mn_2\,H_2\,O_3,$$

and

$$Ac_2\,O_2+2HKO+2Mn\,HO=2KAc\,O+H_2\,O+Mn_2\,H_2\,O_3;$$

also

$$Cl_2+Ba_2\,O_2=2Ba\,Cl+O_2,$$

and

$$Bz_2\,O_2+Ba_2\,O_2=2Ba\,Bz\,O+O_2;$$

also

$$Cl_2+H_2\,O=2H\,Cl+O,$$

and

$$Ac_2\,O_2+H_2\,O=2H\,Ac\,O+O.$$

No parallel can be more complete.

In writing the formula of water and of hydrochloric acid, HO and HCl, it was implicitly assumed that the atom of oxygen stood in the same relation in regard to the molecule of water as does the atom of chlorine in regard to the molecule of hydrochloric acid. This view can no longer be maintained. Peroxide of hydrogen, and not oxygen, is the analogue of chlorine. We have

$$H_2+Cl_2=2H\,Cl,$$
$$H_2+H_2\,O_2=2H_2\,O,$$
$$H_2+Ba_2\,O_2=2H\,Ba\,O,$$
$$H_2+(C_2\,H_3\,O)_2\,O_2=2H(C_2\,H_3\,O)O.$$

The remarkable point in the case of the organic peroxides is not only that this analogy is an analogy of the forms of decomposition, but that the same similarity of properties which exists in hydrochloric acid and acetic acid is found also in chlorine and the peroxide of acetyl; and precisely the same reasons, derived from similarity of chemical properties, which lead us to place in the same class of "*acids*" hydrochloric acid and acetic acid, compel us also to group together chlorine and the peroxide of acetyl. These bodies are the analogues of chlorine in the same sense in which the nitrogen base is the analogue of potash, and in a closer sense than that in which ethyl is the analogue of hydrogen.

The transition is obvious from the peroxide of the acid radical to the peroxide of the basic radical. The question is immediately suggested whether by corresponding processes we may not be able to procure the peroxides of ethyl, of ethylene, of the compound ammoniums. I am yet occupied with this subject, and will now only remark that the peroxide of the glycol series appears undoubtedly to be formed. The bromide of ethylene does not, indeed, decompose the hydrated peroxide of barium; but this peroxide is immediately acted on by the diacetate, and a solution is formed of a most pungent odour, containing no peroxide of hydrogen, but possessing the usual characteristics of the organic peroxides. I have not made many experiments with the compound ammonias. A solution of hydrated oxide of tetramylammonium, evaporated *in vacuo* with a solution of pure peroxide of hydrogen, gave a residue which did not appear to contain even a trace of a substance resembling an alkaline peroxide. But this by no means renders it impossible that the same experiment in other cases may be more successful, for great differences are found in the stability of the peroxides of very analogous metals: the peroxide of potassium is quite decomposed on evaporation *in vacuo*, whereas the hydrated peroxide of sodium can be readily thus obtained.

XIX. *An Account of Experiments on the Change of the Elastic Force of a Constant Volume of Atmospheric Air, between* 32° F. *and* 212° F., *and also on the Temperature of the Melting-point of Mercury. By* BALFOUR STEWART, *M.A., F.R.S.*

Received June 18,—Read June 18, 1863.

IT was some time since proposed by the Kew Committee of the British Association to determine the temperature of the melting-point of mercury, in order if possible to add a third to the two familiar points which have been so long exclusively used in graduating thermometers; and afterwards the sum of £150 was voted for this purpose by the Government Grant Committee of the Royal Society.

In prosecuting this research, the final arrangement of apparatus has cost much labour and time; but the results at length obtained have exhibited a precision which has induced me to present them to the Society in the following communication.

I shall in the first place endeavour to describe the apparatus used, and shall then give an account of the experiments made and deduce results.

Description of Apparatus.

The apparatus employed was very similar in principle and construction to that used by REGNAULT in his fourth set of experiments on the dilatation of elastic fluids,—the coefficient sought being that which denotes the increase for 1° Fahr. of the elastic force of a gas the volume of which is constant.

This apparatus was constructed by Mr. BECKLEY, mechanical assistant at Kew, to whose skill in device and execution I am on this occasion very much indebted; and I would likewise desire to acknowledge the promptness and skill with which the requisite glass-blowing has been executed by Mr. CASELLA.

The atmospheric air upon which it is desired to operate is contained in the glass bulb B (Plate XXIV. fig. 2), and this is connected by means of a capillary tube t with another tube T of larger bore, which is cemented into an iron fitting, D. This fitting is tightly screwed, by means of an india-rubber washer, upon a reservoir R filled with mercury, and there is thus a communication between the mercury of the reservoir and the bore of the tube T. Another tube, T′, similar to T, is attached in the same manner to an iron fitting D′, and by means of it to the reservoir R; but the fitting D′ is furnished with a stopcock, by shutting which the communication between the reservoir and the bore of T′ may be interrupted at pleasure. The upper extremity of T′ communicates with the atmospheric air. The reservoir R, which is made of cast iron, is fitted accurately into a strong slate slab, which is in its turn supported by a solid block of masonry.

S (figs. 1 & 2) is a screw which drives a plunger P up and down, by means of which the capacity of the reservoir R may be enlarged or contracted at pleasure. The consequence is that when the capacity of the reservoir is diminished the mercury will ascend in the tube T, and also in T′ if the stopcock be open, while if the capacity be increased it will descend in these tubes. The inside of the reservoir is so shaped as to push up any small bubble of air that might otherwise have remained in during the process of filling with mercury. There is also a fine screw, S′, with a graduated head, which drives a fine plunger P′; and the change of capacity of the reservoir due to one revolution of this screw requires to be accurately ascertained.

A little above *h*′ there are two side tubes, terminating in bulbs *b*, *b*′, which are attached at an angle to the tube T.

One of these bulbs contains a desiccating substance, such as sulphuric acid, baryta, or anhydrous phosphoric acid, while the other contains a little caustic potash, a substance which has a strong attraction for carbonic acid.

Let us begin by supposing these two bulbs to be attached to the tube T, as in fig. 1, and let us also suppose that they are open to the atmosphere at their extremities. Suppose in fact that the tube T with its appendages has just been screwed on to the reservoir R. Now by means of the screw S drive the mercury up T until it reaches the level *h*′ a little below the opening where the side tubes branch off; and when it has reached this level, seal off the extremities of the two bulbs. All communication between the air in the bulbs and the atmosphere is thus intercepted.

The bulb B must now be heated as often as possible, and each time a portion of its air will be driven into the bulbs *b*, *b*′, and there deprived of moisture and carbonic acid. The tube *t* should also be heated occasionally, and also the tube T above *h*′, including the appendage tubes, but not the bulbs, the object being to drive away any moisture which may cling to the glass; but at the same time care must be taken not to heat the mercury, in case by any possibility some of its vapour may enter into the bulb B.

It is evident that, if this process be continued long enough, the air of the bulb B will be completely deprived of aqueous vapour and of carbonic acid; also by heating B occasionally above 300° F., any ozone which it may at first have contained will be destroyed.

When satisfied that the air of the bulb B has been thoroughly deprived of all these substances, seal off the appendage tubes, thereby detaching the bulbs *b*, *b*′; and then by means of the screw S drive up the mercury to about the level *h*. If the branch tubes are properly shaped, the mercury will now have run down and filled them. We have thus procured a quantity of unexceptionable air, which fills the bulb B and that portion of its attached tube above *h*.

It is well to remark that the tubes T and T′ are supposed to be well cleaned, and the mercury used to be quite pure.

In making an observation, the mercury is driven first by means of the screw S, and afterwards by the more delicate motion of S′ to a fixed level *h*, which is chosen near the top of the tube T, and the reading of the cathetometer C for this point is noted. It is clear that the height of the column of mercury in the long tube T′ (the stopcock at D′ being open) will depend upon the elastic force of the air in the bulb B as compared with that of the atmosphere. If this elasticity be altered by increasing or diminishing the temperature of B, the height of the mercury in T′, as read by the cathetometer, will be altered also; and hence the difference given by the scale of the cathetometer between the fixed level *h*, at which the mercury in T is always set, and the surface of the column of mercury in T′ will afford an indication of the temperature of the bulb B[*]. It is on this principle that the instrument is used as a thermometer.

The mercury used in these experiments was purified in the following manner. It was first heated for some time with dilute nitric acid and then allowed to remain in contact with strong sulphuric acid, being frequently stirred in both cases. It was afterwards well washed, first with a little caustic potash and afterwards with pure water, and was finally mixed with pounded sugar, and well filtered through paper before use.

It was thought desirable to ascertain whether different specimens of mercury so treated were of precisely the same specific gravity; and with this purpose the following experiment was made. The mercury used in the construction of the Kew Standard Barometer, described by JOHN WELSH in the Philosophical Transactions, 1856, page 507, was compared with that used in the construction of another standard barometer, since erected at Kew, and also with the mercury used in the experiments now described; and the following was the result:—

Specific-gravity bottle, filled with mercury from the cistern of old Kew standard	weighed at 62° F. 13975·8 grs.
The same, filled with mercury from the cistern of the new Kew standard	weighed at 62° F. 13976·1 grs.
The same, filled with mercury used in the experiments with air-thermometer	weighed at 62° F. 13976·4 grs.

It will be seen from this how small is the observed difference in specific gravity between these various specimens of mercury, and that even if this were not due to error of observation, yet would the difference between the readings of standard barometers constructed from these different specimens scarcely exceed one thousandth of an inch.

But while these specimens of mercury are sufficiently pure, if the fluid be used in measuring pressure, it might still be doubted whether they would all have the same melting-point. The following experiment will decide this question.

An old Kew standard thermometer (No. 45) was thrust into a beaker which contained eight or ten pounds of mercury, half frozen, half melted. The mercury was not the same as that used in the experiments with the air-thermometer.

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[*] It was ascertained that the strength of the bulb B was sufficient to prevent any sensible change of volume due to increase of pressure within the bulb.

The reading (observed by Dr. W. A. Miller and myself) was . . −37°·75

In an experiment with the mercury used in the air-thermometer in a vessel which contained fifty pounds half frozen, half melted—

The same thermometer read on one occasion . . . . −37·80

The same thermometer read on another occasion . . . . . . −37·70

The reading of this thermometer at the melting-point of ice throughout all these observations was . . . . 32·45

It may therefore be concluded that the melting-point of well-purified mercury for different specimens of the fluid, and for different masses, is practically a point of constant temperature, and that this temperature, as indicated by a Kew standard thermometer (graduated throughout according to the diameter of the bore), is 70°·2 below the freezing-point of water, or is equal to −38°·2 F.

The boiling-point apparatus used in these experiments was that recommended by Regnault. It is represented in fig. 2; and I need only remark that the steam, after passing round the bulb B, flows down by the channel indicated by arrow-heads, and finally escapes into the atmosphere by an orifice near the bottom of the apparatus. The bulb is thus entirely surrounded by steam in a state of motion. This piece of apparatus was compared with two others of the same description but of very different dimensions (one being the small apparatus used by travellers); and the agreement of the three, as tested by a thermometer, was very exact. Distilled water was always used during these experiments. A box with a few small holes bored in its bottom was that used to contain the melting ice, and care was taken that the ice was really in a melting state.

The arrangement for the melting mercury is represented in figs. 3, 4, 5, 6. The vessel for holding it consists of two wooden boxes, the one within the other, with a lining of felt between. When the box was filled with mercury, the bulb, in order to counteract the upward pressure of the fluid, was bound by a string to the bottom of the box. An agitator (figs. 4, 5, 6), made of wire gauze, was made to surround the bulb when the freezing-experiment was in progress, the compartments of which were easily penetrated by fluid mercury. The solid lumps of mercury were introduced outside of the wire gauze; and the agitator served the double purpose of keeping these from contact with the bulb and of promoting currents, by means of which the whole mass was kept at a uniform temperature. For the success of these experiments I am much indebted to Mr. Robert Addams of London; indeed without his ready cooperation it would have been impossible for me to freeze mercury in sufficient quantity.

This gentleman took the trouble to bring cylinders containing liquid carbonic acid to Kew whenever the freezing-experiment was to be performed. I need not here describe how solid carbonic acid is procured from these cylinders, nor how by mixing this with ether a very intense cold is produced; it is sufficient to state that by this process a very large quantity of mercury may be kept frozen for a length of time with great facility.

Experiments and Results.

The following are the formulæ used in the reduction of these experiments:—

Let P denote the elastic force of the air in the bulb B when it is surrounded with melting ice, and let the atmosphere around, including the mercury in the two tubes T, T′, be at the temperature 32°+t. Also let P′ denote the elastic force of the same air when the bulb is at the temperature 32°+T, the atmosphere being supposed to remain at 32°+t.

Further, let V denote the internal volume at 32° of the bulb B, and of that portion of the capillary tube which is subjected to the heating and cooling agents, and let v denote the internal volume at 32° of that portion of the tube T above the mercury which is not subject to the influence of these agents, but which contains air which may be supposed to retain the constant temperature 32°+t throughout the experiment.

Also let k denote the coefficient of expansion for 1° F. of the glass, and let α denote the corresponding coefficient of increase of elastic force of dry air the volume of which remains constant; and, finally, let us denote by unit of mass the air which occupies unit of volume under unit of pressure at the temperature of 32° F.

Then PV denotes the mass of the enclosed air which exists at the temperature of 32° F. when the bulb is surrounded by melting ice; also $\mathrm{P}v\frac{1+kt}{1+\alpha t}$ is that which exists, at the same time, at the temperature of the atmosphere (32°+t).

Hence the whole mass of air operated upon will be denoted by

$$\mathrm{P}\left\{\mathrm{V}+v\frac{1+kt}{1+\alpha t}\right\}. \qquad (1.)$$

Now let the bulb be subjected in like manner to an agent of which the temperature is 32°+T. Hence the mass of air existing at temperature 32°+T will be $\mathrm{P'V}.\frac{1+k\mathrm{T}}{1+\alpha\mathrm{T}}$, while that at the temperature of the atmosphere will be $\mathrm{P'}v.\frac{1+kt}{1+\alpha t}$, and the whole mass will be

$$\mathrm{P'}\left\{\mathrm{V}\frac{1+k\mathrm{T}}{1+\alpha\mathrm{T}}+v\frac{1+kt}{1+\alpha t}\right\}. \qquad (2.)$$

Since the mass of air remains unchanged, we have (1)=(2), or

$$\mathrm{P}\left\{\mathrm{V}+v\frac{1+kt}{1+\alpha t}\right\}=\mathrm{P'}\left\{\mathrm{V}\frac{1+k\mathrm{T}}{1+\alpha\mathrm{T}}+v\frac{1+kt}{1+\alpha t}\right\}.$$

Hence, if we wish to determine α, we shall have

$$1+\alpha\mathrm{T}=\frac{\mathrm{P'}(1+k\mathrm{T})}{\mathrm{P}-(\mathrm{P'}-\mathrm{P})\frac{v}{\mathrm{V}}\frac{1+kt}{1+\alpha t}}. \qquad (3.)$$

Here it may be remarked that $\frac{v}{\mathrm{V}}\frac{1+kt}{1+\alpha t}$ is a small quantity; so that we may in it quite well assume as the value of α that which was previously determined by REGNAULT, even although these experiments should give a slightly different value. Now according to this observer $1+180\alpha=1\cdot3665$. Hence $\alpha=\cdot002036$ nearly.

It must also be noted that in the boiling-water experiment T is determined in conformity with the report presented to the British Government by the Commissioners appointed to construct standard weights and measures, according to which 212° F. is taken to represent at London the temperature of steam at the pressure of 29·905 inches of mercury reduced to 32° F. This is also the value of 212° F., which has been adopted by the Kew Committee of the British Association*.

When it is the freezing-point of mercury which we wish to determine, the formula (3.) must be altered as follows:—

$$\frac{1+\alpha T}{1+kT}=\frac{P''}{P-(P''-P)\frac{v}{V}\cdot\frac{1+kt}{1+\alpha t}}. \quad \ldots\ldots\ldots \quad (4.)$$

Here it is T which we wish to determine, and which will of course appear as a negative quantity. The value of α is in this case supposed to have been previously determined.

Experiments made in order to determine α.—First Series.

In the first set of experiments made for this purpose a flint-glass bulb was used. Its volume and coefficient of expansion were determined by cleaning it, first, with nitric acid, secondly with sulphuric acid, afterwards with water, and, lastly, drying it with alcohol, after which process it admits of being well filled with mercury without any specks. It was then ascertained what weight of this fluid it held at 32°, and also at 212°.

	grs.
The weight of mercury at 32° was ascertained to be	10169·3,
That of mercury at 212° was ascertained to be . .	10011·4,

showing a loss of 157·9 grs. Had the glass not dilated, the loss of weight would have been =181·4 grs. (if we suppose the expansion of mercury between these two points to be =0·018153, which is REGNAULT's determination). Hence the dilatation of the glass envelope of the bulb between 32° and 212° was ·00235, or the coefficient of expansion of the glass for 1° F. =·0000131=k.

Also, assuming 252·5 grs. to be the weight of 1 cubic inch of water at 62°, and having found by experiment 13·584 to be the specific gravity of the mercury used compared with water at 62°, we obtain

	cub. in.
Capacity of the bulb at 62° =	2·957
Capacity of the bulb at 32° =	2·956 nearly.

The bulb having been thus calibrated was sealed on to the capillary tube t, and, along with its appendage bulbs b, b' and tube T, was attached to the reservoir at D in the manner already described. In this experiment the bulb b contained anhydrous phosphoric acid, and b' fused caustic potash; while these tubes were attached, the bulb B was heated and cooled very many times. The potash bulb b' was detached in about a week, but the phosphoric acid bulb was kept attached for at least three weeks.

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* Report of the Kew Committee of the British Association for 1853–54.

It has been already remarked that in these experiments the mercury in the tube T is always brought to a fixed point, determined by its cathetometer-reading, this being as near the top as is conveniently possible. Also, when the boiling-point apparatus is attached, it is arranged so as always to embrace, along with the bulb, the same portion of the capillary tube. If this position be marked, on one side of the mark we shall have air of the temperature of steam, and on the other side air of nearly the same temperature as the atmosphere. In order to estimate the volume of air existing in the tube at the atmospheric temperature when the mercury has been set to its fixed point by means of the cathetometer, shut the stopcock at D′, and estimate, by means of the graduated head, the number of revolutions and parts of a revolution of the fine screw S′ requisite to bring the mercury to that point in the capillary tube *t* which has been marked as that where the temperature of the boiling-water apparatus commences.

It may perhaps be objected to this method of measuring $\frac{v}{V}$, that should there be a small bubble of air lurking in the reservoir R, or at the points of the appendage tubes, after the bulbs *b*, *b*′ have been detached, this air will contract under the additional pressure caused by raising the mercury in T, and will consequently make *v* appear to be greater than it really is. It has, however, been ascertained, by means of pushing the mercury to a fixed point of the capillary tube *t*, with the stopcock at D′ shut, and then increasing the pressure by heating the bulb, that the error arising from this source is inappreciable.

It has also been ascertained, by means of inserting a small thermometer, that the temperature of the air in the tube immediately above the mercury at *h* remains nearly the same as that of the atmosphere without, even when the boiling-water apparatus is in operation.

It requires two observers to work the instrument. When the ice or boiling-water apparatus has been sufficiently long attached to make the observations constant, one observer is stationed at the cathetometer, which is set to the fixed point *h*, while another, by means of the fine screw S′, pushes the mercury (always a very little up) to the proper height for the cathetometer-setting. By making the mercury always rise, a uniform capillary action is secured. The cathetometer-observer then records the height of the mercury in T′. Suppose this to be higher than *h*, the difference between the level of the mercury in the two tubes, added to the barometric pressure, will give us the pressure of the air in the bulb B.

Mr. George Whipple, meteorological assistant at Kew, an exceedingly accurate and delicate observer, took most of the cathetometer- and barometer-readings in these experiments. It has been found that for a length of 30 inches the cathetometer-measurement requires a correction of +·003 inch, or for a difference of 11 inches +·001 nearly. This has been attended to in reducing the experiments. The bore of the tube T′ is about ·25 inch, that of the tube T at *h* is generally smaller; the capillary correction is ascertained after the bulb is detached, and the same atmospheric pressure acts on the surface of mercury in both tubes, by setting the mercury in T to the height h, and then reading by means of the cathetometer the height of the fluid in T'. In the first set of experiments the capillary correction has been found to be insensible. The following Table exhibits the results of this series of experiments.

TABLE I.—Results from first Bulb.

Date.	Number of readings taken		Elasticity of air in inches of mercury, having			Values of				Resulting value of $1+180\alpha$.
1862.	At 32°.	At 212°.	P.	P'.	the temp.	k.	T.	$\frac{v}{V}(1+kt)$.	$1+\alpha t$.	
Sept. 9.	8	4	30·811	41·965	65°	·0000131	180·25	·00555	1·0672	1·36729
Oct. 22.	6	6	30·791	41·868	56	·0000131	179·10	·005646	1·0489	1·36741
29.	7	6	30·777	41·903	52	·0000131	180·04	·005646	1·0407	1·36731
									Mean value of $1+180\alpha$………	1·36733

After the experiments recorded in Table I. were made, the bulb was again carefully examined, and found to be free from mercury, and then a very small portion of this fluid was pushed into the bulb, and the experiments recorded in Table II. were made in order to ascertain the influence of mercurial vapour upon the result obtained.

TABLE II.—Results from first Bulb with a little mercury in it.

Date.	Number of readings taken		Elasticity of air in inches of mercury, having			Values of				Resulting value of $1+180\alpha$.
1862.	At 32°.	At 212°.	P.	P'.	the temp.	k.	T.	$\frac{v}{V}(1+kt)$.	$1+\alpha t$.	
Nov. 21.	6	6	30·757	41·914	44°·0	·0000131	180·45	·00555	1·0244	1·36774
Dec. 2.	6	6	30·768	41·857	45·0	·0000131	179·39	·00555	1·0265	1·36749
8.	8	6	30·769	41·923	50·0	·0000131	180·25	·00575	1·0366	1·36786
13.	6	6	30·768	41·901	47·0	·0000131	180·10	·00565	1·0305	1·36755
									Mean value of $1+180\alpha$………	1·36766

Hence we see that, by forcing a little mercury into the bulb, the value of $1+180\alpha$ is apparently increased by the amount ·00033.

Second Series of Experiments.

On December 22, 1862, the bulb used in the first series of experiments was detached, and another, containing a little anhydrous barytes, was put in its place. In a couple of days the air in this bulb seemed to have become sufficiently dried; while there could be no suspicion of vapour of mercury having in this time distilled over into the bulb through a capillary tube of the length of 9 inches, and while the bulb remained at the same temperature as the other parts of the apparatus.

The bulb also was not calibrated before use, nor did any mercury come in contact with it until after the experiments about to be described. It was finally calibrated in the usual way. The tubes T, T′ were the same as those used in the last experiment, and the capillary correction was inappreciable. The result obtained by this bulb is recorded in the following Table:—

TABLE III.—Results from second Bulb.

Date.	Number of readings taken		Elasticity of air in inches of mercury, having			Values of				Resulting value of $1+180\alpha$.
1862.	At 32°.	At 212°.	P.	P′.	the temp.	k.	T.	$\frac{v}{V}(1+kt)$.	$1+\alpha t$.	
Dec. 24.	6	4	28·454	38·747	44°	·0000130	180·60	·00753	1·0244	1·36730
27.	4	4	28·457	38·766	46	·0000130	180·84	·00753	1·0285	1·36737
29.	4	4	28·460	38·684	48	·0000130	179·28	·00753	1·0326	1·36745
Mean value of $1+180\alpha$										1·36737

After the experiments recorded in Table III. a little mercury was pushed into the bulb; and the results obtained are recorded in the following Table:—

TABLE IV.—Results obtained from second Bulb with a little mercury in it.

Date.	Number of readings taken		Elasticity of air in inches of mercury, having			Values of				Resulting value of $1+180\alpha$.
1863.	At 32°.	At 212°.	P.	P′.	the temp.	k.	T.	$\frac{v}{V}(1+kt)$.	$1+\alpha t$.	
Jan. 12.	6	6	28·444	38·729	41°	·0000130	180·40	·00753	1·0183	1·36762
14.	4	4	28·450	38·749	44	·0000130	180·60	·00753	1·0244	1·36762
Mean value of $1+180\alpha$										1·36762

It appears from a comparison of Tables III. and IV., that the value of $1+180\alpha$ is apparently increased ·00025 by forcing in the mercury. For the first bulb this increase was ·00033; and the difference between these numbers is probably owing to errors of observation: but the agreement is sufficiently close to show that the first bulb cannot have at first contained any vapour of mercury; for, if it had, the difference caused by forcing in mercury, instead of being greater than that for the second bulb, should have been much less.

Third Series of Experiments.

The barytes bulb was now removed, and another bulb of flint glass, along with its own drying-arrangement and tube T, was attached to the apparatus. The bulb was only calibrated after use, and had not come in contact with mercury until the experiments were finished.

Sulphuric acid was used instead of anhydrous phosphoric acid in the bulb b, and potash, as before, in b'.

Unfortunately the capillary correction due to the setting at h was not determined in an unexceptionable manner; but as the third series of experiments is precisely similar $\left(\text{as regards the value of } \frac{v}{V}\right)$ to the fourth, to be hereafter described, we shall apply to the former the capillary correction for the fourth series as probably near the truth. Accordingly, for equal pressures on both tubes, we shall suppose that T would have read ·018 in. lower than T'.

TABLE V.—Results from third Bulb.

Date.	Number of readings taken		Elasticity of air in inches of mercury, having			Values of				Resulting value of $1+180\alpha$.
1863.	At 32°.	At 212°.	P.	P'.	the temp.	k.	T.	$\frac{v}{V}(1+kt)$.	$1+\alpha t$.	
March 3.	6	6	31·887	43·424	56°	·0000141	179·75	·00323	1·0489	1·36729
16.	8	6	31·866	43·413	47	·0000141	180·09	·00323	1·0305	1·36719
31.	14	6	31·879	43·465	54	·0000141	180·55	·00323	1·0448	1·36732
									Mean value of $1+180\alpha$	1·36727

Fourth Series of Experiments.

The third bulb was removed, and another bulb of crown glass put in its place. This bulb had not come in contact with mercury before the experiments were made; it was afterwards calibrated in the usual manner. The capillary correction has been already given. The results with this bulb are embodied in the following Table:—

TABLE VI.—Results from fourth Bulb.

Date.	Number of readings taken		Elasticity of air in inches of mercury, having			Values of				Resulting value of $1+180\alpha$.
1863.	At 32°.	At 212°.	P.	P'.	the temp.	k	T.	$\frac{v}{V}(1+kt)$.	$1+\alpha t$.	
June 1.	6	6	30·681	41·817	62°	·0000141	180·42	·00335	1·0611	1·36713
2.	8	8	30·687	41·804	66	·0000141	180·18	·00335	1·0692	1·36691
5.	8	6	30·681	41·792	64	·0000141	180·01	·00335	1·0651	1·36715
6.	8	6	30·685	41·744	64	·0000141	179·18	·00335	1·0651	1·36705
8.	20 double set.	12 double set.	30·679	41·757	63	·0000141	179·37	·00335	1·0631	1·36736
									Mean value of $1+180\alpha$	1·36716

We have thus, by means of these four series of experiments, four mean values of $1+180\alpha$, as under:—

First bulb, flint glass, dried by anhydrous phosphoric acid, gives .	1·36733
Second bulb, flint glass, dried by anhydrous barytes in bulb, gives	1·36737
Third bulb, flint glass, dried by sulphuric acid, gives	1·36727
Fourth bulb, crown glass, dried by sulphuric acid, gives	1·36716
Mean value of $1+180\alpha$ . .	1·36728

This, therefore, is to be regarded as the coefficient obtained by these experiments. It differs slightly from that found by REGNAULT, who makes it to be 1·3665; and although the difference is not great, I should have preferred to have agreed still more closely with this eminent authority, but I am unable to think of any source of error in these experiments.

Two sets of experiments were made in order to determine the freezing-point of mercury. In these, in order to ensure the greatest possible amount of precision, it was estimated that the air in a portion of the capillary tube near the mercury had a temperature lower than that of the atmosphere. The first set of experiments were made with the first bulb, and the second set with the third bulb.

Here the formula to be employed is

$$\frac{1+\alpha T}{1+kT}=\frac{P''}{P-(P''-P)\frac{v}{V}\frac{1+kt}{1+\alpha t}},$$

and the freezing-point of water now becomes the higher temperature. The result of these experiments is embodied in the following Table:—

TABLE VII.—Experiments made in order to determine the Temperature of Melting Mercury.

Date.	Number of readings taken		Elasticity of air in inches of mercury, having			Values of				Resulting value of T.
	At 32°.	At melting-point of mercury.	P.	P''.	the temp.	α*.	k.	$\frac{v}{V}(1+kt)$.	$(1+\alpha t)$.	
1862. Oct. 22.	6	12	30·791	26·446	56°	·0020404	·0000131	·00652	1·0489	−69·91
1863. Mar. 13.	14	15	31·879	27·368	54	·0020404	·0000141	·00329	1·0448	−69·95
									Mean value of T.........	−69·93

Hence the value on FAHRENHEIT'S scale of the melting-point of mercury, as determined by these experiments, is −37°·93 F., while on a standard Kew mercurial thermometer the reading was −38°·2 F. It may perhaps be gathered from this that mercury, before beginning to freeze, slightly increases the rate at which it contracts; but further experiments must be made with other mercurial thermometers before this can be accurately determined.

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* As determined by the foregoing experiments.

XX. *On some Compounds and Derivatives of Glyoxylic Acid.*

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ORGANIC substances of simple composition, like marsh-gas, ethylene, alcohol, and acetic acid, are deserving of most careful study, not merely on account of their being representative members of numerous and important classes of bodies, but also because they form connecting links between the compounds of inorganic chemistry and the more complicated forms of organic nature.

Glyoxylic acid belongs to this class of bodies, because it bears the same relation to oxalic acid that sulphurous acid does to sulphuric acid, and because it stands to glycolic acid as common aldehyde, $C_2 H_4 O$, does to alcohol, $C_2 H_6 O$. These relations suggested the experiments which will be described in the following pages.

Combinations of Glyoxylates and Sulphites.

a. Glyoxylic Acid and Bisulphite of Soda.

If a concentrated solution of bisulphite of soda be mixed with one-fourth of its volume of nearly anhydrous glyoxylic acid, a white crystalline precipitate will separate from the mixture in the course of a day or two. This precipitate is to be collected on a filter, washed with cold water, and recrystallized from its solution in the smallest possible quantity of hot water.

The substance thus prepared presents itself in small colourless crystals, which dissolve easily in water, and evolve sulphurous acid with sulphuric acid. The aqueous solution is not altered by the addition of potash or ammonia, but yields a copious white precipitate with acetate of lead. The solid substance, heated on a piece of platinum foil, burns without any unusual appearance. Analysis furnished the following results:—

I. 0·361 grm., heated with concentrated sulphuric acid, gave 0·254 grm. of sulphate of soda. Another quantity of sulphuric acid, of the same quality as that used in this experiment, was evaporated in a platinum crucible; the weight of the latter was found to be unchanged.

II. 0·4365 grm., dissolved in hydrochloric acid and evaporated, left, after ignition, 0·2515 grm. of chloride of sodium. This residue furnished, after treatment with sulphuric acid, 0·306 grm. of sulphate of soda.

0·573 grm., oxidized with a mixture of chlorate of potash and hydrochloric acid, and the sulphuric acid precipitated with chloride of barium, gave 0·69 grm. of sulphate of baryta.

0·565 grm. of another preparation, burnt with chromate of lead, gave 0·248 grm. of carbonic acid and 0·051 grm. of water.

The substance therefore contains in 100 parts—

	I.	II.
Carbon	——	11·97
Hydrogen	——	1·00
Sodium	22·8	22·67
Oxygen	——	——
Sulphur	——	16·55

The formula $C_2 H Na O_3 + S H Na O_3$ requires—

Carbon	2	24	12
Hydrogen . . .	2	2	1
Sodium	2	46	23
Oxygen	6	96	48
Sulphur . . .	1	32	16
		200	100

The substance is consequently a compound of glyoxylate of soda with bisulphite of soda, and the above numbers confirm the formula $C_2 H_2 O_3$.

b. Glyoxylate of Lime with Bisulphite of Lime.

Glyoxylate of lime rapidly dissolves in a saturated solution of sulphurous acid. If this solution be concentrated on the water-bath and afterwards allowed to stand in the exsiccator, a fine crop of colourless crystals separates. These crystals have to be purified by recrystallization from water, and dried over sulphuric acid *in vacuo*.

This substance, like the soda compound, easily dissolves in water, and its concentrated solution comports itself with reagents as follows:—Sulphuric acid produces a white precipitate of sulphate of lime and liberates sulphurous acid; lime-water, chloride of barium, and ammonia respectively cause the formation of a white precipitate, whilst carbonate of lime has no decomposing influence. The form of the crystals could not be determined.

Analysis gave the following results:—

0·351 grm. furnished 0·145 grm. carbonate of lime.

0·56 grm. of another preparation, oxidized with a mixture of chlorate of potash and hydrochloric acid, and the sulphuric acid precipitated by chloride of barium, gave 0·555 grm. of sulphate of baryta.

The substance therefore contains in 100 parts—

Calcium	16·52
Sulphur	13·62

These numbers correspond to a double salt of bisulphite and glyoxylate of lime.

The formula $2(C_2HCaO_3+SHCaO_3)+5H_2O$ requires—

Calcium	16·73
Sulphur	13·38

Considering the mode of preparation, the mother-liquor of the salt ought to contain glyoxylic acid. In order to ascertain its presence by experiment, the mother-liquor was evaporated to dryness on the water-bath and the residue exhausted with alcohol. The latter had dissolved an acid which yielded, when treated with carbonate of lime, a salt that contained no sulphur and would not crystallize, but which in other respects comported itself like glyoxylate of lime.

If a compound be the derivative of a substance of an easily changeable nature, its composition, as represented by its rational formula, should always be checked by special experiments. It therefore appeared desirable to regenerate the glyoxylate of lime assumed to form a constituent part of the salt in question. For this purpose an aqueous solution of the latter was mixed with sufficient oxalic acid to precipitate all the lime; the clear filtrate, in order to expel the sulphurous acid, was evaporated on the water-bath till it assumed the consistency of a thin syrup; this residue, which proved to be glyoxylic acid, was dissolved in water and converted, by treatment with chalk, into a crystalline salt that both in form and other properties agreed with glyoxylate of lime.

Glyoxylic acid is a strong acid; it dissolves zinc and expels carbonic acid from carbonates. The experiments described hereafter show that it shares some distinguishing properties with the aldehydes. It may therefore be assumed that, if the acid properties of glyoxylic acid were less marked, it would combine with the bisulphites like hydride of salicyl.

Compound of Glyoxylate and Lactate of Lime.

Like as glyoxylic acid is formed from ethylic alcohol, so will the homologous acid $C_3H_4O_3$ probably be formed from the alcohol C_3H_8O. But since propylic alcohol is only procurable with great difficulty, I attempted the preparation of $C_3H_4O_3$ from lactic acid.

$$\underbrace{C_3H_6O_3}_{\text{Lactic acid.}} - H_2 = \underbrace{C_3H_4O_3}_{\text{Pyroracemic acid.}}.$$

Two experiments made with nitric acid of 1·2 sp. gr. and lactic acid did not give the desired result. At a low temperature no action appeared to take place, and at a higher temperature, or with more concentrated nitric acid, only the formation of oxalic acid could be expected.

It is well known that platinum when alloyed with silver is dissolved by nitric acid. It therefore appeared not unlikely that lactic acid, when mixed with alcohol, would be oxidized at a moderate temperature by nitric acid. The experiment was made with a mixture of equal weights of lactic acid and alcohol and a suitable quantity of nitric acid, on the plan which I employed for the preparation of glyoxylic acid. An acid was thus obtained which, when treated with lime, furnished neither glyoxylate nor lactate of lime. The salt obtained was white and crystalline, required more boiling water for its solution than either $C_2 H Ca O_3$ or $C_3 H_5 Ca O_3$, and, when the hot solution cooled, separated therefrom in white crystalline crusts.

Analysis gave the following results:—

0·241 grm. furnished 0·11 grm. of carbonate of lime.

0·425 grm., burnt with chromate of lead, gave 0·423 grm. of carbonic acid and 0·144 grm. of water.

The substance therefore contains in 100 parts—

Carbon	27·14
Hydrogen	3·76
Calcium	18·25
Oxygen	——

The formula $C_5 H_8 Ca_2 O_7$ requires—

Carbon	5	60	27·27
Hydrogen . . .	8	8	3·63
Calcium . . .	2	40	18·18
Oxygen . . .	7	112	50·92
		220	100·00

According to its formula this substance might be composed as follows:—

Glyoxylate of lime . . .	$C_2 H\ Ca\ O_3$
Lactate of lime	$C_3 H_5 Ca\ O_3$
Water	$H_2\ O$
	$C_5 H_8 Ca_2 O_7$

The following experiments confirm this composition:—The solution of the compound $C_5 H_8 Ca_2 O_7$ yields with lime-water a white precipitate, which immediately after its formation is found to be soluble in acetic acid; this precipitate, however, after the lapse of some time, or by exposure to a temperature of 212°, becomes insoluble in this acid. The same property distinguishes the soluble glyoxylates.

$$2(\underbrace{C_2 H Ca O_3}_{\text{Glyoxylate of lime.}}) + Ca H O = \underbrace{C_2 Ca_2 O_4}_{\text{Oxalate of lime.}} + \underbrace{C_2 H_3 Ca O_3}_{\text{Glycolate of lime.}}.$$

If the formula $C_2 H Ca O_3 + C_3 H_5 Ca O_3 + H_2 O$ be the correct expression of the composition of the salt in question, then a quantity represented by the above formula ought to be decomposed by boiling lime-water into one atom of lactate, half an atom of glycolate, and half an atom of oxalate of lime; the quantity of the latter may be easily determined.

0·76 grm. of the substance was dissolved in water and boiled with an excess of clear lime-water until the decomposition was complete; in order to prevent carbonate of lime from mixing with the precipitate, a slight excess of acetic acid was added; the precipitated oxalate of lime was collected on a filter and converted in the usual manner into carbonate of lime; the weight of the latter was found to be 0·168 grm., which corresponds to 28·3 per cent. of oxalate of lime. The formula $C_5 H_8 Ca_2 O_7$ requires 29·0 per cent. of oxalate. In order to confirm the conclusion to be drawn from this experiment, the substance $C_5 H_8 Ca_2 O_7$ was prepared from lactate and glyoxylate of lime. For this purpose equivalent quantities of $C_3 H_5 Ca O_3$ and $C_2 H Ca O_3$ were respectively dissolved in the least possible quantity of boiling water, and the solutions were then mixed. From the cooling liquid a large quantity of crystals separated, which were identical in properties with the body $C_5 H_8 Ca_2 O_7$. Two determinations of the solubility in water of I., the substance prepared as just described, and of II., the body obtained by the oxidation of a mixture of lactic acid and alcohol by means of nitric acid, placed the identity of the two substances beyond a doubt.

I. 5·003 grms. of the solution, saturated at 18° C., left after evaporation at 100° C. 0·0265 grm. of residue.

II. 3·151 grms. of the solution, treated in the same manner as I., left 0·017 grm. of residue. Therefore at 18° C.

One part of I. dissolves in 187 parts of water.
One part of II. dissolves in 184 parts of water.

The formula $C_5 H_8 Ca_2 O_7 = (C_2 H Ca O_3 + C_3 H_5 Ca O_3 + H_2 O)$ may therefore be considered the true expression of the composition of the compound. The last member of this formula may be eliminated by heat; it appears, however, that the remainder, $C_2 H Ca O_3 + C_3 H_5 Ca O_3$, suffers some change by this process.

The tendency to form this double salt is perhaps the reason why the oxidating influence of the nitric acid only affects the alcohol and does not extend to the lactic acid.

Ammoniacal Compounds of some Glyoxylates.

a. Glyoxylate of Lime and Ammonia.

Pure and well-crystallized glyoxylate of ammonia was dissolved in the least possible quantity of hot water, and the solution thus obtained was divided into two equal parts. One part, on being mixed with chloride of calcium, assumed the appearance of a transparent jelly, which resembled silicic acid when precipitated by hydrochloric acid from a concentrated solution of silicate of potash; the vessel, in fact, could be inverted without any portion of its contents being lost; after the lapse of a few hours white opaque points were observed in the jelly, which gradually increased in number and magnitude, until at last the whole of the jelly-like substance was converted into a fine crop of prismatic crystals; these were found to agree in form and other properties with glyoxylate of lime.

To the other part of the solution of glyoxylate of ammonia a mixture of chloride of calcium and acetate of ammonia was added. In this instance no immediate change took place, but in the course of twenty-four hours a white precipitate, a compound of glyoxylate of lime and ammonia, made its appearance; the precipitate was collected on a filter, washed with cold water, and dried over sulphuric acid.

A similar result was obtained by precipitating a solution of glyoxylate of ammonia with acetate of lime, just sufficient ammonia being at the same time added to keep the liquid neutral; the compound thus prepared is colourless or slightly yellow, dissolves sparingly in water, but is easily soluble in dilute acetic acid; it becomes highly electrical when rubbed in a mortar, and burns like tinder on a piece of heated platinum foil, leaving carbonate of lime. The aqueous solution possesses an alkaline reaction, and shows no change on the addition of lime-water.

The following experimental results determined its composition:—

0·139 grm. furnished 0·061 grm. of carbonate of lime.

0·236 grm., burnt with soda-lime, gave 0·293 grm. of ammonio-chloride of platinum.

0·26 grm., burnt with chromate of lead, gave 0·206 grm. of carbonic acid and 0·088 grm. of water.

The substance therefore contains in 100 parts—

Carbon	21·6
Hydrogen	3·76
Calcium	17·7
Nitrogen	7·8
Oxygen	——

According to the formula $3(C_2HCaO_3), 2NH_3, H_2O$, we obtain

Carbon . . .	6	72	21·75
Hydrogen . .	11	11	3·32
Calcium . . .	3	60	18·12
Nitrogen . . .	2	28	8·45
Oxygen . . .	10	160	——
		331	

The calculated numbers do not agree very well with those found by experiment; the differences, however, are not greater than those which might be expected from the circumstance of the analysed substance having been obtained in the form of a precipitate.

If, therefore, glyoxylate of ammonia be decomposed by acetate of lime, acetate of ammonia and glyoxylate of lime are formed; the latter, however, withdraws a part of the ammonia from the acetate of ammonia, and thus produces the compound which has just been mentioned. Some grammes of this substance were mixed with oxalic acid solution sufficient to convert two-thirds of its lime into oxalate. If the composition ascribed to the compound be correct, the filtrate from the oxalate of lime ought to contain glyoxylates of lime and ammonia; the filtered liquid was evaporated over sulphuric acid *in vacuo*, when a crust, consisting of small prisms arranged round a common centre, remained. This substance was repeatedly recrystallized from water; and in each process the crystals which form first, were separated from those which made their appearance after a half or three-quarters of the solution had been evaporated. In this manner two kinds of crystals were obtained: those which separated first, were found to be identical with glyoxylate of lime; and those which formed last, possessed the properties of glyoxylate of ammonia. This experiment, therefore, confirms the formula $3(C_2 H Ca O_3), 2N H_3 + H_2 O$, which was deduced from analytical results.

A compound of a similar nature may be directly obtained from glyoxylate of lime and ammonia. For this purpose a hot and concentrated solution of $C_2 H Ca O_3 + H_2 O$ is to be mixed with a few drops of ammonia, and filtered; ammonia is then to be added to the filtrate so long as a precipitate forms; the latter is to be collected and washed with cold water. The compound, thus prepared, possesses the same properties as the one obtained from glyoxylate of ammonia and acetate of lime.

The analytical results were as follow:—

0·161 grm. gave 0·079 grm. of carbonate of lime.

0·227 grm., burnt with soda-lime, gave 0·316 grm. of ammonio-chloride of platinum.

0·488 grm., burnt with chromate of lead, furnished 0·402 grm. of carbonic acid and 0·126 grm. of water.

Therefore 100 parts of the substance contain—

Carbon	22·47
Hydrogen	2·86
Calcium	19·62
Nitrogen	8·74
Oxygen	——

and the formula $3(C_2 H Ca O_3), 2N H_3$ requires—

Carbon . . .	6	72	23·00
Hydrogen . . .	9	9	2·87
Calcium . . .	3	60	19·16
Nitrogen . . .	2	28	8·94
Oxygen . . .	9	144	——

This compound, therefore, differs from the preceding one by containing one atom of water less.

If a quantity of this substance be treated with oxalic acid solution, sufficient to convert its lime into oxalate, the supernatant liquid comports itself like a solution of glyoxylic acid and glyoxylate of ammonia.

b. Glyoxylate of Silver and Ammonia.

A concentrated solution of glyoxylate of ammonia yields with nitrate of silver a crystalline precipitate of glyoxylate of silver. But if the solution of glyoxylate of ammonia contain other ammoniacal compounds, such as nitrate of ammonia, the precipitate is found to contain ammonia besides the other components already mentioned.

A specimen prepared under the last-mentioned circumstances and afterwards dried *in vacuo*, was found to contain in 100 parts—

Nitrogen	5·8
Silver	59·01

These numbers were calculated from the following determinations:—

0·442 grm., burnt with soda-lime, gave 0·408 grm. of ammonio-chloride of platinum.

0·238 grm., after treatment with hydrochloric acid, furnished 0·172 grm. of chloride of silver and 0·011 grm. of metallic silver.

The formula $4(C_2 H Ag O_3), 3N H_3 - 3H_2 O$ requires—

Nitrogen	5·82
Silver	59·9

The absence of nitric acid in this salt was proved by a special experiment with protosulphate of iron and sulphuric acid. The silver could not be determined by simple ignition, because heat decomposes the compound with explosive violence.

c. Glyoxylate of Lead and Ammonia.

Acetate of lead produced, in a liquid containing glyoxylate and acetate of ammonia, a heavy white precipitate.

0·409 grm. of this precipitate, dried over sulphuric acid, furnished 0·243 grm. of metallic lead.

0·589 grm., burnt with soda-lime, gave 0·209 grm. of ammonio-chloride of platinum.

The compound contains, therefore, in 100 parts—

Nitrogen	2·22
Lead	59·4

The formula $7(C_2 H Pb O_3), 2N H_3 - 2H_2 O$ requires—

Nitrogen	2·27
Lead	58·7

This substance, on being pounded in a mortar, became highly electrical.

A solution of crystallized glyoxylate of ammonia in ammonia, when raised to temperatures below 212°, turns brown, and forms derivatives of an acid character; neither these substances, however, nor their salts could be obtained in crystals.

The compounds of ammonia with the glyoxylates are easily decomposed by heat, and by nitric acid and other reagents; the products of decomposition could not be examined, because their physical properties precluded their preparation in a pure state. An experiment, which showed the attraction between glyoxylates and ammonia, may be mentioned here. If a quantity of $3(C_2 H Ca O_3), 2N H_3$ be boiled with caustic potash, a part of the ammonia is very slowly expelled; if the liquid be evaporated to dryness, and the residue be raised to a higher temperature, it assumes a beautiful purple colour, and at the same time emits streams of ammonia. This red substance is very changeable, and is formed, even under the most favourable circumstances, in but very small quantities.

Action of Hydriodic Acid on Glyoxylates.

Hydriodic acid and glyoxylate of lime were heated together for several days in sealed glass tubes, the temperature varying from 100° C. to 110° C. In order to decompose a part or the whole of the hydriodic acid, the contents of the tubes were exposed for some time to the influence of the atmosphere, and finally saturated with carbonate of lime. The whole was then boiled and filtered, and the filtrate mixed with alcohol. A precipitate was formed which proved to be a quantity of glyoxylate of lime, little inferior to that which was originally taken for the experiment.

Action of Sulphuretted Hydrogen on Glyoxylates.

Through a concentrated solution of glyoxylic acid a current of sulphuretted hydrogen was passed until the liquid appeared to be completely saturated; no perceptible action took place, and even after twenty-four hours' contact the liquid seemed to be unchanged. The solution was now evaporated, at first over pieces of hydrate of potash, and afterwards over sulphuric acid *in vacuo*. As soon as most of the water was gone, small needles, radiating from a common centre, began to form, and at last the whole of the contents of the evaporating-basin appeared one mass of crystals, which were found to be soaked with a syrupy mother-liquor, and to be so easily soluble in the ordinary means of solution, that all attempts at further purification were abandoned. A solution of glyoxylate of lime comports itself like glyoxylic acid when acted upon by sulphuretted hydrogen; it apparently remains unchanged; but on allowing a part of it to evaporate at ordinary temperatures *in vacuo*, a brittle, transparent and amorphous compound is obtained.

In order to avoid the inconvenience of evaporation *in vacuo*, the rest of the glyoxylate-of-lime solution, after having been treated with sulphuretted hydrogen, was mixed with a little more than its bulk of alcohol. Nearly the whole of the new compound separated as a precipitate, which was collected on a filter and washed with spirit of wine.

Thus prepared, the substance easily dissolved in water, forming a solution of a pale pink colour. The liquid, after the evaporation of the water *in vacuo*, left a transparent amorphous and nearly colourless substance. This compound is the lime-salt of an acid which bears a similar relation to glyoxylic acid that thiacetic acid does to acetic acid. In the state of powder it exhibits a striking property when brought in contact with water; it becomes as viscous as glass when rendered red-hot by means of a Bunsen's burner, and may be drawn out in long threads. By degrees the water dissolves the viscous mass, and forms a solution which shows the following properties with reagents:—A white precipitate is formed by the addition of acetate of zinc or corrosive sublimate; acetate of lead or nitrate of silver throws down a yellow precipitate; and sulphate of copper causes the immediate separation of a black substance, which is probably sulphuret of copper. Both the silver and the lead precipitates turn black, the silver after the lapse of some time at ordinary temperatures, and the lead at once on exposure to a temperature of 100° C. Hydrochloric acid produces no perceptible change in the aqueous solution of the compound; ammonia causes the formation of a white precipitate; and lime-water the same result as it does with glyoxylate of lime. The brown colour of iodine immediately disappears, as it does in solutions of other sulphur compounds, such as xanthate of potash. Sesquichloride of iron acts like iodine. The solution of this sulphur compound, when boiled, decomposes; it turns yellow; a crystalline powder of oxalate of lime separates; and a lime-salt, which could not be obtained in crystals, remains in solution. The compound burns on a hot piece of platinum foil like tinder, and evolves, when heated in a glass tube which is sealed at one end, an odour like that of mercaptan.

In order to obtain some guarantee for the homogeneous nature of the substance, a powdered quantity of it was well mixed and digested for a long time with such a quantity of very dilute spirit of wine as was required to dissolve about half of it. The undissolved portion will be called S, the dissolved part S′. S′ was obtained by evaporating the solution wherein it was contained; both S and S′ were prepared for analysis by being dried over sulphuric acid *in vacuo*.

The analysis of S gave the following results:—

0·731 grm., burnt with chromate of lead, gave 0·508 grm. of carbonic acid and 0·186 grm. of water.

0·423 grm., boiled with nitric acid and precipitated with chloride of barium, gave 0·39 grm. of sulphate of baryta.

The analysis of S′ gave the following results:—

0·274 grm., burnt with chromate of lead, yielded 0·192 grm. of carbonic acid and 0·074 grm. of water.

The results of the analysis of S+S′ were as follows:—

0·557 grm. from another preparation furnished 0·381 grm. of carbonic acid.

0·4 grm., oxidized with chlorate of potash and hydrochloric acid, gave 0·357 grm. of sulphate of baryta.

0·49 grm., precipitated with oxalate of ammonia, gave 0·192 grm. of carbonate of lime.

According to these determinations, 100 parts contain—

	S.	S′.	S+S′.
Carbon . . .	18·95	19·11	18·65
Hydrogen . .	2·82	3·00	——
Calcium . .	——	——	15·67
Oxygen . . .	——	——	——
Sulphur . .	12·67	——	12·26

The formula $C_4 H_2 Ca_2 \left.\begin{matrix} O_5 \\ S \end{matrix}\right\} + 3H_2 O$ requires—

Carbon . . .	4	48	18·75
Hydrogen . .	8	8	3·12
Calcium . . .	2	40	15·62
Oxygen . . .	8	128	——
Sulphur . . .	1	32	12·5

The homogeneity of the substance may be considered as proved by the identity in composition of S and S′; and its formation would be represented by the following equation:—

$$\underbrace{2(C_2 H Ca O_3 + H_2 O)}_{\text{Glyoxylate of lime.}} + H_2 S = C_4 H_2 Ca_2 \left.\begin{matrix} O_5 \\ S \end{matrix}\right\} + 3H_2 O.$$

The crystalline derivative from sulphuretted hydrogen and glyoxylic acid just mentioned is probably represented by the formula $C_4 H_4 \left.\begin{matrix} O_5 \\ S \end{matrix}\right\}$.

Neither the salts prepared from this acid, nor the compounds obtained by exchanging the calcium in $C_4 H_2 Ca_2 \left.\begin{matrix} O_5 \\ S \end{matrix}\right\} + 3H_2 O$ for other metals could be obtained in crystals; and therefore I did not pursue the investigation of these bodies.

Action of Zinc on Glyoxylic Acid.

A concentrated solution of this acid dissolves pure zinc without evolution of hydrogen gas, and the liquid at the same time becomes perceptibly warm; the reaction very soon ceases. Even if metal and acid be left in contact for a day or two at ordinary temperatures, the liquid will still contain a considerable portion of unchanged glyoxylic acid. The process may be accelerated, and may in fact be completed in the course of eight or ten hours, by exposing the reacting substances to a temperature of about 80° C. The previously colourless liquid will then have assumed a yellow colour, and does no longer contain glyoxylic acid or glyoxylate of zinc. The pieces of undissolved zinc are found to be covered with a small quantity of a white crystalline powder. Glyoxylic acid thus saturated with zinc was mixed with pure carbonate of lime and treated with sulphuretted hydrogen: the zinc precipitated as sulphuret of zinc, and the liberated acid dissolved carbonate of lime and formed a lime-salt. The filtrate from the $Zn_2 S$ and the excess of $Ca_2 O, CO_2$ furnished, on evaporation, only one kind of crystals, which possessed the form of glycolate of lime. After purification by recrystallization from boiling water, they were dried over sulphuric acid and analysed, with the following results:—

I. 0·371 grm. lost at 125° C. 0·078 grm. of water, and gave 0·154 grm. of carbonate of lime.

II. 0·194 grm. of another preparation lost at 108° C. 0·041 grm. of water, and yielded by the usual treatment 0·08 grm. of carbonate of lime.

The compound therefore contained in 100 parts—

	I.	II.
Calcium	16·5	16·49
Water	21·02	21·13

The formula of glycolate of lime, $2(C_2H_3CaO_3)+3H_2O$, requires—

Calcium	16·3
Water	22·1

The percentage of water as found by experiment differs to the amount of nearly 1 per cent. from the theoretical number. I always found the amount of water contained in glycolate of lime which has been dried over sulphuric acid to be a little below the calculated quantity. If, however, from the quantity of carbonate of lime found in the above analysis the amount of calcium contained in 100 parts of the anhydrous substance be calculated, the number thus obtained agrees with the theoretical percentage of calcium in anhydrous glycolate of lime.

	I.	II.	Theory.
Calcium	21·03	20·9	21·05

The substance is therefore glycolate of lime. In order to prove the identity of the glycolic acid obtained by the oxidation of alcohol by means of nitric acid with the acid formed by reducing glyoxylic acid with zinc, the solubility in water of their respective lime-salts was determined.

A = lime-salt of the glycolic acid prepared from alcohol.

B = lime-salt prepared from the acid obtained by the action of zinc on glyoxylic acid.

A. 3·073 grm. of solution, saturated at 12° C., evaporated at 100° C., left 0·027 grm. of residue.

B. 1·767 grm. of solution, saturated at the same temperature, left 0·016 grm. of residue.

3·206 grm. of solution, prepared at 7° C., gave 0·028 grm. of residue.

100 parts of water therefore dissolve—

	12° C.	7° C.
A.	0·886 . . .	——
B.	0·913 . . .	0·881

The lime-salts A and B, and consequently the acids contained therein, may therefore be considered to be identical—a conclusion which is supported by the other properties of A and B.

Glycolic acid can be obtained from glyoxylic acid by at least two different ways.

1. By direct addition of hydrogen:—

$$\underbrace{C_2H_2O_3}_{\text{Glyoxylic acid.}} + H_2 = \underbrace{C_2H_4O_3}_{\text{Glycolic acid.}}$$

2. By the decomposition represented in the following equation:—

$$2(C_2H_2O_3)+H_2O=\underbrace{C_2H_2O_4}_{\text{Oxalic acid.}}+\underbrace{C_2H_4O_3}_{\text{Glycolic acid.}}\text{. . . . .} \quad (1.)$$

Under the influence of zinc the following reactions could take place:—

$$3(C_2H_2O_3)+Zn_4=\underbrace{2(C_2H_3ZnO_3)}_{\text{Glycolate of zinc.}}+\underbrace{C_2Zn_2O_3}_{\text{Glyoxylate of zinc*.}}, \text{ . . . . } \quad (2.)$$

or

$$C_2H_2O_3+Zn_2=\underbrace{C_2Zn_2O_3}_{\text{Glyoxylate of zinc.}}+H_2.$$

Glyoxylates which contain more than one atom of metal decompose easily at the temperatures at which the experiments were made, and form compounds of glycolic and oxalic acids.

$$\underbrace{2(C_2Zn_2O_3)}_{\text{Glyoxylate of zinc.}}+2H_2O=\underbrace{C_2H_3ZnO_3}_{\text{Glycolate of zinc.}}+\underbrace{C_2Zn_2O_4}_{\text{Oxalate of zinc.}}+\left.\begin{matrix}Zn\\H\end{matrix}\right\}O.$$

By combining this and the preceding equation, the action of zinc on glyoxylic acid may at once be represented by the following equation:—

$$2(C_2H_2O_3)+2H_2O+Zn_4=\underbrace{C_2H_3ZnO_3}_{\text{Glycolate of zinc.}}+\underbrace{C_2Zn_2O_4}_{\text{Oxalate of zinc.}}+H_4+\left.\begin{matrix}Zn\\H\end{matrix}\right\}O. \text{ . . } \quad (3.)$$

If the reaction takes place according to (2.), then by the side of five atoms of glycolate only one atom of oxalate of zinc can be produced; but if according to (3.), for every five atoms of glycolate five atoms of oxalate of zinc ought to be formed, and twenty atoms of hydrogen be liberated.

The relative quantities of oxalic acid and glycolic acid formed by the action of zinc on glyoxylic acid were therefore first determined. An unknown quantity of the acid was treated with zinc until the action was complete, and the filtrate from the undissolved metal was divided into two parts. In one part the oxalic acid was determined by the usual method, and the glycolic acid in the other by converting the glycolate of zinc into glycolate of lime, and by evaporating and drying the latter at 100° C.; this could be done because no other substance was present in the liquid. In this way 0·98 grm. of glycolate of lime[†], and 0·005 grm. of carbonate of lime from the oxalate were obtained.

A white crystalline powder has been mentioned which settles on the zinc during its action on glyoxylic acid; this powder would contain oxalate of zinc. From its solution in ammonia a precipitate of oxalate of lime was obtained, which gave after ignition 0·095 grm. of carbonate of lime. The whole of the Ca_2O, CO_2 obtained from the

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[*] Proceedings of the Royal Society, vol. ix. p. 711. LIEBIG, Ann. vol. cx. p. 326.

[†] This glycolate of lime contained 21·4 per cent. of calcium, whereas theory requires 21·05 per cent.

oxalate weighed therefore 0·1 grm., which corresponds to 0·09 grm. of oxalic acid, $C_2H_2O_4$; the above-mentioned 0·98 grm. of glycolate of lime contain 0·784 grm. of glycolic acid, and therefore for every atom of oxalic acid we obtain 10·3 of glycolic acid.

Another experiment conducted on the same plan gave 0·122 grm. of carbonate of lime and 0·876 grm. of anhydrous glycolate of lime, or

1 atom of oxalic acid : 7·5 atoms of glycolic acid.

A third experiment yielded nearly the same result as the second.

The action of zinc on glyoxylic acid takes place therefore according to equation (2.).

0·906 grm. of glyoxylic acid gave 0·876 grm. of glycolate of lime, corresponding to 0·7 grm. of glycolic acid, and 0·122 grm. of carbonate of lime, corresponding to 0·109 grm. of $C_2H_2O_4$. If we suppose none of the glyoxylate of zinc, formed according to (2.), to be decomposed, then no oxalic acid and only 0·62 grm. of glycolic acid ought to have been obtained; if, however, all the glyoxylate of zinc had been decomposed into glycolate and oxalate of zinc, 0·775 grm. of glycolic acid and 0·183 grm. of oxalic acid ought to have been found. The actual numbers are intermediate between these two extremes, and consequently half of the glyoxylate of zinc formed from 0·906 grm. of glyoxylic acid, according to equation (2.), underwent the decomposition represented in equation (1.). The action of zinc on glyoxylic acid may therefore be explained as follows:—zinc dissolves and forms glyoxylate of zinc, $C_2Zn_2O_3$; the hydrogen instead of being liberated combines with other glyoxylic acid and zinc and produces glycolate of zinc; a quantity of $C_2Zn_2O_3$, dependent on time and temperature, decomposes into glycolate and oxalate of zinc.

The following experiment proves that no hydrogen is liberated in this reaction. 0·906 grm. of glyoxylic acid and a few pieces of zinc were placed in a flask and the vessel nearly filled with water; a perforated cork holding a bent glass tube was then attached to the mouth of the flask, the other end of the tube being placed under a graduated receiver filled with mercury. The flask was then warmed to nearly 100° C., and kept at this temperature for ten hours, after which time no more gas was given off. The apparatus was found to be air-tight both before and after the experiment. The quantity of the gas which was collected, when measured at 11° C., was 10 cub. centims.; it possessed the properties of atmospheric air. If one atom of hydrogen had been set free for each atom of glyoxylic acid which was taken, the 0·906 grm. of acid ought to have liberated 0·0122 grm. of hydrogen; this quantity measures 135 cub. centims. at 0° C. and 0·76 in. pressure.

Some years ago I directed attention to the following considerations[*].

Among the products formed by the action of nitric acid on alcohol, which remain after the more volatile substances have been evaporated on the water-bath, are three acids, glycolic acid, glyoxylic acid, and oxalic acid. These bodies show the same differences in composition as benzylic alcohol, oil of bitter almonds, and benzoic acid.

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[*] Quarterly Journal of the Chemical Society, vol. xii. p. 234.

$$\underbrace{C_7H_8O}_{\text{Benzylic alcohol.}} \qquad \underbrace{C_2H_4O_3}_{\text{Glycolic acid.}}$$

$$\underbrace{C_7H_6O}_{\text{Oil of bitter almonds.}} \qquad \underbrace{C_2H_2O_3}_{\text{Glyoxylic acid.}}$$

$$\underbrace{C_7H_6O_2}_{\text{Benzoic acid.}} \qquad \underbrace{C_2H_2O_4}_{\text{Oxalic acid.}}$$

They comport themselves with different reagents like the members of the benzoyl series. Hydrate of potash decomposes glyoxylic acid into glycolic and oxalic acids, and oil of bitter almonds into benzylic alcohol and benzoic acid; dilute nitric acid oxidizes glyoxylic acid to oxalic acid, and oil of bitter almonds to benzoic acid.

a. Decomposition with potash:—

$$2(\underbrace{C_7H_6O}_{\text{Oil of bitter almonds.}}) + H_2O = \underbrace{C_7H_8O}_{\text{Benzylic alcohol.}} + \underbrace{C_7H_6O_2}_{\text{Benzoic acid.}}$$

$$2(\underbrace{C_2H_2O_3}_{\text{Glyoxylic acid.}}) + H_2O = \underbrace{C_2H_4O_3}_{\text{Glycolic acid.}} + \underbrace{C_2H_2O_4}_{\text{Oxalic acid.}}$$

b. Oxidation with nitric acid:—

$$\underset{\text{Oil of bitter almonds.}}{C_7H_6O} + O = \underset{\text{Benzoic acid.}}{C_7H_6O_2}$$

$$\underbrace{C_2H_2O_3}_{\text{Glyoxylic acid.}} + O = \underbrace{C_2H_2O_4}_{\text{Oxalic acid.}}$$

According to FRIEDEL, oil of bitter almonds unites with hydrogen and forms benzylic alcohol; and the experiments described in this paper show glyoxylic acid to enter into combination with this element, forming glycolic acid.

$$\underbrace{C_7H_6O}_{\text{Oil of bitter almonds.}} + H_2 = \underbrace{C_7H_8O}_{\text{Benzylic alcohol.}}$$

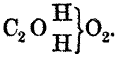
$$\underbrace{C_2H_2O_3}_{\text{Glyoxylic acid.}} + H_2 = \underbrace{C_2H_4O_3}_{\text{Glycolic acid.}}$$

Glycolic acid may therefore be termed the alcohol of oxalic acid, and glyoxylic acid the aldehyde of both; in fact glyoxylic acid possesses other properties which are generally only found in connexion with aldehydes. Amongst these are the great affinity of glyoxylates for sulphites (whereby well-defined and beautifully crystallizing compounds are formed), the exchange of the oxygen of the acid for the sulphur of sulphuretted hydrogen, and, finally, the production of compounds with ammonia.

Glyoxylic acid may likewise be compared with various other bodies; it has, for instance, in many respects a resemblance to sulphurous acid.

KEKULÉ'S interesting experiments with fumaric and maleic acids induced me to examine the action of bromine on a solution of glyoxylic acid. If the two bodies be left in contact at ordinary temperatures in a closed vessel, the colour of the bromine disappears in the course of a few days. Only hydrobromic and oxalic acids, however, resulted from this reaction.

If a rational formula be required for glyoxylic acid, the following expression may be adopted:—



$$C_2O\left.\begin{matrix}H\\H\end{matrix}\right\}O_2.$$

It indicates the diatomic nature of the acid as well as its intermediate position between glycolic and oxalic acids. Two atoms of hydrogen added to the radical C_2O produce the radical of glycolic acid, and one atom of oxygen the corresponding radical of oxalic acid.

XXI. *On Skew Surfaces, otherwise Scrolls.* *By* ARTHUR CAYLEY, *F.R.S.*

Received February 3,—Read March 5, 1863.

IT may be convenient to mention at the outset that, in the paper "On the Theory of Skew Surfaces"[*], I pointed out that upon any skew surface of the order n there is a singular (or nodal) curve meeting each generating line in $(n-2)$ points, and that the class of the circumscribed cone (or, what is the same thing, the class of the surface) is equal to the order n of the surface. In the paper "On a Class of Ruled Surfaces"[†], Dr. SALMON considered the surface generated by a line which meets three curves of the orders m, n, p respectively: such surface is there shown to be of the order $=2mnp$; and it is noticed that there are upon it a certain number of double right lines (nodal generators); to determine the number of these, it was necessary to consider the skew surface generated by a line meeting a given right line and a given curve of the order m twice; and the order of such surface is found to be $=\frac{1}{2}m(m-1)+h$, where h is the number of apparent double points of the curve. The theory is somewhat further developed in Dr. SALMON's memoir "On the Degree of a Surface reciprocal to a given one"[‡], where certain minor limits are given for the orders of the nodal curves on the skew surface generated by a line meeting a given right line and two curves of the orders m and n respectively, and on that generated by a line meeting a given right line and a curve of the order m twice. And in the same memoir the author considers the skew surface generated by a line the equations whereof are $(a, \ldots \between t, 1)^m=0$ $(a', \ldots \between t, 1)^n=0$, where $a, \ldots a', \ldots$ are any linear functions of the coordinates, and t is an arbitrary parameter. And the same theories are reproduced in the 'Treatise on the Analytic Geometry of Three Dimensions'[§]. I will also, though it is less closely connected with the subject of the present memoir, refer to a paper by M. CHASLES, "Description des Courbes à double courbure de tous les ordres sur les surfaces réglées du troisième et du quatrième ordre"[||].

The present memoir (in the composition of which I have been assisted by a correspondence with Dr. SALMON) contains a further development of the theory of the skew surfaces generated by a line which meets a given curve or curves: viz. I consider, 1st, the surface generated by a line which meets each of three given curves of the orders m, n, p respectively; 2nd, the surface generated by a line which meets a given curve of the order m twice, and a given curve of the order n once; 3rd, the surface which meets a given curve of the order m three times; or, as it is very convenient to express it, I consider the skew surfaces, or say the "Scrolls," $S(m, n, p)$, $S(m^2, n)$, $S(m^3)$. The chief results are embodied in the Table given after this introduction, at the commencement of the memoir. It is to be noticed that I attend throughout to the general theory, not considering otherwise than incidentally the effect of any singularity in the system of the given curves, or in the given curves separately: the memoir contains however some remarks as to what are the singularities material to a complete theory; and, in particular as regards the surface $S(m^3)$, I am thus led to mention an entirely new kind of singularity of a curve in space—viz. such a curve has in general a determinate number of "lines through four points" (lines which meet the curve in four points); it may happen that, of the lines through three points which can be drawn through any point whatever of the curve, a certain number will unite together and form a line through four (or more) points, the number of the lines through four points (or through a greater number of points) so becoming infinite.

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[*] Cambridge and Dublin Math. Journ. vol. vi. pp. 171–173 (1852).

[†] Ibid. vol. viii. pp. 45, 46 (1853).

[‡] Trans. Royal Irish Acad. vol. xxiii. pp. 461–488 (read 1855).

[§] Dublin, 1862.

[||] Comptes Rendus, t. liii. (1861, 2e Sem.), pp. 884–889.

Notation and Table of Results, Articles 1 to 10.

1. In the present memoir a letter such as m denotes the order of a curve in space. It is for the most part assumed that the curve has no actual double points or stationary points, and the corresponding letter M denotes the class of the curve taken negatively and divided by 2; that is, if h be the number of apparent double points, then $M=-\frac{1}{2}[m]^2+h$: here and elsewhere $[m]^2$, &c. denote factorials, viz. $[m]^2=m(m-1)$, $[m]^3=m(m-1)(m-2)$, &c. It is to be noticed that for the system of two curves m, m', if h, h' represent the number of apparent double points of the two curves respectively, then for the system the number of apparent double points is $=mm'+h+h'$, and the corresponding value of M is therefore $-\frac{1}{2}[m+m']^2+mm'+h+h'$, which is $=-\frac{1}{2}[m]^2+h-\frac{1}{2}[m']^2+h'$, which is $=M+M'$.

2. The use of the combinations (m, n, p, q), (m^2, n, p), &c. hardly requires explanation; it may however be noticed that $G(m, n, p, q)$ denoting the lines which meet the curves m, n, p, q (that is, curves of these orders) each of them once, $G(m^2, n, p)$ will denote the lines which meet the curve m twice and the curves n and p each of them once; and so in all similar cases.

3. The letters G, S, ND, NG, NR, NT (read Generators, Scroll, Nodal Director, Nodal Generator, Nodal Residue, and Nodal Total) are in the nature of functional symbols, used (according to the context) to denote geometrical forms, or else the orders of these forms. Thus $G(m, n, p, q)$ denotes either the lines meeting the curves m, n, p, q each of them once, or else it denotes the order of such system of lines, that is, the number of lines. And so $S(m, n, p)$ denotes the Skew Surface or Scroll generated by a line which meets the curves m, n, p each once, or else it denotes the order of such surface.

4. $G(m, n, p, q)$: the signification is explained above.

5. $S(m, n, p)$: the signification has just been explained; but as the surfaces $S(m, n, p)$,

S(m^2, n), S(m^3) are in fact the subject of the present memoir, I give the explanation in full for each of them, viz. S(m, n, p) is the surface generated by a line which meets the curves m, n, p each once; S(m^2, n) is the surface generated by a line which meets the curve m twice and the curve n once; S(m^3) the surface generated by the line which meets the curve m thrice. As already mentioned, these surfaces and their orders are represented by the same symbols respectively.

6. ND(m, n, p). The directrix curves m, n, p of the scroll S(m, n, p) are nodal (multiple) curves on the surface, viz. m is an np-tuple curve, and so for n and p. Reckoning each curve according to its multiplicity, viz. the curve m being reckoned $\tfrac{1}{2}[np]^2$ times, or as of the order $m.\tfrac{1}{2}[np]^2$, and so for the curves n and p, the aggregate, or sum of the orders, gives the Nodal Director ND(m, n, p).

7. NG(m, n, p). The scroll S(m, n, p) has the nodal generating lines G(m^2, n, p), G(m, n^2, p), G(m, n, p^2). Each of these is a mere double line, to be reckoned once only, and we have thus the Nodal Generator

$$\text{NG}(m, n, p)=\text{G}(m^2, n, p)+\text{G}(m, n^2, p)+\text{G}(m, n, p^2).$$

But to take another example, the scroll S(m^2, n) has the nodal generating lines G(m^3, n), each of which is a triple line to be reckoned $\tfrac{1}{2}[3]^2$, that is, three times, and also the nodal generating lines G(m^2, n^2), each of them a mere double line to be reckoned once only; whence here NG$(m^2, n)=3$G$(m^3, n)+$G(m^2, n^2). And so for the scroll S(m^3), this has the nodal generating lines G(m^4), each of them a quadruple line to be reckoned $\tfrac{1}{2}[4]^2$, that is, six times; or we have NG$(m^3)=6$G(m^4).

8. NR(m, n, p). The scroll S(m, n, p) has besides the directrix curves m, n, p or Nodal Director, and the nodal generating lines or Nodal Generator, a remaining nodal curve or Nodal Residue, the locus of the intersections of two non-coincident generating lines meeting in a point not situate on any one of the directrix curves. This Nodal Residue, as well for the scroll S(m, n, p) as for the scrolls S(m^2, n) and S(m^3) respectively, is a mere double curve to be reckoned once only; and such curve or its order is denoted by NR, viz. for the scroll S(m, n, p), the Nodal Residue is NR(m, n, p).

9. NT(m, n, p). The Nodal Director, Nodal Generator, and Nodal Residue of the scroll S(m, n, p) form together the Nodal Total NT(m, n, p), that is, we have

$$\text{NT}(m, n, p)=\text{ND}(m, n, p)+\text{NG}(m, n, p)+\text{NR}(m, n, p);$$

and similarly for the scrolls S(m^2, n) and S(m^3).

10. I remark that the formulæ are best exhibited in an order different from that in which they are in the sequel obtained, viz. I collect them in the following

Table.

$$\begin{aligned}
&\text{G}(m, n, p, q)=2mnpq,\\
&\text{G}(m^2, n, p) &&=np([m]^2+\text{M}),\\
&\text{G}(m^2, n^2) &&=\tfrac{1}{2}[m]^2[n]^2+\text{M}.\tfrac{1}{2}[n]^2+\text{N}.\tfrac{1}{2}[m]^2+\text{MN},\\
&\text{G}(m^3, n) &&=n\left(\tfrac{1}{3}[m]^3+\text{M}(m-2)\right),\\
&\text{G}(m^4) &&=\tfrac{1}{12}[m]^4+m+\text{M}(\tfrac{1}{2}[m]^2-2m+\tfrac{11}{2})+\text{M}^2.\tfrac{1}{2},
\end{aligned}$$

$$\begin{aligned}
\mathrm{S}(m, n, p) &= 2mnp,\\
\mathrm{ND}(m, n, p) &= \tfrac{1}{2}mnp(mn+mp+np-3),\\
\mathrm{NG}(m, n, p) &= mnp(m+n+p-3)+\mathrm{M}np+\mathrm{N}mp+\mathrm{P}mn,\\
\mathrm{NR}(m, n, p) &= \tfrac{1}{2}mnp\big(4mnp-(mn+mp+np)-2(m+n+p)+5\big),\\
{}^{*}\mathrm{NT}(m, n, p) &= \tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\mathrm{M}np+\mathrm{N}mp+\mathrm{P}mn\\
&= 2mnp(mnp-1)+\mathrm{M}np+\mathrm{N}mp+\mathrm{P}mn;
\end{aligned}$$

included in which we have

$$\begin{aligned}
\mathrm{S}(1, 1, m) &= 2m,\\
\mathrm{ND}(1, 1, m) &= [m]^2,\\
\mathrm{NG}(1, 1, m) &= [m]^2+\mathrm{M},\\
\mathrm{NR}(1, 1, m) &= 0,\\
\mathrm{NT}(1, 1, m) &= \tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\mathrm{M}\\
&= 2[m]^2+\mathrm{M},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{S}(1, m, n) &= 2mn,\\
\mathrm{ND}(1, m, n) &= \tfrac{1}{2}mn(mn+m+n-3),\\
\mathrm{NG}(1, m, n) &= mn(m+n-2)+\mathrm{M}n+\mathrm{N}m,\\
\mathrm{NR}(1, m, n) &= \tfrac{3}{2}[m]^2[n]^2,\\
\mathrm{NT}(1, m, n) &= \tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\mathrm{M}n+\mathrm{N}m\\
&= 2[mn]^2+\mathrm{M}n+\mathrm{N}m.
\end{aligned}$$

Moreover

$$\begin{aligned}
\mathrm{S}(m^2, n) &= n\,\big([m]^2+\mathrm{M}\big),\\
\mathrm{ND}(m^2, n) &= n\,\big(\tfrac{1}{8}[m]^4+[m]^3+\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{1}{2})+\mathrm{M}^2.\tfrac{1}{2}\big)\\
&\quad +[n]^2(\tfrac{1}{2}[m]^3+\tfrac{1}{2}[m]^2),\\
\mathrm{NG}(m^2, n) &= n\,\big([m]^3+\mathrm{M}.3(m-2)\big)\\
&\quad +[n]^2(\tfrac{1}{2}[m]^2+\tfrac{1}{2}\mathrm{M})\\
&\quad +\mathrm{N}\,(\tfrac{1}{2}[m]^2+\mathrm{M}),\\
\mathrm{NR}(m^2, n) &= n\,\big(\tfrac{3}{8}[m]^4+\mathrm{M}(\tfrac{1}{2}[m]^2-2m+3)\big)\\
&\quad +[n]^2\big(\tfrac{1}{2}[m]^4+\tfrac{3}{2}[m]^3+[m]^2+\mathrm{M}([m]^2-\tfrac{1}{2})+\mathrm{M}^2.\tfrac{1}{2}\big),\\
\mathrm{NT}(m^2, n) &= \tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+n\mathrm{M}(m-\tfrac{5}{2})+\mathrm{N}(\tfrac{1}{2}[m]^2+\mathrm{M})\\
&= n\,\big(\tfrac{1}{2}[m]^4+2[m]^3+\mathrm{M}([m]^2+m-\tfrac{7}{2})+\mathrm{M}^2.\tfrac{1}{2}\big)\\
&\quad +[n]^2(\tfrac{1}{2}[m]^4+2[m]^3+[m]^2+\mathrm{M}.[m]^2+\mathrm{M}^2.\tfrac{1}{2})\\
&\quad +\mathrm{N}\,(\tfrac{1}{2}[m]^2+\mathrm{M});
\end{aligned}$$

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[*] In the first of the two expressions for NT(m, n, p), S stands for S(m, n, p); and so in the first of the two expressions for NT(m^2, n), &c., S stands for S(m^2, n), &c.

included in which we have

$$\begin{aligned}
\mathrm{S}(1, m^2)&=[m]^2+\mathrm{M},\\
\mathrm{ND}(1, m^2)&=\tfrac{1}{8}[m]^4+[m]^3+\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{1}{2})+\mathrm{M}^2.\tfrac{1}{2},\\
\mathrm{NG}(1, m^2)&=[m]^3+\mathrm{M}.3(m-2),\\
\mathrm{NR}(1, m^2)&=\tfrac{3}{8}[m]^4+\mathrm{M}(\tfrac{1}{2}[m]^2-2m+3),\\
\mathrm{NT}(1, m^2)&=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\mathrm{M}(m-\tfrac{5}{2})\\
&=\tfrac{1}{2}[m]^4+2[m]^3+\mathrm{M}([m]^2+m-\tfrac{7}{2})+\mathrm{M}^2.\tfrac{1}{2}\,;
\end{aligned}$$

and finally

$$\begin{aligned}
\mathrm{S}(m^3)&=\tfrac{1}{3}[m]^3+(m-2)\mathrm{M},\\
\mathrm{ND}(m^3)&=\tfrac{1}{8}[m]^5+\tfrac{1}{2}[m]^4+\tfrac{1}{2}[m]^3+\mathrm{M}(\tfrac{1}{2}[m]^3+\tfrac{1}{2}[m])+\mathrm{M}^2.\tfrac{1}{2}m,\\
\mathrm{NG}(m^3)&=\tfrac{1}{2}[m]^4+6m+\mathrm{M}(3[m]^2-12m+33)+\mathrm{M}^2.3,\\
\mathrm{NR}(m^3)&=\tfrac{1}{18}[m]^6+\tfrac{3}{8}[m]^5-\tfrac{1}{2}[m]^3+3m\\
&\quad+\mathrm{M}(\tfrac{1}{3}[m]^4-\tfrac{1}{6}[m]^3-\tfrac{5}{2}[m]^2+8m-20)+\mathrm{M}^2(\tfrac{1}{2}[m]^2-2m),\\
\mathrm{NT}(m^3)&=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+3m+\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{5}{2}m+11)+\mathrm{M}^2\\
&=\tfrac{1}{18}[m]^6+\tfrac{1}{2}[m]^5+[m]^4+3m\\
&\quad+\mathrm{M}(\tfrac{1}{3}[m]^4+\tfrac{1}{3}[m]^3+\tfrac{1}{2}[m]^2-\tfrac{7}{2}m+13)\\
&\quad+\mathrm{M}^2(\tfrac{1}{2}[m]^2-\tfrac{3}{2}m+3).
\end{aligned}$$

The formulæ are investigated in the following order, ND, G, NG, S, NR, and NT.

The ND *formulæ*, Articles 11 to 13.

11. ND(m, n, p).—Taking any point on the curve m, this is the vertex of two cones passing through the curves n, p respectively; the cones are of the orders n, p respectively, and they intersect therefore in np lines, which are the generating lines through the point on the curve m; hence this curve is an np-tuple line on the scroll S(m, n, p), and we have thus the term $m.\tfrac{1}{2}[np]^2$ of ND. Whence

$$\begin{aligned}
\mathrm{ND}(m, n, p)&=m.\tfrac{1}{2}[np]^2+n.\tfrac{1}{2}[mp]^2+p.\tfrac{1}{2}[mn]^2\\
&=\tfrac{1}{2}mnp(mn+mp+np-3).
\end{aligned}$$

12. ND(m^2, n).—Taking first a point on the curve m, this is the vertex of a cone of the order $m-1$ through the curve m, and of a cone of the order n through the curve n; the two cones intersect in $(m-1)n$ lines, which are the generating lines through the point on the curve m; that is, the curve m is a $(m-1)n$-tuple line on the scroll S(m^2, n); and we have thus the term $m.\tfrac{1}{2}[(m-1)n]^2$ of ND. Taking next a point on the curve n, this is the vertex of a cone of the order m through the curve m; such cone has ($h=$)$\tfrac{1}{2}[m]^2+\mathrm{M}$ double lines, which are the generating lines through the point on the curve n; hence this curve is a ($\tfrac{1}{2}[m]^2+\mathrm{M}$)tuple line on the surface, and we have thus the term $n.\tfrac{1}{2}\big[\tfrac{1}{2}[m]^2+\mathrm{M}\big]^2$ in ND. And therefore

$$\begin{aligned}
\mathrm{ND}(m^2, n)&=m.\tfrac{1}{2}[(m-1)n]^2+n.\tfrac{1}{2}\big[\tfrac{1}{2}[m]^2+\mathrm{M}\big]^2\\
&=n\ \big(\tfrac{1}{8}[m]^4+[m]^3+\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{1}{2})+\mathrm{M}^2.\tfrac{1}{2}\big)\\
&\quad+[n]^2(\tfrac{1}{2}[m]^3+\tfrac{1}{2}[m]^2).
\end{aligned}$$

13. ND(m^3).—Taking a point on the curve m, this is the vertex of a cone of the order $m-1$ through the curve m; such cone has $(h-m+2=)\frac{1}{2}[m]^2-m+2+\mathrm{M}$ double lines, or the curve m is a $(\frac{1}{2}[m]^2-m+2+\mathrm{M})$tuple line on the scroll $\mathrm{S}(m)^3$). Hence we have

$$\begin{aligned}\mathrm{ND}(m)^3&=m.\tfrac{1}{2}\left[\tfrac{1}{2}[m]^2-m+2+\mathrm{M}\right]^2\\&=\tfrac{1}{8}[m]^5+\tfrac{1}{2}[m]^4+\tfrac{1}{2}[m]^3+\mathrm{M}(\tfrac{1}{2}[m]^3+\tfrac{1}{2}[m])+\mathrm{M}^2.\tfrac{1}{2}m.\end{aligned}$$

Preparatory remarks in regard to the G *formulæ, the hypertriadic singularities of a curve in space*, Articles 14 to 22.

14. It is to be remarked that the generating line of any one of the scrolls $\mathrm{S}(m, n, p)$, $\mathrm{S}(m^2, n)$, $\mathrm{S}(m^3)$ satisfies three conditions; and that it cannot in anywise happen that one of these conditions is implied in the other two. Thus, for instance, as regards the scroll $\mathrm{S}(m, n, p)$, if the curves m, n are given, and we take the entire series of lines meeting each of these curves, these lines form a double series of lines, all of them passing of course through the curves m, n, but not all of them passing through any other curve whatever; that is, there is no curve p such that every line passing through the curves m and n passes also through the curve p. And the like as regards the scrolls $\mathrm{S}(m^2, n)$ and $\mathrm{S}(m^3)$.

15. But (in contrast to this) if the three conditions are satisfied, it may very well happen that a fourth condition is satisfied *ipso facto*. To see how this is, imagine a curve q on the scroll $\mathrm{S}(m, n, p)$, or, to meet an objection which might be raised, say a curve q the complete intersection of the scroll $\mathrm{S}(m, n, p)$ by a plane or any other surface. Every line whatever which meets the curves m, n, p is a generating line of the scroll $\mathrm{S}(m, n, p)$, and as such will meet the curve q; that is, in the case in question, $\mathrm{G}(m, n, p, q)$, the lines which meet the curves m, n, p, q, are the entire series of generating lines of the scroll $\mathrm{S}(m, n, p)$, and they are thus infinite in number; so that in such case the question does not arise of finding the number of the lines $\mathrm{G}(m, n, p, q)$. The like remarks apply to the lines $\mathrm{G}(m^2, n, p)$, $\mathrm{G}(m^2, n^2)$, $\mathrm{G}(m^3, n)$, and $\mathrm{G}(m^4)$; but I will develope them somewhat more particularly as regards the lines $\mathrm{G}(m^4)$.

16. Given a curve m, then (as in fact mentioned in the investigation for ND(m^3)) through *any point whatever* of the curve there can be drawn

$$(h-m+2=)[m]^2+m-2+\mathrm{M}$$

lines meeting the curve in two other points, or say $[m]^2+m-2+\mathrm{M}$ lines through three points. But in general no one of these lines meets the curve in a fourth point; that is, we cannot through every point of the curve m draw a line through four points; there are, however, on the curve m a *certain* number $(=4\mathrm{G}(m^4))$ of points through which can be drawn a line through four points, or line $\mathrm{G}(m^4)$.

17. But the curve m may be such that through every point of the curve there passes a line through four points. In fact, assume any skew surface or scroll whatever, and upon this surface a curve meeting each generating line in four points (*e. g.* the intersection of the scroll by a quartic surface). Taking the curve in question for the curve m, then it is clear that through every point of this curve there passes a line (the generating line of the assumed scroll) which is a line through four points, or line $\mathrm{G}(m^4)$.

18. It is to be noticed, moreover, that if we take on the curve m any point whatever, then of the $[m]^2+m-2+\mathrm{M}$ lines through three points which can be drawn through this point, three will unite together in the generating line of the assumed scroll (for if 0 be the point on the curve m, and 1, 2, 3 the other points in which the generating line of the assumed scroll meets the curve m, then such generating line unites the three lines 012, 013, 023, each of them a line through three points); and there will be besides $\frac{1}{2}[m]^2+m-5+\mathrm{M}$ mere lines through three points. The line through four points generates the assumed scroll taken ($\frac{1}{2}[3]^2=$) 3 times, or considered as three coincident scrolls; the remaining lines generate a scroll $\mathrm{S}'(m^3)$, which is such that the curve m is on this scroll a ($\frac{1}{2}[m]^2+m-5+\mathrm{M}$)tuple line; the assumed scroll three times and the scroll $\mathrm{S}'(m^3)$ make up the entire scroll $\mathrm{S}(m^3)$ derived from the curve m, or say $\mathrm{S}(m^3)=$ 3 (assumed scroll) $+\mathrm{S}'(m^3)$.

19. The case just considered is that of a curve m such that through every point of it there passes a line through four points counting as ($\frac{1}{2}[3]^2=$) 3 lines through three points, and that all the other lines through three points are mere lines through three points. But it is clear that we may in like manner have a line through p points counting as $\frac{1}{2}[p-1]^2$ lines through three points; and more generally if $p, q, \ldots$ are numbers all different and not <3, and if

$$\tfrac{1}{2}[m]^2-m+2+\mathrm{M}=\alpha\,.\,\tfrac{1}{2}[p-1]^2+\beta\,.\,\tfrac{1}{2}[q-1]^2+\ldots,$$

then we may have a curve m such that through every point of it there pass α lines each through p points and counting as $\frac{1}{2}[p-1]^2$ lines, β lines each through q points and counting as $\frac{1}{2}[q-1]^2$ lines, &c. ...: the case $p=3$ gives of course α lines each through three points and counting as a single line. It is to be added that, in the case just referred to, the α lines will generate a scroll $\mathrm{S}'(m^3)$ taken $\frac{1}{6}[p]^3$ times, the β lines will generate a scroll $\mathrm{S}''(m^3)$ taken $\frac{1}{6}[q]^3$ times, &c., which scrolls together make up the scroll $\mathrm{S}(m^3)$, or say

$$\mathrm{S}(m^3)=\tfrac{1}{6}[p]^3\,.\,\mathrm{S}'(m^3)+\tfrac{1}{6}[q]^3\,.\,\mathrm{S}''(m^3)+\ \&\text{c.};$$

it may however happen that, *e. g.* of the α lines, any set or sets or even each line will generate a distinct scroll or scrolls—that is, that the scroll $\mathrm{S}'(m^3)$ will itself break up into scrolls of inferior orders.

20. A good illustration is afforded by taking for the curve m a curve on the hyperboloid or quadric scroll*; such curves divide themselves into species; viz. we have say the (p, q) curve on the hyperboloid, a curve of the order $p+q$ meeting each generating line of the one kind in p points, and each generating line of the other kind in q points; here

$$m=p+q,\ (h=\tfrac{1}{2}[p]^2+\tfrac{1}{2}[q]^2,\ \text{and}\ \therefore)\mathrm{M}=-pq.$$

Assuming for the moment that p, q are each of them not less than 3, it is clear that the lines through three points which can be drawn through any point of the curve are the generating line of the one kind counting as $\frac{1}{2}[p-1]^2$ lines through three points, and the generating line of the other kind counting as $\frac{1}{2}[q-1]^2$ lines through three points, so that

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* It is hardly necessary to remark that (*reality* being disregarded) any quadric surface whatever is a hyperboloid or quadric scroll.

$$\tfrac{1}{2}[m]^2+m-2+\mathrm{M}=\tfrac{1}{2}[p-1]^2+\tfrac{1}{2}[q-1]^2.$$

The complete scroll $\mathrm{S}(m^3)$ is made up of the hyperboloid considered as generated by the generating lines of the one kind taken $\frac{1}{6}[p]^3$ times, and the hyperboloid considered as generated by the generating lines of the other kind taken $\frac{1}{6}[q]^3$ times (so that there is in this case the speciality that the surfaces $\mathrm{S}'(m^3)$, $\mathrm{S}''(m^3)$ are in fact the same surface). And hence we have

$$\mathrm{S}(m^3)=\left(2(\tfrac{1}{6}[p]^3+\tfrac{1}{6}[q]^3)=\right)\tfrac{1}{3}[p]^3+\tfrac{1}{3}[q]^3.$$

21. I notice also the case of a system of m lines. Taking here a point on one of the lines, the $(h-m+2=)[m]^2-m+2$ lines through three points which can be drawn through this point are the $\frac{1}{2}[m-1]^2$ lines which can be drawn meeting a pair of the other $(m-1)$ lines, and besides this the line itself counting as one line through three points $(\frac{1}{2}[m-1]^2+1=[m]^2-m+2)$; the line itself, thus counting as a single line through three points, is not to be reckoned as a line through four or more points drawn through the point in question, that is, the system is not to be regarded as a curve through every point of which there passes a line through four points: each of the lines is nevertheless to be counted as a single line through four points, and (since there are besides two lines which may be drawn meeting each four of the m lines) the total number of lines through four points is $=\frac{1}{12}[m]^4+m$.

22. In the following investigations for $\mathrm{G}(m, n, p, q)$, &c., the foregoing special cases are excluded from consideration; it may however be right to notice how it is that the formulæ obtained are inapplicable to these special cases; for instance, as will immediately be seen, the number of the lines $\mathrm{G}(m, n, p, q)$ is obtained as the number of intersections of the surface $\mathrm{S}(m, n, p)$ by the curve q, $=2mnp\times q=2mnpq$; but if the curve q lie on the surface $\mathrm{S}(m, n, p)$, then $\mathrm{G}(m, n, p, q)$ is no longer $=2mnpq$.

The G *formulæ*, Articles 23 to 34.

23. $\mathrm{G}(m, n, p, q)$.—Considering the scroll $\mathrm{S}(m, n, p)$ generated by a line which meets each of the curves m, n, p, this meets the curve q in $q\,\mathrm{S}(m, n, p)$ points through each of which there passes a line $\mathrm{G}(m, n, p, q)$; that is, we have

$$\mathrm{G}(m, n, p, q)=q\ \mathrm{S}(m, n, p).$$

But from this equation we have

$$\mathrm{S}(m, n, p)=\mathrm{G}(1, m, n, p)=p\ \mathrm{S}(1, m, n);$$

thence also

$$\mathrm{S}(1, m, n)=\mathrm{G}(1, 1, m, n)=n\ \mathrm{S}(1, 1, n),$$

and

$$\mathrm{S}(1, 1, m)=\mathrm{G}(1, 1, 1, m)=m\ \mathrm{S}(1, 1, 1);\ \mathrm{S}(1, 1, 1)=\mathrm{G}(1, 1, 1, 1)=2,$$

since 2 is the number of lines which can be drawn meeting each of four given right lines. Hence ultimately

$$G(m, n, p, q)=mnpqG(1, 1, 1, 1)=2mnpq.$$

24. $G(m^2, n, p)$.—In a precisely similar manner we find

$$G(m^2, n, p)=npG(1, 1, m^2)=npS(1, m^2),$$

and it is the same question to find $G(1, 1, m^2)$ and to find $S(1, m^2)$. I investigate $G(1, 1, m^2)$ by considering the particular case where the curve m is a plane curve having n double points. The plane of the curve meets the two lines 1, 1 in two points, and the line through these two points meets each of the lines 1, 1, and meets the curve in m points; combining the last-mentioned m points two and two together, the line in question is to be considered as $\frac{1}{2}[m]^2$ coincident lines, each of them meeting the lines 1, 1, and also meeting the curve m twice. But we may also through any double point of the curve draw a line meeting each of the lines 1, 1; such line, inasmuch as it passes through a double point, meets the curve twice; and we have h such lines. This gives for the case in question $G(1, 1, m^2)=h+\frac{1}{2}[m]^2$; or, introducing in the place of h the quantity $M(=h-\frac{1}{2}[m]^2)$, so that $h=\frac{1}{2}[m]^2+M$, we have

$$G(1, 1, m^2)=[m]^2+M.$$

And, to the double points of the plane curve, there correspond in the general case the apparent double points of the curve m. Admitting the correctness of the result just obtained, we then have

$$G(m^2, n, p)=np([m]^2+M).$$

25. $G(m^2, n^2)$.—I investigate the value by a process similar to that employed for $G(1, 1, m^2)$. Suppose that the curves m and n are plane curves having respectively h and k double points; then the line of intersection of the two planes meets the curve m in m points, and the curve n in n points; or, combining in every manner the m points two and two together, and the n points two and two together, the line in question is to be considered as $\frac{1}{2}[m]^2.\frac{1}{2}[n]^2$ coincident lines, each meeting the curves m, n, each curve twice. There are besides the hk lines joining each double point of the curve m with each double point of the curve n. This gives in all $\frac{1}{4}[m]^2[n]^2+hk$ lines; or, writing $h=\frac{1}{2}[m]^2+M$, $k=\frac{1}{2}[n]^2+N$, the number is

$$=\tfrac{1}{2}[m]^2[n]^2+M.\tfrac{1}{2}[n]^2+N.\tfrac{1}{2}[m]^2+MN;$$

which is the value of $G(m^2, n^2)$ given by the investigation.

26. $G(m^3, n)$.—We have

$$G(m^3, n)=nG(1, m^3)=nS(m^3),$$

and it is in fact the same question to find $G(1, m^3)$ and to find $S(m^3)$. I assume for the present that the value is $=\frac{1}{3}[m]^3+M(m-2)$; and we then have

$$G(m^3, n)=n\left(\tfrac{1}{3}[m]^3+M(m-2)\right).$$

27. Before going further, I observe that there are certain functional conditions which must be satisfied by the G formulæ. Thus if the curve m be replaced by the system of the two curves m, m', instead of M we have $M+M'$. Let $G(m)$ denote any one of the functions $G(m, n, p, q)$, $G(m, n^2, p)$, $G(m, n^3)$, we must have

$$G(m+m') = G(m) + G(m').$$

Similarly, if $G(m^2)$ denote either of the functions $G(m^2, n, p)$, $G(m^2, n^2)$, we must have

$$G(m+m')^2 = G(m^2) + G(m, m') + G(m'^2);$$

and so if $G(m^3)$ stand for $G(m^3, n)$, then

$$G(m+m')^3 = G(m^3) + G(m^2, m') + G(m, m'^2) + G(m'^3);$$

and finally

$$G(m+m')^4 = G(m^4) + G(m^3, m') + G(m^2, m'^2) + G(m, m'^3) + G(m'^4).$$

28. The first three equations may be at once verified by means of the above given values of the G functions. But conversely, at least on the assumption that $G(m)$, $G(m^2)$, &c., in so far as they respectively depend on the curve m, are functions of m and M only, we may, by the solution of the functional equations, obtain the values of the G functions. It is to be observed that the first equation is of the form

$$\phi(m+m') = \phi(m) + \phi(m'),$$

the general solution whereof is

$$\phi m = \alpha m + \beta M;$$

the second equation, supposing that $G(m, m')$ is known—the third equation, supposing that $G(m^2, m')$ and $G(m, m'^2)$ are known—and the fourth equation, supposing that $G(m^3, m')$, $G(m^2, m'^2)$, $G(m, m'^3)$ are known, are respectively of the form

$$\phi(m+m') = \phi m + \phi m' + \text{funct.}\ (m, m');$$

and hence if a particular solution be given, the general solution is

$$\phi(m) = \text{Particular Solution} + \alpha m + \beta M.$$

The values of the constants must in each case be determined by special considerations.

29. The value of $G(m, n, p, q)$ was obtained strictly; that of $G(m^2, n, p)$ was reduced to depend on $G(1, 1, m^2)$, and that of $G(m^3, n)$ on $G(1, m^3)$. I apply therefore the functional equations to the confirmation of the values of $G(1, 1, m^2)$, $G(m^2, n^2)$, and $G(1, m^3)$, and to the determination of the value of $G(m^4)$.

30. First, if $G(m^2)$ denote $G(1, 1, m^2)$, then $G(m, m')$ denotes $G(1, 1, m, m')$, which is $=2mm'$; hence

$$G(m+m')^2 - G(m^2) - G(m'^2) = 2mm',$$

which is satisfied by $G(m^2) = [m]^2$. This gives

$$G(1, 1, m^2) = [m]^2 + \alpha m + \beta M.$$

But if the curve m be a system of m lines ($m=m$, $M=0$), then $G(1, 1, m^2) = [m]^2$; and again, if the curve m be a conic ($m=2$, $M=-1$), then $G(1, 1, m^2)=1$. This gives $\alpha=0$, $\beta=1$, and therefore

$$G(1, 1, m^2)=[m]^2+M.$$

31. Next, if $G(m^2)$ denote $G(m^2, n^2)$, then $G(m, m')$ denotes $G(m, m', n^2)$, which is $=mm'([n]^2+N)$. The functional equation is

$$G(m+m')^2-G(m^2)-G(m'^2)=mm'([n]^2+N),$$

which is satisfied by

$$G(m^2)=\tfrac{1}{2}[m]^2([n]^2+N).$$

Hence we have

$$G(m^2, n^2)=\tfrac{1}{2}[m]^2([n]^2+N)+\alpha m+\beta M,$$

where α, β are functions of n, N; and observing that $G(m^2, n^2)$ must be symmetrical in regard to the curves m and n, it is easy to see that we may write

$$G(m^2, n^2)=\tfrac{1}{2}[m]^2[n]^2+M\,.\,\tfrac{1}{2}[n]^2+N\,.\,\tfrac{1}{2}[m]^2+\alpha mn+\beta(mN+nM)+\gamma MN,$$

where α, β, γ are absolute constants. To determine them, if the curve m be a pair of lines ($m=2$, $M=0$), then

$$G(m^2, n^2)=G(1, 1, n^2)=[n]^2+N;$$

and if each of the curves m, n be a conic ($m=2$, $M=-1$, $n=2$, $N=-1$), then

$$G(m^2, n^2)=1.$$

These cases give $\alpha=\beta=0$, $\gamma=1$, and therefore

$$G(m^2, n^2)=\tfrac{1}{2}[m]^2[n]^2+M\,.\,\tfrac{1}{2}[n]^2+N\,.\,\tfrac{1}{2}[m]^2+MN.$$

32. Again, $G(m^3)$ standing for $G(1, m^3)$, then $G(m^2, m')$ and $G(m, m'^2)$ will stand for $G(1, m^2, m')$ and $G(1, m, m'^2)$, the values whereof are $m'([m]^2+M)$ and $m([m']^2+M')$ respectively. We thus have

$$G(m+m')^3-G(m^3)-G(m'^3)=m'([m]^2+M)+m([m']^2+M'),$$

a solution of which is $G(m^3)=\tfrac{1}{3}[m]^3+mM$. Hence we have

$$G(1, m^3)=\tfrac{1}{3}[m]^3+mM+\alpha m+\beta M.$$

Suppose first that the curve m is a system of lines ($m=m$, $M=0$), then $G(1, m^3)=\tfrac{1}{3}[m]^3$; and next that the curve m is a cubic in space or skew cubic ($m=3$, $M=-2$), then $G(1, m^3)=0$, since a line can meet the curve in two points only. We thus find $\alpha=0$, $\beta=-2$, and thence

$$G(1, m^3)=\tfrac{1}{3}[m]^3+M(m-2).$$

33. Hence, substituting for $G(m^3, m')$, $G(m^2, m'^2)$, $G(m, m'^3)$ their values

$$m'\left(\tfrac{1}{3}[m]^3+M(m-2)\right),\ \tfrac{1}{2}[m]^2[m']^2+M.\tfrac{1}{2}[m']^2+M'.\tfrac{1}{2}[m]^2+MM',\ \text{and}\ m\left(\tfrac{1}{3}[m']^3+M'(m'-2)\right)$$

respectively, we find

$$\begin{aligned}G(m+m')^4-G(m^4)-G(m'^4)= \quad & m'\left(\tfrac{1}{3}[m]^3+M(m-2)\right)\\ & +\tfrac{1}{2}[m]^2[m']^2+M.\tfrac{1}{2}[m']^2+M'.\tfrac{1}{2}[m]^2+MM'\\ & + \quad m\left(\tfrac{1}{3}[m']^3+M'(m'-2)\right),\end{aligned}$$

and thence, obtaining first a particular solution, the general solution is

$$G(m^4)=\tfrac{1}{12}[m]^4+M(\tfrac{1}{2}[m]^2-2m)+M^2.\tfrac{1}{2}+\alpha m+\beta M.$$

34. To determine the constants, suppose first that the curve m is a system of lines ($m=m$, $M=0$), we must have $G(m^4)=\tfrac{1}{12}[m]^4+m$, and thence $\alpha=0$. Next, if the curve m be a conic ($m=2$, $M=-1$), we must have $G(m^4)=0$; and this gives $\beta=\tfrac{11}{2}$, and consequently

$$G(m^4)=\tfrac{1}{12}[m]^4+m+M(\tfrac{1}{2}[m]^2-2m+\tfrac{11}{2})+M^2.\tfrac{1}{2}.$$

The NG *formulæ*, Article 35.

35. The NG formulæ are now at once obtained, viz. we have

$$\begin{aligned}NG(m, n, p)&=G(m^2, n, p)+G(m, n^2, p)+G(m, n, p^2),\\ NG(m^2, n) &=3G(m^3, n) +G(m^2, n^2),\\ NG(m^3) &=6G(m^3),\end{aligned}$$

which give the values in the Table.

The S *formulæ, particular cases*, Articles 36 to 40.

36. The S formulæ have in fact been obtained in the investigation of the G formulæ: we have

$$\begin{aligned}S(m, n, p)&=2mnp,\\ S(m^2, n) &=n([m]^2 +M),\\ S(m^3) &=\tfrac{1}{3}[m]^3+M(m-2).\end{aligned}$$

37. In confirmation of the formula $S(1, m^2)=[m]^2+M$, it is to be remarked that if we take through the line 1 an arbitrary plane, this meets the curve m in m points, and joining these two and two together we have $\tfrac{1}{2}[m]^2$ lines, each of them meeting the curve m twice and also meeting the line 1; that is, the lines in question are generating lines of the scroll $S(1, m^2)$. The line 1 is, as already mentioned, an $(h=)(\tfrac{1}{2}[m]^2+M)$tuple line on the scroll; the section by the arbitrary plane is therefore the line 1 taken $(\tfrac{1}{2}[m]^2+M)$ times, together with the before-mentioned $\tfrac{1}{2}[m]^2$ lines; that is, the order of the surface is $[m]^2+M$, as it should be. This is in fact the mode in which the order of the scroll $S(1, m^2)$ was originally obtained by Dr. SALMON.

38. As regards the formula $S(m^3)=\tfrac{1}{3}[m]^3+M(m-2)$, suppose that the curve m is a (p, q) curve on the hyperboloid, we have as before $m=p+q$, $M=-pq$, and the formula becomes

$$S(m^3)=\tfrac{1}{3}[p+q]^3-pq(p+q-2),$$

which is

$$=\tfrac{1}{3}[p]^3+\tfrac{1}{3}[q]^3,$$

viz. as already remarked, the surface is in this case the hyperboloid taken $\frac{1}{6}[p]^3+\frac{1}{6}[q]^3$ times.

39. It is to be noticed also that if the curve m be a system of lines ($m=m$, $\mathrm{M}=0$), then the formula gives

$$\mathrm{S}(m^3)=\tfrac{1}{3}[m]^3,$$

which is right, since in this case the scroll is made up of the $\frac{1}{6}[m]^3$ hyperboloids, generated each of them by a line which meets three out of the m lines.

In the case of a curve m, which is such that the coordinates of any point of the curve are proportional to rational and integral functions of the order m of an arbitrary parameter θ, or say the case of a *unicursal* curve of the order m, we have

$$(h=\tfrac{1}{2}[m-1]^2 \text{ and } \therefore)\mathrm{M}=-(m-1),$$

and the formula gives

$$\mathrm{S}(m^3)=\tfrac{1}{3}[m-1]^3,$$

for a direct investigation of which see *post*, Annex No. 1.

40. In the case of a curve m, which is the complete intersection of two surfaces of the orders p and q respectively, or say a complete $(p\times q)$ intersection, we have

$$m=pq,\ \big(h=\tfrac{1}{2}pq(p-1)(q-1) \text{ and } \therefore\big)\mathrm{M}=-\tfrac{1}{2}pq(p+q-2);$$

and we find

$$\begin{aligned}\mathrm{S}(m^3)&=\tfrac{1}{6}pq(pq-2)(2pq-3p-3q+4)\\&=\tfrac{1}{6}\beta(\beta-2)(2\beta-3\alpha+4)\end{aligned}$$

if $\alpha=pq$, $\beta=p+q$. The mode of obtaining this result by a direct investigation was pointed out to me by Dr. SALMON; see *post*, Annex No. 2.

Particular cases of the formula for $\mathrm{G}(m^4)$, Articles 41 & 42.

41. In the case of a (p, q) curve on the hyperboloid, putting as before $m=p+q$, $\mathrm{M}=-pq$, we find

$$\mathrm{G}(m^4)=\tfrac{1}{12}[p+q]^4+p+q-pq\big(\tfrac{1}{2}[p+q]^2-2(p+q)+\tfrac{11}{2}\big)+\tfrac{1}{2}p^2q^2,$$

which is

$$=\tfrac{1}{12}([p]^4+[q]^4)-2q[p-1]^3-2p[q-1]^3,$$

vanishing if p, q are neither of them greater than 3: this is as it should be, since there is then no line which meets the curve four times. The curves for which the condition is satisfied are (1, 1) the conic, (1, 2) the cubic, (2, 2) the quadriquadric, (1, 3) the excubo-quartic, (2, 3) the excubo-quintic (viz. the quintic curve, which is the partial intersection of a quadric surface and a cubic surface having a line in common), and (3, 3) the quadri-cubic, or complete intersection of a quadric surface and a cubic surface. If either p or q exceeds 3, we have the case of a curve through every point whereof there can be drawn a line or lines through four or more points, and the formula is inapplicable.

42. In the case of a complete $(p\times q)$ intersection, we have as before $m=pq$, $M=-\frac{1}{2}pq(p+q-2)$, and the formula for $G(m^4)$ becomes

$$G(m^4)=\tfrac{1}{24}\beta\left\{\begin{array}{l}\quad -66\alpha+144\\ +\beta(3\alpha^2+18\alpha-26)\\ +\beta^2.\ -6\alpha\\ +\beta^3.\,2,\end{array}\right\}$$

a formula the direct verification whereof is due to Dr. SALMON; see *post*, Annex No. 3.

The formulæ for NR(1, *m*, *n*) *and* NR(1, *m*²), Articles 43 to 46.

43. $NR(1, m, n)$.—Through the line 1 take any plane meeting the curve m in m points and the curve n in n points; then if m_1, m_2 be any two of the m points, and n_1, n_2 any two of the n points, the lines m_1n_1 and m_2n_2 are generating lines of the scroll $S(1, m, n)$, and these lines intersect in a point which belongs to the Nodal Residue NR; and in like manner the lines m_1n_2 and m_2n_1 are generating lines of the scroll, and they intersect on a point of NR; we have thus

$$(2\,.\tfrac{1}{2}[m]^2.\tfrac{1}{2}[n]^2=)\tfrac{1}{2}[m]^2[n]^2$$

points on NR, that is, the arbitrary plane through the line 1 cuts NR in $\frac{1}{2}[m]^2[n]^2$ points. But the plane also cuts NR in certain points lying on the line 1, and if the number of these be (a), then

$$NR(1, m, n)=\tfrac{1}{2}[m]^2[n]^2+\mathrm{a}.$$

44. The points (a) are included among the cuspidal points on the line 1. Taking for a moment $x=0$, $y=0$ for the equations of the line 1 (which, as we have seen, is a mn-tuple line on the scroll), the equation of the scroll is of the form $(A,\ldots\between x, y)^{mn}=0$, where $A,\ldots$ are functions of the coordinates of the degree mn. The entire number of cuspidal points on the line 1 is thus $=2[mn]^2$; but these include different kinds of cuspidal points, viz. we have

$$2[mn]^2=2\mathrm{a}+2\alpha+2\alpha'+\mathrm{R},$$

if (a) be the number of points in which the line 1 meets NR,
„ α „ „ „ „ $S(m^2, n)$,
„ α' „ „ „ „ $S(m, n^2)$,
„ R „ „ „ „ $\text{Torse}(m, n)$,

where by $\text{Torse}(m, n)$ I denote the developable surface or "Torse" generated by a line which meets each of the curves m and n. The order of the Torse in question is

$$\mathrm{R}=\big(n([m]^2-2h)+m([n]^2-2k)=\big)-2(n\mathrm{M}+m\mathrm{N}),$$

see *post*, Annex No. 4. And then observing that we have

$$\begin{aligned}\alpha\ &=S(m^2, n)=n\,([m]^2+\mathrm{M}),\\ \alpha'&=S(m, n^2)=m([n]^2+\mathrm{N}\,),\end{aligned}$$

these values give

$$2\alpha+2\alpha'+\mathrm{R}=2n[m]^2+2m[n]^2,$$

and we have

$$\begin{aligned}\mathrm{a}&=\tfrac{1}{2}(2[mn]^2-2\alpha-2\alpha'-\mathrm{R})\\&=[mn]^2-n[m]^2-m[n]^2\\&=[m]^2[n]^2,\end{aligned}$$

and thence

$$\mathrm{NR}(1,\ m,\ n)=\tfrac{3}{2}[m]^2[n]^2.$$

45. NR$(1, m^2)$.—Through the line 1 take any arbitrary plane meeting the curve m in m points; if m_1, m_2, m_3, m_4 be any four of these, then the lines m_1m_2 and m_3m_4 are generating lines of the scroll S$(1, m^2)$, and their intersection is a point of the nodal residue NR; but in like manner the lines m_1m_3 and m_2m_4 are generating lines of the scroll, and their intersection is a point of NR; and so the lines m_1m_4 and m_2m_3 are generating lines of the scroll, and their intersection is a point of NR. We have thus $(3\times\frac{1}{24}[m]^4=)\frac{1}{8}[m]^4$ points of NR on the arbitrary plane through the line 1. But there are besides the points of NR which lie on the line 1; and if the number of these be (a), then

$$\mathrm{NR}(1,\ m^2)=\tfrac{1}{8}[m]^4+\mathrm{a}.$$

46. The points (a) are included among the cuspidal points of the scroll lying on the line 1. Supposing for a moment that $x=0$, $y=0$ are the equations of the line 1, then this line being a $(\frac{1}{2}[m]^2+\mathrm{M})$tuple line on the scroll, the equation of the scroll is of the form $(\mathrm{A},\ldots)(x,\ y)^{\frac{1}{2}[m]^2+\mathrm{M}}=0$, where A, ... are functions of the coordinates of the degree $\frac{1}{2}[m]^2$: the number of cuspidal points on the line 1 is thus

$$\big(2\,.\,\tfrac{1}{2}[m]^2(\tfrac{1}{2}[m]^2-1+\mathrm{M})=\big)[m]^2(\tfrac{1}{2}[m]^2-1+\mathrm{M}).$$

But these include cuspidal points of several kinds, viz., we have

$$[m]^2(\tfrac{1}{2}[m]^2-1+\mathrm{M})=2\mathrm{a}+3\beta+\mathrm{R}'$$

if (a) be the number of points in which the line 1 meets NR,
,, β ,, ,, ,, ,, S(m^3),
,, R$'$,, ,, ,, ,, Torse (m^2),

where Torse (m^2) denotes the developable surface or Torse generated by a line which meets the curve m twice. The order of the Torse in question is

$$\mathrm{R}'=-2(m-3)\mathrm{M},$$

(see *post*, Annex No. 5); and then since

$$\beta=\mathrm{S}(m^3)=\tfrac{1}{3}[m]^3+\mathrm{M}(m-2),$$

we find

$$\begin{aligned}2\mathrm{a}&=[m]^2(\tfrac{1}{2}[m]^2-1+\mathrm{M})-3\big(\tfrac{1}{3}[m]^3+\mathrm{M}(m-2)\big)+2\mathrm{M}(m-3)\\&=\tfrac{1}{2}[m]^4+[m]^3+\mathrm{M}([m]^2-m],\end{aligned}$$

and thence

$$\mathrm{NR}(1,\ m^2)=\tfrac{3}{8}[m]^4+\tfrac{1}{2}[m]^3+\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{1}{2}m).$$

But I have not succeeded in finding by a like direct investigation the values of

$$\mathrm{NR}(m, n, p),\ \mathrm{NR}(m^2, n),\ \mathrm{NR}(m^3).$$

Formulæ for NT(1, m, n), NT(1, m^2), Articles 47 & 48.

47. We have

$$\begin{array}{lll} \mathrm{NT}(1, m, n)= & \mathrm{NG}(1, m, n)= & mn(m+n-2)+m\mathrm{N}n\mathrm{M} \\ & +\mathrm{ND}(1, m, n) & +\tfrac{1}{2}mn(mn+m+n-3) \\ & +\mathrm{NR}(1, m, n) & +\tfrac{3}{2}[m]^2[n]^2, \end{array}$$

which is

$$=2[mn]^2+m\mathrm{N}+n\mathrm{M}$$
$$=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+m\mathrm{N}+n\mathrm{M},$$

where $\mathrm{S}=\mathrm{S}(1, m, n)=2mn$.

48. And moreover

$$\begin{array}{llll} \mathrm{NT}(1, m^2)= & \mathrm{ND}(1, m^2)= & \tfrac{1}{8}[m]^4+[m]^3+\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{1}{2})+\mathrm{M}^2\,.\,\tfrac{1}{2} & \\ & +\mathrm{NG}(1, m^2) & \qquad +[m]^3+\mathrm{M}(3m-6) & \\ & +\mathrm{NR}(1, m^2) & +\tfrac{3}{8}[m]^4 \qquad +\mathrm{M}(\tfrac{1}{2}[m]^2-2m+3), & \end{array}$$

which is

$$=\tfrac{1}{2}[m]^4+2[m]^3+\mathrm{M}([m]^2+m-\tfrac{7}{2})+\mathrm{M}^2\,.\,\tfrac{1}{2}$$
$$=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\mathrm{M}(m-\tfrac{5}{2})$$

if $\mathrm{S}=\mathrm{S}(1, m^2)=[m]^2+\mathrm{M}$.

The NT *and* NR *formulæ*, Articles 49 to 58.

49. I proceed to find NT(m, n, p), &c. by a functional investigation, such as was employed for finding G(1, 1, m^2), &c. Writing S(m) to denote either of the scrolls S(m, n, p), S(m, n^2), and supposing that in place of the curve m we have the aggregate of the two curves m, m'; then the scroll S($m+m'$) breaks up into the scrolls Sm, Sm', and the intersection of these is part of the nodal total NT($m+m'$); that is, we have

$$\mathrm{NT}(m+m')=\mathrm{NT}(m)+\mathrm{NT}(m')+\mathrm{S}(m)\,.\,\mathrm{S}(m');$$

and in like manner, if S(m^2) stands for S(m^2, n), then

$$\mathrm{NT}(m+m')^2=\mathrm{NT}(m^2)+\mathrm{NT}(m, m')+\mathrm{NT}(m'^2)+\mathrm{C}_2\big(\mathrm{S}(m^2), \mathrm{S}(m, m'), \mathrm{S}(m'^2)\big),$$

where C_2 denotes the sum of the combinations two and two together; and so also

$$\mathrm{NT}(m+m')^3=\mathrm{NT}(m^3)+\mathrm{NT}(m^2, m')+\mathrm{NT}(m, m'^2)+\mathrm{NT}(m'^3)$$
$$+\mathrm{C}_2\big(\mathrm{S}(m^3), \mathrm{S}(m^2, m'), \mathrm{S}(m, m'^2), \mathrm{S}(m'^3)\big).$$

50. Instead of assuming

$$\mathrm{NT}=\tfrac{1}{2}\mathrm{S}^2+\varphi,$$

it is the same thing, and it is rather more convenient, to assume

$$\mathrm{NT}=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\varphi,$$

viz. $\text{NT}(m)=\frac{1}{2}\big(\text{S}(m)\big)^2-\text{S}(m)+\phi(m)$, &c. Then observing that

$$\text{S}(m+m')=\text{S}(m)+\text{S}(m'), \text{ \&c.},$$

the foregoing equations for NT give

$$\begin{aligned}&\phi(m+m')=\phi(m\ )+\phi(m'),\\&\phi(m+m')^2=\phi(m^2)+\phi(m,\ m')+\phi(m'^2),\\&\phi(m+m')^3=\phi(m^3)+\phi(m^2,\ m')+\phi(m,\ m'^2)+\phi(m'^3);\end{aligned}$$

and if in the second equation $\phi(m,\ m')$ and in the third equation $\phi(m^2,\ m')$ and $\phi(m,\ m'^2)$ are regarded as known, these are all of them of the form

$$f(m+m')-f(m)-f(m')=\text{Funct.}\ (m,\ m');$$

so that, a particular solution being obtained, the general solution is $f(m)=$ Particular Solution $+\alpha m+\beta\text{M}$, at least on the assumption that $f(m)$, in so far as it depends on the curve m, is a function of m and M only.

51. First, if $\phi(m)$ stands for $\phi(m,\ n,\ p)$, we obtain $\phi(m,\ n,\ p)=\alpha m+\beta\text{M}$, or observing that $\phi(m,\ n,\ p)$ must be symmetrical in regard to the curves m, n, and p, we may write

$$\phi(m,\ n,\ p)=\alpha mnp+\beta(\text{M}np+\text{N}mp+\text{P}mn)+\gamma(m\text{NP}+n\text{MP}+p\text{MN})+\delta\text{MNP},$$

and then

$$\begin{aligned}\text{NT}(m,\ n,\ p)&=\tfrac{1}{2}\text{S}^2-\text{S}+\phi(m,\ n,\ p)\\&=2mnp(mnp-1)+\phi(m,\ n,\ p).\end{aligned}$$

But for $p=1$ this should reduce itself to the known value of $\text{NT}(1,\ m,\ n)$; this gives $\alpha=0$, $\beta=1$, $\gamma=0$; we in fact have, as will be shown, *post*, Art. 55, $\delta=0$; and hence

$$\begin{aligned}\text{NT}(m,\ n,\ p)&=\tfrac{1}{2}\text{S}^2-\text{S}\ +(\text{M}np+\text{N}mp+\text{P}mn)\\&=2[mnp]^2+(\text{M}np+\text{N}mp+\text{P}mn).\end{aligned}$$

52. Next, if $\phi(m^2)$ stand for $\phi(m^2,\ n)$, then $\phi(m,\ m')$ stands for $\phi(m,\ m',\ n)$, which is $=\text{N}mm'+n(m\text{M}'+m'\text{M})$, and the equation is

$$\phi(m+m')^2-\phi(m^2)-\phi(m'^2)=\text{N}mm'+n(m\text{M}'+m'\text{M}).$$

A particular solution is $\phi(m^2)=\frac{1}{2}[m]^2\text{N}+nm\text{M}$, and we have therefore

$$\phi(m^2,\ n)=\tfrac{1}{2}[m]^2\text{N}+nm\text{M}+\alpha m+\beta\text{M};$$

or observing that $\phi(m^2,\ n)$ considered as a function of n, satisfies the equation

$$\phi(n+n')=\phi(n)+\phi(n'),$$

and is therefore a linear function of n and N, we may write

$$\phi(m^2,\ n)=\tfrac{1}{2}[m]^2\text{N}+nm\text{M}+\alpha nm+\beta n\text{M}+\gamma m\text{N}+\delta\text{MN};$$

we then have

$$\text{NT}(m^2,\ n)=\tfrac{1}{2}\text{S}^2-\text{S}+\phi(m^2,\ n),$$

where

$$\text{S}=\text{S}(m^2,\ n)=n([m]+\text{M}).$$

And then putting $n=1$, and comparing with the known value of $\mathrm{NT}(1, m^2)$, we find $\alpha=0$, $\beta=-\frac{5}{2}$. It will be shown, *post*, Art. 55, that $\gamma=0, \delta=0$; and we have therefore

$$\phi(m^2, n)=n\mathrm{M}(m-\tfrac{5}{2})+\mathrm{N}(\tfrac{1}{2}[m]^2+\mathrm{M}),$$

and thence

$$\begin{aligned}\mathrm{NT}(m^2, n)&=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\phi(m^2, n)\\&=n\left(\tfrac{1}{2}[m]^4+2[m]^3+\mathrm{M}([m]^2+m-\tfrac{7}{2})+\mathrm{M}^2\,.\,\tfrac{1}{2}\right)\\&\quad+[n]^2(\tfrac{1}{2}[m]^4+2[m]^3+[m]^2+\mathrm{M}[m]^2+\mathrm{M}^2\,.\,\tfrac{1}{2})\\&\quad+\mathrm{N}(\tfrac{1}{2}[m]^2+\mathrm{M}).\end{aligned}$$

53. Next for $\phi(m^3)$, substituting for $\phi(m^2, m')$ and $\phi(m, m'^2)$ their values, we have

$$\begin{aligned}\phi(m+m')^3-\phi(m^3)-\phi(m'^3)=m'\mathrm{M}(m-\tfrac{5}{2})&+\mathrm{M}'(\tfrac{1}{2}[m]^2+\mathrm{M})\\+m\mathrm{M}'(m'-\tfrac{5}{2})&+\mathrm{M}(\tfrac{1}{2}[m']^2+\mathrm{M}'),\end{aligned}$$

which is satisfied by

$$\phi(m^3)=\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{5}{2}m)+\mathrm{M}^2,$$

and the general value then is

$$\phi(m^3)=\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{5}{2}m)+\mathrm{M}^2+\alpha m+\beta\mathrm{M},$$

and we have

$$\mathrm{NT}(m^3)=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\phi(m^3),$$

where

$$\mathrm{S}=\mathrm{S}(m^3)=\tfrac{1}{3}[m]^3+\mathrm{M}(m-2).$$

54. Taking for the curve m the (p, q) curve on the hyperboloid $(m=p+q,\ \mathrm{M}=-pq)$, $\mathrm{S}(m^3)$ becomes the hyperboloid taken k times, if $k=\frac{1}{6}[p]^3+\frac{1}{6}[q]^3$; that is, $\mathrm{S}(m^3)=2k$, and $\mathrm{NT}(m^3)=4\,.\,\frac{1}{2}[k]^2+\phi(m^3)$; $\phi(m^3)$ must vanish if p and q are each not greater than 3, this implies $\alpha=3$, $\beta=11$, for with these values the formula gives

$$\phi(m^3)=-\tfrac{1}{2}(q[p-1]^3+p[q-1]^3).$$

55. I assume the correctness of the value

$$\phi(m^3)=3m+\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{5}{2}m+11)+\mathrm{M}^2$$

so obtained, as being in fact verified by means of the six several curves (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3); and I remark that if the foregoing value of $\phi(m, n, p)$ had been increased by $6\alpha\mathrm{MNP}$, then it would have been necessary to increase the value of $\phi(m^2, n)$ by $3\alpha\mathrm{M}^2\mathrm{N}$, and that of $\phi(m^3)$ by $\alpha\mathrm{M}^3$; and moreover that if the foregoing value of $\phi(m^2, n)$ had been increased by $\gamma m\mathrm{N}+\delta\mathrm{MN}$, then it would have been necessary to increase the value of $\phi(m^3)$ by $\gamma m\mathrm{M}+\delta\mathrm{M}^2$; this is easily seen by writing down the values

$$\begin{aligned}\phi(m^3)\quad&=\gamma m\mathrm{M}\ +\delta\mathrm{M}^2\ +\ \alpha\mathrm{M}^3\ ,\\\phi(m^2, m')&=\gamma m\mathrm{M}'\ +\delta\mathrm{MM}'+3\alpha\mathrm{M}^2\mathrm{M}',\\\phi(m, m'^2)&=\gamma m'\mathrm{M}\ +\delta\mathrm{MM}'+3\alpha\mathrm{MM}'^2,\\\phi(m'^3)\quad&=\gamma m'\mathrm{M}'+\delta\mathrm{M}'^2\ +\ \alpha\mathrm{M}'^3\ ,\end{aligned}$$

the sum of which is

$$=\gamma(m+m')(\mathrm{M}+\mathrm{M}')+\delta(\mathrm{M}+\mathrm{M}')^2+\alpha(\mathrm{M}+\mathrm{M}')^3,$$

the corresponding term of $\phi(m^3)$; hence the value of $\phi(m^3)$ being correct without the foregoing addition, we must have $\gamma=0$, $\delta=0$, $\alpha=0$; which confirms the foregoing values of $\phi(m, n, p)$, $\phi(m^2, n)$.

56. The equation

$$\mathrm{NT}(m^3)=\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+\phi(m^3)$$

gives

$$\begin{aligned}\mathrm{NT}(m^3)=&\tfrac{1}{2}\mathrm{S}^2-\mathrm{S}+3m+\mathrm{M}(\tfrac{1}{2}[m]^2-\tfrac{5}{2}m+11)+\mathrm{M}^2\\ &=\tfrac{1}{18}[m]^6+\tfrac{1}{2}[m]^5+[m]^4+3m\\ &\quad+\mathrm{M}(\tfrac{1}{3}[m]^4+\tfrac{1}{3}[m]^3+\tfrac{1}{2}[m]^2-\tfrac{7}{2}m+13)\\ &\quad+\mathrm{M}^2(\tfrac{1}{2}[m]^2-\tfrac{3}{2}m+3).\end{aligned}$$

57. We have

$$\begin{aligned}\mathrm{NR}(m^2, n)=&\ \mathrm{NT}(m^2, n)-\mathrm{ND}(m^2, n)-\mathrm{NG}(m^2, n)\\ =&\ \ n\ \left(\tfrac{3}{8}[m]^4+\mathrm{M}(\tfrac{1}{2}[m]^2-2m+3)\right)\\ &+[n]^2\left(\tfrac{1}{2}[m]^4+\tfrac{3}{2}[m]^3+[m]^2+\mathrm{M}([m]^2-\tfrac{1}{2})+\mathrm{M}^2.\tfrac{1}{2}\right).\end{aligned}$$

58. And moreover,

$$\begin{aligned}\mathrm{NR}(m^3)=&\ \mathrm{NT}(m^3)-\mathrm{ND}(m^3)-\mathrm{NG}(m^3)\\ =&\ \tfrac{1}{18}[m]^6+\tfrac{3}{8}[m]^5-\tfrac{1}{2}[m]^3+3m\\ &+\mathrm{M}(\tfrac{1}{3}[m]^4-\tfrac{1}{6}[m]^3-\tfrac{5}{2}[m]^2+8m-20)+\mathrm{M}^2(\tfrac{1}{2}[m]^2-2m):\end{aligned}$$

and the investigation of the series of results given in the Table is thus concluded.

Intersections of a generating line with the Nodal Total, Articles 59 to 63.

59. We may for the scrolls S(1, m, n) and S(1, m^2) verify the theorem that each generating line meets the Nodal Total in a number of points$=$S-2.

In fact for the scroll S(1, m, n), the directrix curves are respectively multiple curves of the orders mn, n, m, and a generating line meets each of these in a single point, counting for the three curves respectively as $mn-1$, $n-1$, and $m-1$ points respectively. Moreover the construction (*ante*, Art. 43) for the Nodal Residue NR(1, m, n) shows that a generating line meets this curve in $(m-1)(n-1)$ points; and since the curve is merely a double curve, these count each as a single point; and the generating line does not meet the Nodal Generator NG(1, m, n). The number of intersections therefore is

$$mn-1+(m-1)+(n-1)+(m-1)(n-1),$$

which is

$$=2mn-2,\quad =\mathrm{S}-2.$$

60. Similarly for the scroll S(1, m^2); the directrix curves are multiple curves, viz. the line 1 is a $(\frac{1}{2}[m]^2+\mathrm{M})$tuple curve, and the curve m a $(m-1)$tuple curve; the

generating line meets the former in a single point, counting as $\tfrac{1}{2}[m]^2+\mathrm{M}-1$ points, and the latter in two points, each counting as $(m-2)$ points. The construction (*ante*, Art. 45) for the Nodal Residue NR$(1, m^2)$ shows that the generating line meets this curve in $\tfrac{1}{2}[m-2]^2$ points; and since the curve is merely a double curve, these count each as a single point. Finally, the generating line does not meet the Nodal Generator NG$(1, m^2)$. The number of intersections thus is

$$\tfrac{1}{2}[m]^2-1+\mathrm{M}+2(m-2)+\tfrac{1}{2}[m-2]^2,$$

which is

$$=[m]^2-2+\mathrm{M}, \quad =\mathrm{S}-2.$$

In the remaining cases we may use the theorem to find the number of points in which the generating line meets the Nodal Residue. Using Π as the symbol for the points in question $\big(\Pi(m, n, p)$ for the scroll $\mathrm{S}(m, n, p)$, &c.$\big)$, we find

61. For the scroll $\mathrm{S}(m, n, p)$,

$$(mn-1)+(np-1)+(mp-1)+\Pi(m, n, p)=\mathrm{S}-2=2mnp-2,$$

which gives

$$\Pi(m, n, p)=2mnp-mn-mp-np+1.$$

This includes the before-mentioned case

$$\Pi(1, m, n)=(m-1)(n-1),$$

and the more particular one

$$\Pi(1, 1, m)=0.$$

62. For the scroll $\mathrm{S}(m^2, n)$,

$$\tfrac{1}{2}[m]^2-1+\mathrm{M}+2\big((m-1)n-1\big)+\Pi(m^2, n)=\mathrm{S}-2=n([m]^2+\mathrm{M})-2,$$

which gives

$$\begin{aligned}\Pi(m^2, n)=n([m]^2-2m+2+\mathrm{M})&\\ -\tfrac{1}{2}[m]^2+1-\mathrm{M}.&\end{aligned}$$

This includes the before-mentioned particular case

$$\Pi(1, m^2)=\tfrac{1}{2}[m-2]^2.$$

63. And lastly for the scroll $\mathrm{S}(m^3)$,

$$3(\tfrac{1}{2}[m]^2-m+1+\mathrm{M})+\Pi(m^3)=\mathrm{S}-2=\tfrac{1}{3}[m]^3+(m-2)\mathrm{M}-2,$$

which gives

$$\Pi(m^3)=\tfrac{1}{3}[m]^3-\tfrac{3}{2}[m]^2+3m-5+\mathrm{M}(m-5).$$

The foregoing expressions for Π might with propriety have been inserted in the Table.

Annex No. 1.—*Investigation of the formula for* S(m^3) *in the case of the unicursal curve* (referred to, Art. 39).

Consider the unicursal m-thic curve the equations whereof are $x:y:z:w=\mathrm{A}:\mathrm{B}:\mathrm{C}:\mathrm{D}$, where A, B, C, D are rational and integral functions of a parameter θ. And let it be required to find the equation of a plane meeting the curve in such manner that three of the points of intersection are *in lineâ*. Taking for the equation of the plane

$$\xi x+\eta y+\zeta z+\omega w=0,$$

we find between $(\xi, \eta, \zeta, \omega)$ an equation of a certain degree in $(\xi, \eta, \zeta, \omega)$, which is the equation in plane-coordinates of the scroll $S(m^3)$, the degree of the equation is therefore equal to the class of the scroll; but as the class of a scroll is equal to its order, the degree of the equation is equal to the order of the scroll, or say $=S(m^3)$.

Proceeding with the investigation, if θ be determined by the equation

$$\xi A+\eta B+\zeta C+\omega D=0,$$

then the roots $\theta_1, \theta_2, \ldots \theta_m$ of this equation belong to the points of intersection of the plane and curve; and the corresponding coordinates of these points are (A_1, B_1, C_1, D_1), &c.

Suppose that the points 1, 2, 3 are *in lineâ*, and let $\lambda, \mu, \nu, \varrho$ be the coordinates of an arbitrary point, then the four points are *in plano*, that is, we have

$$\begin{vmatrix} \lambda\;, & \mu\;, & \nu\;, & \varrho \\ A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \end{vmatrix}=0\;;$$

and if we form the equation

$$\Pi\begin{vmatrix} \lambda\;, & \mu\;, & \nu\;, & \varrho \\ A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \end{vmatrix}=0,$$

where Π denotes the product of the terms belonging to all the triads of the m roots, the result will be symmetrical in regard to all the roots; and replacing the symmetrical functions of the roots by their values in terms of the coefficients, we have the required relation between $(\xi, \eta, \zeta, \omega)$.

Π contains $\frac{1}{6}[m]^3$ terms, whereof $\frac{1}{2}[m-1]^2$ contain the m-thic functions (A_1, B_1, C_1, D_1) of the root θ_1; that is, the form of Π is

$$(\lambda, \mu, \nu, \varrho)^{\frac{1}{6}[m]^3}(\theta_1, 1)^{\frac{1}{2}[m]^3}(\theta_2, 1)^{\frac{1}{2}[m]^3}\ldots\;;$$

or, when the symmetrical functions are expressed in terms of the coefficients, the form is

$$(\lambda, \mu, \nu, \varrho)^{\frac{1}{6}[m]^3}(\xi, \eta, \zeta, \omega)^{\frac{1}{2}[m]^3}.$$

Now the above-mentioned determinant is divisible by $(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_2-\theta_3)$, or Π is divisible by $\Pi(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_2-\theta_3)$; and since this product contains $(3\times\frac{1}{6}[m]^3=)\frac{1}{2}[m]^3$ linear factors, and the product $\zeta(\theta_1, \theta_2, \ldots \theta_m)$ of the squared differences of the roots contains $(2\times\frac{1}{2}[m]^2=)[m]^2$ linear factors, so that we have

$$\Pi(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_2-\theta_3)=\{\zeta(\theta_1, \theta_2, \ldots \theta_m)\}^{\frac{1}{2}(m-2)},$$

where

$$\zeta(\theta_1, \theta_2, .. \theta_m) = \text{Disct.} = (\xi, \eta, \zeta, \omega)^{2(m-1)},$$

and consequently

$$\Pi(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_2-\theta_3) = (\xi, \eta, \zeta, \omega)^{[m-1]^2},$$

so that, omitting this factor, the remaining factor of Π is of the form

$$(\lambda, \mu, \nu, \varrho)^{\frac{1}{6}[m]^3}(\xi, \eta, \zeta, \omega)^{\frac{1}{2}[m]^3-[m-1]^2};$$

but the determinant vanishes if

$$\lambda, \mu, \nu, \varrho = (A_1, B_1, C_1, D_1), \quad (A_2, B_2, C_2, D_2), \quad (A_3, B_3, C_3, D_3),$$

or say if

$$(\lambda, \mu, \nu, \varrho) = (A, B, C, D), \quad \theta = \theta_1, \theta_2, \text{ or } \theta_3;$$

it follows that the product Π contains the factor

$$(\lambda\xi + \mu\eta + \nu\zeta + \varrho\omega)^{\frac{1}{6}[m]^3};$$

or omitting this factor, and observing that

$$\tfrac{1}{2}[m]^3-[m-1]^2-\tfrac{1}{6}[m]^3 = \tfrac{1}{3}[m]^3-[m-1]^2 = \tfrac{1}{3}[m-1]^3,$$

the remaining factor is of the form

$$(\xi, \eta, \zeta, \omega)^{\frac{1}{3}[m-1]^3};$$

or we have finally

$$S(m^3) = \tfrac{1}{3}[m-1]^3,$$

which is the required expression.

I give the following investigation of the expression $\frac{1}{2}[m-1]^2$ for the number of apparent double points. Imagine through the point ($x=0$, $y=0$, $z=0$) a line cutting the curve in the two points corresponding to the values θ_1, θ_2 of the parameter. We have

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2},$$

which equations determine θ_1 and θ_2.

Writing the equations under the form

$$\frac{A_1B_2-A_2B_1}{\theta_1-\theta_2} = 0, \quad \frac{A_1C_2-A_2C_1}{\theta_1-\theta_2} = 0,$$

and treating θ_1 and θ_2 as coordinates, each of these equations belongs to a curve of the order $2(m-1)$, having a $(m-1)$thic point at infinity on each of the axes. The number of intersections thus is

$$= 4(m-1)^2-(m-1)^2-(m-1)^2, \quad = 2(m-1)^2.$$

But among these are included points not belonging to the original system, viz. the points for which ($A_1=0$, $A_2=0$) other than those for which $\theta_1=\theta_2$; the points so included are in number $=m^2-m$; and omitting them, the number is

$$\big(2(m-1)^2-m(m-1)\big) = [m-1]^2,$$

which is the number of points θ_1 lying *in lineâ* with the origin and another point θ_2; the number of apparent double points is the half of this, or $h=\frac{1}{2}[m-1]^2$. And thence

$$M=(-\tfrac{1}{2}[m]^2+h=)-(m-1).$$

I investigate also the number of lines through two points which meet two arbitrary lines; this is in fact $=S(1, m^2)$, which for the curve in question is

$$=\left(\tfrac{1}{2}[m]^2-(m-1)=\right)(m-1)^2.$$

Let the equations of the two lines be $(x=0, y=0)$ and $(z=0, w=0)$; then the conditions to be satisfied are

$$\frac{A_1}{A_2}=\frac{B_1}{B_2},\quad \frac{C_1}{C_2}=\frac{D_1}{D_2};$$

or writing these under the form

$$\frac{A_1B_2-A_2B_1}{\theta_1-\theta_2}=0,\quad \frac{C_1D_2-C_2D_1}{\theta_1-\theta_2}=0,$$

and treating θ_1, θ_2 as coordinates, the number of intersections of these two curves is $=2(m-1)^2$, the same as in the two curves last above considered. And the number of the lines in question is one half of this, or $=(m-1)^2$.

Lemma employed in the following Annexes 2 and 3. *Formulæ for the order and weight of certain systems of equations.*

Let $\alpha_{x'}$ denote a function of the degree α in the *order* variables $(x, y, ..)$, and of the degree α' in the *weight* variables $(x', y', ..)$, and so in other cases; and consider first the equation

$$\begin{vmatrix} \alpha_{\alpha'}, & (\alpha+A)_{\alpha'+A'}, & \dots \\ \beta_{\beta'}, & (\beta+A)_{\beta'+A'}, & \\ \cdot & & \\ \cdot & & \\ \cdot & & \end{vmatrix}=0,$$

where the matrix is a square; then

$$\text{Order}\ =\Sigma\alpha+\Sigma A,$$
$$\text{Weight}=\Sigma\alpha'+\Sigma A'.$$

Consider next the system

$$\begin{Vmatrix} \alpha_{\alpha'}, & (\alpha+A)_{\alpha'+A'}, & (\alpha+B)_{\alpha'+B'}, & \dots \\ \beta_{\beta'}, & (\beta+A)_{\beta'+A'}, & (\beta+B)_{\beta'+B'}, & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{Vmatrix}=0,$$

where the matrix is a square $+1$, that is, the number of columns exceeds by 1 the number of lines; then

$$\text{Order}\ =\Sigma AB-\Sigma\alpha\beta+\Sigma\alpha(\Sigma A+\Sigma\alpha),$$
$$\text{Weight}=(\Sigma A+\Sigma\alpha)(\Sigma A'+\Sigma\alpha')-\Sigma AA'+\Sigma\alpha\alpha'.$$

And again, the system

$$\left\| \begin{matrix} \alpha_{\alpha'}, & (\alpha+\text{A})_{\alpha'+\text{A}'}, & (\alpha+\text{B})_{\alpha'+\text{B}'}, & (\alpha+\text{C})_{\alpha'+\text{C}'}, & \ldots \\ \beta_{\beta'}, & (\beta+\text{A})_{\beta'+\text{A}'}, & (\beta+\text{B})_{\beta'+\text{B}'}, & (\beta+\text{C})_{\beta'+\text{C}'}, & \\ \vdots & & & & \end{matrix} \right\| =0,$$

where the matrix is a square $+2$, that is, the number of columns exceeds by 2 the number of lines; then

$$\begin{aligned} \text{Order } &= \Sigma\text{ABC}+\Sigma\alpha\beta\gamma+\Sigma\alpha(\Sigma\text{AB}-\Sigma\alpha\beta)+\big((\Sigma\alpha)^2-\Sigma\alpha\beta\big)(\Sigma\text{A}+\Sigma\alpha), \\ \text{Weight} &= \{\Sigma\text{AB}-\Sigma\alpha\beta+\Sigma\alpha(\Sigma\text{A}+\Sigma\alpha)\}(\Sigma\text{A}'+\Sigma\alpha')-(\Sigma\text{A}+\Sigma\alpha)(\Sigma\text{AA}'-\Sigma\alpha\alpha') \\ &\qquad +\Sigma\text{A}^2\text{A}'+\Sigma\alpha^2\alpha'. \end{aligned}$$

The last formula, for the weight of the square $+2$ system, was communicated to me by Dr. SALMON, the others are all in effect given in the Appendix, "On the Order of Systems of Equations," to his Treatise on the Analytic Geometry of Three Dimensions; and in the investigation in the following Annexes 2 and 3, the route which I have followed was completely traced out for me by him, so that I have only supplied the details of the work.

Annex No. 2.—*Investigation of the formula for* S(m^3), *when the curve* m *is the* pq *complete intersection, viz. when it is the intersection of two surfaces of the orders* p *and* q *respectively* (referred to, Art. 40).

Let U=0, V=0 be the equations of the two surfaces of the orders p and q respectively. Take (x, y, z, w) the coordinates of a point on the curve, so that for these coordinates we have U=0, V=0; and in the equations of the two curves respectively, write for the coordinates $x+\varrho x'$, $y+\varrho y'$, $z+\varrho z'$, $w+\varrho w'$; then putting for shortness

$$\Delta=x'\partial_x+y'\partial_y+z'\partial_z+w'\partial_w,$$

the resulting equations may be represented by

$$(\Delta\text{U}, \Delta^2\text{U}, \ldots \Delta^p\text{U} \,)\!(\, 1, \varrho)^{p-1}=0,$$

$$(\Delta\text{V}, \Delta^2\text{V}, \;..\, \Delta^q\text{V} \,)\!(\, 1, \varrho)^{q-1}=0,$$

where it is to be noticed that besides the expressed literal coefficients there are numerical coefficients (not as the notation usually denotes, the binomial coefficients, but) $=\frac{1}{1}, \frac{1}{1.2}, \frac{1}{1.2.3}$, &c.

Supposing that (x', y', z', w') are the current coordinates of a point on the line drawn through the point (x, y, z, w) to meet the curve in two other points, the equations in ϱ must have two common roots, and this gives a system equivalent to two equations, or say a plexus of two equations. If from the plexus and the two equations U=0, V=0 we eliminate (x, y, z, w), we obtain an equation S'=0 in (x', y', z', w'), which is in fact the equation of the scroll S(m^3), taken (as is easily seen to be the case) thrice; that is, S(m^3)= $\frac{1}{3}$Degree of S'. But observing that the coordinates (x', y', z', w') enter into the plexus only and not into the functions U, V, and treating (x', y', z', w') as *weight* variables, Degree of S′=Weight of System (U=0, V=0, Plexus)=Deg. U×Deg. V×Weight of Plexus, $=pq\times$Weight of Plexus; or, writing $pq=\beta$,

$$\mathrm{S}(m^3)=\tfrac{1}{3}\beta\times\text{Weight of Plexus}.$$

The plexus in question is the square +1 system,

$$\left\Vert\begin{array}{l}
\Delta\mathrm{U},\ \Delta^2\mathrm{U},\ldots \\
\qquad\Delta\mathrm{U},\ldots \\
\cdot \\
\cdot \\
\Delta\mathrm{V},\ \Delta^2\mathrm{V},\ldots \\
\qquad\Delta\mathrm{V},\ldots \\
\cdot \\
\cdot
\end{array}\qquad\qquad\right\Vert=0,$$

$p+q-3$ columns, $(q-2)+(p-2)=(p+q-4)$ lines; or representing the terms according to their order and weight, that is, degree in (x, y, z, w) and (x', y', z', w') respectively (the order and weight of the evanescent terms being fixed so as that they may form a regular series with the other terms), the system is

$$\left\Vert\begin{array}{ll}
 & \overbrace{\qquad\qquad\qquad\qquad\qquad}^{p+q-3\text{ columns.}} \\
(q-2)\text{ lines.}\left\{\vphantom{\begin{array}{l}a\\a\\a\\a\end{array}}\right. & \begin{array}{l}(p-1)_1,\ (p-2)_2,\ \ldots \\ \quad p_0\quad,\ (p-1)_1, \\ \quad\cdot \\ \quad\cdot\end{array} \\
(p-2)\text{ lines.}\left\{\vphantom{\begin{array}{l}a\\a\\a\\a\end{array}}\right. & \begin{array}{l}(q-1)_1,\ (q-2)_2, \\ \quad q_0\quad,\ (q-1)_1, \\ \quad\cdot \\ \quad\cdot\end{array}
\end{array}\right\Vert=0,$$

so that

$$\begin{array}{llll}
\alpha,\beta,\ldots= & p-1,\ p,\ \ldots\ p+q-4, & q-1,\ q,\ \ldots & p+q-4, \\
\alpha',\beta',\ldots= & 1,\ 0,\ \ldots\ -q+4, & 1,\ 0,\ \ldots & -p+4, \\
\mathrm{A},\mathrm{B},\ldots= & -1,-2, & & \ldots-(p+q-4), \\
\mathrm{A}',\mathrm{B}',\ldots= & 1,\ 2, & & p+q-4,
\end{array}$$

or, as regards the first two lines,

$$\left.\begin{array}{lll}
\alpha,\beta,\ldots= & p-2+\theta, & q-2+\phi \\
\alpha',\beta',\ldots= & 2-\theta, & 2-\phi
\end{array}\right\}\theta=1\text{ to }q-2,\text{ and }\phi=1\text{ to }p-2.$$

We then find

$$\begin{aligned}
\Sigma\alpha= & \ (q-2)(p-2)+\tfrac{1}{2}(q-2)(q-1) \\
& +(p-2)(q-2)+\tfrac{1}{2}(p-2)(p-1), \\
\Sigma\alpha'= & \ 2(q-2)-\tfrac{1}{2}(q-2)(q-1) \\
& +2(p-2)-\tfrac{1}{2}(p-2)(p-1),
\end{aligned}$$

$$\begin{aligned}
\Sigma\mathrm{A} &= -\Sigma\mathrm{A}' = -\tfrac{1}{2}(p+q-4)(p+q-3),\\
\Sigma\alpha\alpha' &= 2(p-2)(q-2)-(p-4)\,.\,\tfrac{1}{2}(q-2)(q-1)-\tfrac{1}{6}(q-2)(q-1)(2q-3)\\
&\quad +2(q-2)(p-2)-(q-4)\,.\,\tfrac{1}{2}(p-2)(p-1)-\tfrac{1}{6}(p-2)(p-1)(2p-3),\\
\Sigma\mathrm{AA}' &= -\tfrac{1}{6}(p+q-4)(p+q-3)(2p+2q-7),
\end{aligned}$$

which putting therein $p+q=\alpha$, $pq=\beta$, give

$$\begin{aligned}
\Sigma\alpha &= \beta+\tfrac{1}{2}\alpha^2-\tfrac{11}{2}\alpha+10,\\
\Sigma\alpha' &= \beta-\tfrac{1}{2}\alpha^2+\tfrac{7}{2}\alpha-10,\\
\Sigma\mathrm{A} &= -\Sigma\mathrm{A}' = -\tfrac{1}{2}\alpha^2+\tfrac{7}{2}\alpha-6,\\
\Sigma\alpha\alpha' &= \tfrac{1}{2}\alpha\beta-\tfrac{1}{3}\alpha^3+\tfrac{7}{2}\alpha^2-\tfrac{103}{6}\alpha+26,\\
\Sigma\mathrm{AA}' &= -\tfrac{1}{3}\alpha^3+\tfrac{7}{2}\alpha^2-\tfrac{73}{6}\alpha+14,
\end{aligned}$$

and thence

$$\begin{aligned}
\Sigma\mathrm{A}+\Sigma\alpha &= \beta-2\alpha+4,\\
\Sigma\mathrm{A}'+\Sigma\alpha' &= \beta-4,\\
\Sigma\mathrm{AA}'-\Sigma\alpha\alpha' &= -\tfrac{1}{2}\alpha\beta+5\alpha-12,
\end{aligned}$$

and therefore

$$\begin{aligned}
\text{Weight} &= (\beta-2\alpha+4)(\beta-4)+\tfrac{1}{2}\alpha\beta-5\alpha+12\\
&= \beta^2-\tfrac{3}{2}\alpha\beta+6\alpha-8\\
&= \tfrac{1}{2}(\beta-2)(2\beta-3\alpha+4),
\end{aligned}$$

and consequently

$$\begin{aligned}
\mathrm{S}(m^3) &= \tfrac{1}{3}\beta\times\text{weight}\\
&= \tfrac{1}{6}\beta(\beta-2)(2\beta-3\alpha+4),
\end{aligned}$$

which is right.

Annex No. 3.—*Investigation of* G(m^4) *in the case where the curve m is a pq complete intersection* (referred to, Art. 42).

Suppose, as before, that $\mathrm{U}=0$, $\mathrm{V}=0$ are the equations of the two surfaces of the orders p and q respectively; taking also (x, y, z, w) as the coordinates of a point on the curve, and substituting in the equations $x+\varrho x'$, $y+\varrho y'$, $z+\varrho z'$, $w+\varrho w'$ in place of the coordinates, then if $\Delta=x'\partial_x+y'\partial_y+z'\partial_z+w\partial_w$, we have as before

$$(\Delta\mathrm{U}, \Delta^2\mathrm{U}, \ldots \Delta^p\mathrm{U})(1, \varrho)^{p-1}=0,$$
$$(\Delta\mathrm{V}, \Delta^2\mathrm{V}, \ldots \Delta^q\mathrm{V})(1, \varrho)^{q-1}=0,$$

where the numerical coefficients $\frac{1}{1}$, $\frac{1}{1.2}$, $\frac{1}{1.2.3}$, &c. are to be understood as before.

Suppose now that (x, y, z, w) are the coordinates of a point on the curve, through which point there passes a line through three other points, or line G(m^4); and that (x', y', z', w') are the current coordinates of a point on such line: the two equations in ϱ must have three equal roots; or we must have a system equivalent to three equations, or say a plexus of three equations. The coordinates (x', y', z', w'), although four in number, are in fact eliminable from this plexus; or what is the same thing, combining with the plexus the equation

$$\alpha x'+\beta y'+\gamma z'+\delta w'=0$$

of an arbitrary plane, and then eliminating (x', y', z', w'), the result is of the form

$$(\alpha x+\beta y+\gamma z+\delta w)^{\theta}\square=0,$$

where $\square$ is a function of $(x, y, z. w)$ only; and considering (x, y, z, w) as weight variables, $\theta=$Order of Plexus. But degree in (x, y, z, w) of $(\alpha x+\beta y+\gamma z+\delta w)^{\theta}\square$ is $=$Weight of Plexus, and therefore Degree of $\square$ is $=$ Weight of Plexus$-\theta$, $=$(Weight$-$Order) of Plexus.

The equations $\mathrm{U}=0$, $\mathrm{V}=0$, $\square=0$ then give the coordinates (x, y, z, w) of the points through which may be drawn a line $\mathrm{G}(m^4)$; viz. they give (as it is easy to see) these points four times over. And we therefore have

$$\begin{aligned}\mathrm{G}(m^4)&=\tfrac{1}{4}\text{ Order of }(\mathrm{U}=0,\ \mathrm{V}=0,\ \square=0)\\ &=\tfrac{1}{4}\text{ Deg. U. Deg. V. Deg. }\square\\ &=\tfrac{1}{4}\beta\times(\text{Weight}-\text{Order})\text{ of Plexus.}\end{aligned}$$

The Plexus is here the square $+2$ system

$$\left\|\begin{array}{lll} \Delta\mathrm{U}, & \Delta^2\mathrm{U}, & \ldots \\ . & \Delta\,\mathrm{U}, & \\ . & & \\ . & & \\ \Delta\mathrm{V}, & \Delta^2\mathrm{V}, & \\ & \Delta\,\mathrm{V}, & \\ . & & \\ . & & \end{array}\right\|=0,$$

$\big(p+q-4$ columns, $(q-3)+(p-3)=p+q-6$ lines$\big)$. Or representing the terms by their order and weight $\big($the weight variables being in the present case (x, y, z, w), and the order variables $(x', y', z', w')\big)$, and attributing as before an order and weight to the evanescent terms, the system is

$$\left\|\begin{array}{rl} & \overbrace{\hphantom{1_{p-1},\quad 2_{p-2},\quad \ldots}}^{p+q-3\text{ columns.}} \\ q-3\text{ lines.}\left\{\begin{array}{l}\\ \\ \\ \\ \end{array}\right. & \begin{array}{lll} 1_{p-1}, & 2_{p-2}, & \ldots \\ 0_p\ , & 1_{p-1}, & \\ \cdot & & \\ \cdot & & \end{array} \\ p-3\text{ lines.}\left\{\begin{array}{l}\\ \\ \\ \\ \end{array}\right. & \begin{array}{lll} 1_{q-1}, & 2_{q-2}, & \\ 0_q\ , & 1_{q-1}, & \\ \cdot & & \\ \cdot & & \end{array} \end{array}\right\|=0,$$

so that we have

$$\begin{array}{ll} \alpha\,,\beta\,,\,.. = & 1,\,0,\;-1,\,..\;\;-(q-5),\,1,\,0,\,-1,\;\;..\,-(p-5), \\ \alpha'\,,\beta',\,.. = & p-1,\,p,\,p+1,\,..\,p+q-5,\;q-1,\,q,\,q+1,\,..\,q+p-5, \\ \mathrm{A},\mathrm{B},\,.. = & 1,\;\;2,\,\ldots \qquad\qquad\qquad\qquad p+q-5, \\ \mathrm{A}',\mathrm{B}',\,.. = & -1,\,-2,\,\ldots \qquad\qquad\quad -(p+q-5), \end{array}$$

or, as regards the first two lines,

$$\left.\begin{array}{l} \alpha,\beta,\,.. = \;\; 2-\theta,\,2-\phi \\ \alpha',\beta',\,.. = p-2+\theta,\,p-2+\phi \end{array}\right\}\theta=1 \text{ to } q-3,\ \phi=1 \text{ to } p-3.$$

We then find

$$\begin{aligned}
\Sigma\alpha =&\ 2(q-3)-\tfrac{1}{2}(q-3)(q-2)\\
&+2(p-3)-\tfrac{1}{2}(p-3)(p-2),\\
\Sigma\alpha' =&\ (p-2)(q-3)+\tfrac{1}{2}(q-3)(q-2)\\
&+\ (q-2)(p-3)+\tfrac{1}{2}(p-3)(p-2),\\
\Sigma\alpha^2 =&\ 4(q-3)-4\,.\,\tfrac{1}{2}(q-3)(q-2)+\tfrac{1}{6}(q-3)(q-2)(2q-5)\\
&+4(p-3)-4\,.\,\tfrac{1}{2}(p-3)(p-2)+\tfrac{1}{6}(p-3)(p-2)(2p-5),\\
\Sigma\alpha^3 =&\ 8(q-3)-12\,.\,\tfrac{1}{2}(q-3)(q-2)+6\,.\,\tfrac{1}{6}(q-3)(q-2)(2q-5)-\tfrac{1}{4}(q-3)^2(q-2)^2\\
&+8(p-3)-12\,.\,\tfrac{1}{2}(p-3)(p-2)+6\,.\,\tfrac{1}{6}(p-3)(p-2)(2p-5)-\tfrac{1}{4}(p-3)^2(p-2)^2,\\
\Sigma\alpha\alpha' =&\ 2(p-2)(q-3)-(p-4)\,.\,\tfrac{1}{2}(q-3)(q-2)-\tfrac{1}{6}(q-3)(q-2)(2q-5)\\
&+2(q-2)(p-3)-(q-4)\,.\,\tfrac{1}{2}(p-3)(p-2)-\tfrac{1}{6}(p-3)(p-2)(2p-5),\\
\Sigma\alpha^2\alpha' =&\ 4(p-2)(q-3)-4(p-3)\,.\,\tfrac{1}{2}(q-3)(q-2)+(p-6)\,.\,\tfrac{1}{6}(q-3)(q-2)(2q-5)+\tfrac{1}{4}(q-3)^2(q-2)\\
&+4(q-2)(p-3)-4(q-3)\,.\,\tfrac{1}{2}(p-3)(p-2)+(q-6)\,.\,\tfrac{1}{6}(p-3)(p-2)(2p-5)+\tfrac{1}{4}(p-3)^2(p-2)\\
\Sigma\mathrm{A} =&\ \tfrac{1}{2}(p+q-5)\,(p+q-4),\\
\Sigma\mathrm{A}^2 =&-\Sigma\mathrm{AA}'=\tfrac{1}{6}(p+q-5)\,(p+q-4)(2p+2q-9),\\
\Sigma\mathrm{A}^3 =&-\Sigma\mathrm{A}^2\mathrm{A}'=\tfrac{1}{4}(p+q-5)^2(p+q-4)^2,
\end{aligned}$$

which, putting therein $p+q=\alpha$, $pq=\beta$, and from the reduced expressions obtaining the values of $\Sigma\alpha\beta$, &c., give

$$\begin{aligned}
\Sigma\alpha\ \ &=\beta-\tfrac{1}{2}\alpha^2+\tfrac{9}{2}\alpha-18,\\
\Sigma\alpha^2\ \ &=\beta\,(-\alpha+9)+\tfrac{1}{3}\alpha^3-\tfrac{9}{2}\alpha^2+\tfrac{121}{6}\alpha-58,\\
\Sigma\alpha^3\ \ &=\beta^2(-\tfrac{1}{2})+\beta(\alpha^2-\tfrac{27}{2}\alpha+\tfrac{121}{2})-\tfrac{1}{4}\alpha^4+\tfrac{9}{2}\alpha^3-\tfrac{121}{4}\alpha^2+90\alpha-198,\\
\Sigma\alpha\beta\ &=\beta^2(\tfrac{1}{2})+\beta(-\tfrac{1}{2}\alpha^2+5\alpha-\tfrac{45}{2})+\tfrac{1}{8}\alpha^4-\tfrac{29}{12}\alpha^3+\tfrac{171}{8}\alpha^2-\tfrac{1093}{12}\alpha+191,\\
\Sigma\alpha\beta\gamma&=\beta^3(\tfrac{1}{6})+\beta^2(-\tfrac{1}{4}\alpha^2+\tfrac{11}{4}\alpha-\tfrac{41}{3})+\beta(\tfrac{1}{8}\alpha^4-\tfrac{8}{3}\alpha^3+\tfrac{629}{24}\alpha^2-\tfrac{749}{6}\alpha+\tfrac{1753}{6})\\
&\qquad -\tfrac{1}{48}\alpha^6+\tfrac{31}{48}\alpha^5-\tfrac{445}{48}\alpha^4+\tfrac{3617}{48}\alpha^3-\tfrac{8969}{24}\alpha^2+1071\alpha-1560,\\
\Sigma\alpha'\ \ &=\beta+\tfrac{1}{2}\alpha^2-\tfrac{15}{2}\alpha+18,\\
\Sigma\alpha\alpha'\ &=\beta(-\tfrac{1}{2}\alpha)-\tfrac{1}{3}\alpha^3+\tfrac{9}{2}\alpha^2-\tfrac{175}{6}\alpha+58,\\
\Sigma\alpha^2\alpha'&=\beta^2(-\tfrac{1}{6})+\beta(-\tfrac{2}{3}\alpha^2+9\alpha-\tfrac{121}{6})+\tfrac{1}{4}\alpha^4-\tfrac{9}{2}\alpha^3+\tfrac{121}{4}\alpha^2-119\alpha+198,
\end{aligned}$$

$$\begin{aligned}
\Sigma\mathrm{A} &= \tfrac{1}{2}\alpha^2-\tfrac{9}{2}\alpha+10,\\
\Sigma\mathrm{A}^2 &= -\Sigma\mathrm{AA}'=\tfrac{1}{3}\alpha^3-\tfrac{9}{2}\alpha^2+\tfrac{121}{6}\alpha-30,\\
\Sigma\mathrm{A}^3 &= -\Sigma\mathrm{A}^2\mathrm{A}'=\tfrac{1}{4}\alpha^4-\tfrac{9}{2}\alpha^3+\tfrac{121}{4}\alpha^2-90\alpha+100,\\
\Sigma\mathrm{AB} &= \tfrac{1}{8}\alpha^4-\tfrac{29}{12}\alpha^3+\tfrac{139}{8}\alpha^2-\tfrac{661}{12}\alpha+65,\\
\Sigma\mathrm{ABC} &= \tfrac{1}{48}\alpha^6-\tfrac{31}{48}\alpha^5+\tfrac{397}{48}\alpha^4-\tfrac{2689}{48}\alpha^3+\tfrac{5081}{24}\alpha^2-\tfrac{1270}{3}\alpha+\tfrac{1050}{3},
\end{aligned}$$

we then find

$$\begin{aligned}
\Sigma\mathrm{A}\ +\Sigma\alpha\ &=\beta-8,\\
\Sigma\mathrm{A}'\ +\Sigma\alpha'\ &=\beta-3\alpha+8,\\
\Sigma\mathrm{AB}\ -\Sigma\alpha\beta\ &=\beta^2(-\tfrac{1}{2})+\beta(\tfrac{1}{2}\alpha^2-5\alpha+\tfrac{45}{2})-4\alpha^2+36\alpha-126,\\
\Sigma\mathrm{AA}'\ -\Sigma\alpha\alpha'\ &=\beta(-\tfrac{1}{2}\alpha)+9\alpha-28,\\
\Sigma\mathrm{ABC}+\Sigma\alpha\beta\gamma &=\beta^3(\tfrac{1}{6})+\beta^2(-\tfrac{1}{4}\alpha^2+\tfrac{11}{4}\alpha-\tfrac{41}{3})+\beta(\tfrac{1}{8}\alpha^4-\tfrac{8}{3}\alpha^3+\tfrac{629}{24}\alpha^2-\tfrac{749}{6}\alpha+\tfrac{1753}{6})\\
&\quad-\alpha^4+\tfrac{58}{3}\alpha^3-162\alpha^2+\tfrac{1943}{3}\alpha-1210,\\
\Sigma\mathrm{A}^2\mathrm{A}'+\Sigma\alpha^2\alpha' &=\beta^2(-\tfrac{1}{6})+\beta(-\tfrac{2}{3}\alpha^2+9\alpha-\tfrac{121}{6})-29\alpha+98;
\end{aligned}$$

and then also

$$\begin{aligned}
&\Sigma\alpha(\Sigma\mathrm{A}+\Sigma\alpha) =\beta^2+\beta(-\tfrac{1}{2}\alpha^2+\tfrac{9}{2}\alpha-26)+4\alpha^2-36\alpha+144,\\
&(\Sigma\mathrm{AB}-\Sigma\alpha\beta)+\Sigma\alpha(\Sigma\mathrm{A}+\Sigma\alpha) =\beta^2(\tfrac{1}{2})+\beta(-\tfrac{1}{2}\alpha-\tfrac{7}{2})+18,\\
&\{(\Sigma\mathrm{AB}-\Sigma\alpha\beta)+\Sigma\alpha(\Sigma\mathrm{A}+\Sigma\alpha)\}(\Sigma\mathrm{A}'+\Sigma\alpha')=\\
&\qquad\beta^3(\tfrac{1}{2})+\beta^2(-2\alpha+\tfrac{1}{2})+\beta(\tfrac{3}{2}\alpha^2+\tfrac{13}{2}\alpha-10)-54\alpha+144,\\
&-(\Sigma\mathrm{A}+\Sigma\alpha)(\Sigma\mathrm{AA}'-\Sigma\alpha\alpha')=\\
&\qquad\beta^2(\tfrac{1}{2}\alpha)+\beta(-13\alpha+28)+72\alpha-224,
\end{aligned}$$

and

$$\begin{aligned}
&\Sigma\mathrm{A}^2\mathrm{A}'+\Sigma\alpha^2\alpha'=(\textit{ut suprà})\\
&\qquad\beta^2(-\tfrac{1}{6})+\beta(-\tfrac{2}{3}\alpha^2+9\alpha-\tfrac{121}{6})-29\alpha+98;
\end{aligned}$$

whence, adding the last three expressions, we find

$$\text{Weight}=\beta^3(\tfrac{1}{2})+\beta^2(-\tfrac{3}{2}\alpha+\tfrac{1}{3})+\beta(\tfrac{5}{6}\alpha^2+\tfrac{5}{2}\alpha-\tfrac{13}{6})-11\alpha+18;$$

and for the order we have

$$\begin{aligned}
(\Sigma\alpha)^2-\Sigma\alpha\beta= &\ \beta^2(\tfrac{1}{2})+\beta(-\tfrac{1}{2}\alpha^2+4\alpha-\tfrac{27}{2})\\
&+\tfrac{1}{3}\alpha^4-\tfrac{25}{12}\alpha^3+\tfrac{135}{8}\alpha^2-\tfrac{851}{12}\alpha+133;
\end{aligned}$$

and then

$$\begin{aligned}
&\Sigma\mathrm{ABC}+\Sigma\alpha\beta\gamma=(\textit{ut suprà})\\
&\qquad\beta^3(\tfrac{1}{6})+\beta^2(-\tfrac{1}{4}\alpha^2+\tfrac{11}{4}\alpha-\tfrac{41}{3})+\beta(\tfrac{1}{8}\alpha^4-\tfrac{8}{3}\alpha^3+\tfrac{629}{24}\alpha^2-\tfrac{749}{6}\alpha+\tfrac{1753}{6})\\
&\qquad\quad-\alpha^4+\tfrac{58}{3}\alpha^3-162\alpha^2+\tfrac{1943}{3}\alpha-1210,\\
&(\Sigma\mathrm{AB}-\Sigma\alpha\beta)\Sigma\alpha=\\
&\qquad\beta^3(-\tfrac{1}{2})+\beta^2(\tfrac{3}{4}\alpha^2-\tfrac{29}{4}\alpha+\tfrac{63}{2})+\beta(-\tfrac{1}{4}\alpha^4+\tfrac{19}{4}\alpha^3-\tfrac{187}{4}\alpha^2+\tfrac{909}{4}\alpha-531)\\
&\qquad\quad+2\alpha^4-36\alpha^3+297\alpha^2-1215\alpha+2268,\\
&\big((\Sigma\alpha)^2-\Sigma\alpha\beta\big)(\Sigma\mathrm{A}+\Sigma\alpha)=\\
&\qquad\beta^3(\tfrac{1}{2})+\beta^2(-\tfrac{1}{2}\alpha^2+4\alpha-\tfrac{35}{2})+\beta(\tfrac{1}{8}\alpha^4-\tfrac{25}{12}\alpha^3+\tfrac{167}{8}\alpha^2-\tfrac{1235}{12}\alpha+241)\\
&\qquad\quad-\alpha^4+\tfrac{50}{3}\alpha^3-135\alpha^2+\tfrac{1702}{3}\alpha-1064;
\end{aligned}$$

whence, adding these three expressions,

$$\text{Order}=\beta^3(\tfrac{1}{6})+\beta^2(-\tfrac{1}{2}\alpha+\tfrac{1}{3})+\beta(\tfrac{1}{3}\alpha^2-\tfrac{1}{2}\alpha+\tfrac{13}{6})-6\,;$$

and by means of the foregoing expression for the weight, we then have

$$\text{Weight}-\text{Order}=\beta^3(\tfrac{1}{3})+\beta^2(-\alpha\qquad)+\beta(\tfrac{1}{2}\alpha^2+3\alpha-\tfrac{13}{3})-11\alpha+24\,;$$

and therefore

$$\begin{aligned}\text{G}(m^4)&=\tfrac{1}{4}\beta\times(\text{Weight}-\text{Order})\\&=\tfrac{1}{24}\beta\{2\beta^3+\beta^2(-6\alpha)+\beta(3\alpha^2+18\alpha-26)-66\alpha+144\},\end{aligned}$$

which is right.

Annex No. 4.—*Order of* Torse (m, n) (referred to, Art. 44).

We have to find the order of the developable or Torse generated by a line meeting two curves of the orders m, n respectively; viz. representing by μ, ν the classes of the two curves respectively, it is to be shown that the expression for the Order is

$$\text{Torse }(m, n)=m\nu+n\mu.$$

I remark, in the first place, that, given two surfaces of the orders p and q respectively, the curve of intersection is of the order pq and class $pq(p+q-2)$, or as this may be written, class $=qp(p-1)+pq(q-1)$. Reciprocally for two surfaces of the classes p and q respectively, the Torse enveloped by their common tangent planes is of the class pq and order $qp(p-1)+pq(q-1)$. Now, in the same way that a surface of the order p may degenerate into a Torse of the order p, so a surface of the class p may degenerate into a curve of the class p; and the class of a curve being p, then (disregarding singularities) its order is $=p(p-1)$. Hence replacing p and $p(p-1)$ by μ and m respectively, and in like manner q and $q(q-1)$ by ν and n respectively, we have $m\nu+n\mu$ as the order of the Torse generated by the tangent planes of the curves of the orders m and n respectively; where by tangent plane of a curve is to be understood a plane passing through a tangent line of the curve. The intersection of two consecutive tangent planes is a line meeting the two curves, which line is the generating line of the Torse, and such Torse is therefore the Torse (m, n) in question.

The foregoing investigation is not very satisfactory, but I confirm it by considering the case of two plane curves, orders m and n, and classes μ and ν, respectively. The tangents of the two curves can, it is clear, only meet on the line of intersection of the planes of the curves; and the construction of the Torse is in fact as follows: from any point of the line of intersection draw a tangent to m and a tangent to n, then the line joining the points of contact of these tangents is a generating line of the Torse. The order of the Torse is equal to the number of generating lines which meet an arbitrary line; and taking for the arbitrary line the line of intersection of the two planes, it is easy to see that the only generating lines which meet the line of intersection are those for which one of the points of contact lies on the line of intersection; that is, they are the generating lines derived from the points in which the line of intersection meets one or other of the two curves; they are therefore in fact the tangents drawn to the curve n from the points in which the line of intersection meets the curve m, together with the tangents drawn to the curve m from the points in which the line of intersection meets the curve n. Now the line meets the curve n in n points, and from each of these there are μ tangents to the curve m; and it meets the curve m in m points, and from each of these there are ν tangents to the curve n; hence the entire number of the tangents in question is $=n\mu+m\nu$, which confirms the theorem.

Annex No. 5.—*Order of* Torse (m^2) (referred to, Art. 46).

We have here to find the order of the developable or Torse generated by a line meeting a curve of the order m twice, viz., the class of the curve being μ, it is to be shown that we have

$$\text{Torse } (m^2)=(m-3)\mu.$$

I deduce the expression from the formula given p. 424 of Dr. SALMON's 'Geometry of Three Dimensions;' viz. putting in his formula $\beta=0$, and μ for his r, we have

$$\text{Order}=m(\mu-4)-\tfrac{1}{2}\alpha=m\mu-(4m+\tfrac{1}{2}\alpha),$$

where (see p. 234 *et seq.*)

$$\mu=m(m-1)-2h,$$
$$\tfrac{1}{2}\alpha=(n-m)=3m(m-2)-6h-m,$$

and thence

$$3\mu-\tfrac{1}{2}\alpha=4m, \text{ or } 4m+\tfrac{1}{2}\alpha=3\mu,$$

so that we have

$$\text{Order }=(m-3)\mu.$$

A more complete discussion of the Torses (m, n) and (m^2) is obviously desirable; but as they are only incidentally connected with the subject of the present memoir, I have contented myself with obtaining the required results in the way which most readily presented itself.

XXII. *On the Differential Equations of Dynamics. A sequel to a Paper on Simultaneous Differential Equations.*

By George Boole, *F.R.S., Professor of Mathematics in Queen's College, Cork.*

Received December 22, 1862,—Read January 22, 1863.

Jacobi, in a posthumous memoir* which has only this year appeared, has developed two remarkable methods (agreeing in their general character, but differing in details) of solving non-linear partial differential equations of the first order, and has applied them in connexion with that theory of the Differential Equations of Dynamics which was established by Sir W. R. Hamilton in the Philosophical Transactions for 1834–35. The knowledge, indeed, that the solution of the equations of a dynamical problem is involved in the discovery of a single central function, defined by a single partial differential equation of the first order, does not appear to have been hitherto (perhaps it will never be) very fruitful in practical results. But in the order of those speculative truths which enable us to perceive unity where it was unperceived before, its place is a high and enduring one.

Given a system of dynamical equations, it is possible, as Jacobi had shown, to construct a partial differential equation such that from any complete primitive of that equation, *i. e.* from any solution of it involving a number of constants equal to the number of the independent variables, all the integrals of the dynamical equations can be deduced by processes of differentiation. Hitherto, however, the discovery of the complete primitive of a partial differential equation has been supposed to require a previous knowledge of the integrals of a certain auxiliary system of ordinary differential equations; and in the case under consideration that auxiliary system consisted of the dynamical equations themselves. Jacobi's new methods do not require the preliminary integration of the auxiliary system. They require, instead of this, the solution of certain systems of simultaneous linear partial differential equations. To this object therefore the method developed in my recent paper on Simultaneous Differential Equations† might be applied. But the systems of equations in question are of a peculiar form. They admit, in consequence of this, of a peculiar analysis. And Jacobi's methods of solving them are in fact different from the one given by me, though connected with it by remarkable relations. He does indeed refer to the general problem of the solution of simultaneous partial differential equations, and this in language which does not even suppose the condition of linearity. He says, "Non ego hic immorabor quæstioni generali quando et quomodo duabus compluribusve æquationibus differentialibus partialibus una eademque functione satisfieri possit, sed ad casum propositum investigationem restringam. Quippe quo præclaris uti licet artificiis ad integrationem expediendam commodis." But he does not, as far as I have been able to discover, discuss any systems of equations more general than those which arise in the immediate problem before him.

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* Nova methodus æquationes differentiales partiales primi ordinis inter numerum variabilium quemcunque propositas integrandi (Crelle's Journal, Band lx. p. 1).

† Philosophical Transactions for 1862.

It is only very lately that I have come to understand the nature of the relation between the general method of solving simultaneous partial differential equations, published in my recent memoir, and the particular methods of JACOBI. But in arriving at this knowledge I have been led to perceive how by a combination of my own method with one of those of JACOBI, the problem may be solved in a new and perhaps better, certainly a remarkable way. This new way forms the subject of the present paper*. Before proceeding to explain it, it will be necessary to describe JACOBI'S methods, to refer to my own already published, and to point out the nature of the connexion between them.

The system of linear partial differential equations being given, and it being required to find a simultaneous solution of them, JACOBI, according to his first method, transforms these equations by a change of variables; he directs that an integral of the first equation of the system be found; he shows that, in virtue of the form of the equations and the relation which connects the first and second of them, other integrals of the first equation may be derived by mere processes of differentiation from the integral already found; and he shows how, by means of such integrals of the first equation, a *common* integral of the first and second equations of the system may be found. This common integral is a function of the above integrals of the first equation, and of certain variables, and its form is obtained by the solution of a differential equation between two variables—a differential equation which is in general non-linear, and of an order equal to the total number of integrals previously found.

An integral of the first two equations of the given system having been obtained, JACOBI shows that by a second process of derivation, followed by the solution of a second differential equation, an integral which will satisfy simultaneously the first three equations of the system may be found; and thus he proceeds by alternate processes of derivation and integration till an integral satisfying all the equations of the given system together is obtained. In these alternations, it is the function of the processes of derivation to give new integrals of the equations already satisfied; it is the function of the processes of integration to determine the functional forms by which the remaining equations may in their turn be satisfied.

JACOBI'S second method does not require a preliminary transformation of the equations; but the process of derivation, by which from an integral of the first equation other integrals are derived by virtue of the relation connecting the first and second equations, is carried further than in his first method. It is indeed carried on until no new integrals arise. The difference of result is, that the common integral of the first and second partial differential equations is determined as a function solely of the integrals known, and not as a mixed function of integrals and variables. But its form is determined, as before, by the solution of a differential equation. All the subsequent processes of derivation and integration are of a similar nature.

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* It was stated by me, but without demonstration, at the Meeting of the British Association in Cambridge in October of the present year (1862).

On the other hand, the method of my former paper applied to the same problem leads, by a certain process of derivation, to a system of ordinary differential equations equal in number to the number of possible integrals, and, without being individually exact, susceptible of combination into exact differential equations. The integration of these would give all the common integrals of the given system.

All these methods possess, with reference to the requirements of the actual case, a superfluous generality. A single common integral of the system is all that is required.

Now the chief result to be established in this paper is the following:—

If, with JACOBI, according to his second method, we suppose one integral of the untransformed first partial differential equation to be found, if by means of this we construct according to a certain type a new partial differential equation, if to the system thus increased we apply the process of my former paper, continually deriving new partial differential equations until, no more arising, the system is *complete*, then, under a certain condition hereafter to be explained, a common integral of all the equations of the complete system, and therefore of the original system which is contained in it, may be found by the integration of a *single* differential equation susceptible of being made integrable by means of a factor.

When the condition referred to is not satisfied, the results obtained may be applied to the transforming of the original system of equations into an equivalent system of the same character, but containing one equation less than before. To this system we may apply the same process as to the former, and shall arrive at the same final alternation, viz. either the satisfying of the system by a function determined by the solution of a single differential equation susceptible of being made exact by a factor, or the power of reducing it to an equivalent system containing still one equation less. In the most unfavourable case the common integral sought will be ultimately given by the solution of a single final partial differential equation.

The condition in question is grounded on the theoretical connexion which exists between the process of derivation of partial differential equations developed in my former paper, and the process of derivation of integrals involved in JACOBI'S methods. In the actual problem, and in virtue of the peculiar form of the partial differential equations employed, these two processes are coordinate, and it may even be said equivalent. The equations of that problem, if expressed in the symbolical form

$$\Delta_1 P=0; \ \Delta_2 P=0, \ .. \ \Delta_m P=0,$$

satisfy identically the condition

$$(\Delta_i \Delta_j - \Delta_j \Delta_i)P=0.$$

Each of the given equations is moreover of the form

$$\Sigma\left(\frac{dH}{dx_i}\frac{dP}{dp_i}-\frac{dH}{dp_i}\frac{dP}{dx_i}\right)=0,$$

H being a given function of the independent variables $x_1, x_2, .. x_n, p_1, p_2, .. p_n$.

It is usual to represent the first member of the above equation in the form [H, P]. If we adopt this notation, the entire system of equations may be expressed in the form

$$[H_1, P]=0, [H_2, P]=0, .. [H_m, P]=0.$$

Lastly, though this is not a new condition, being already implied in the former ones, the functions $H_1, H_2, .. H_m$ are all common integrals of the system. It is the object of the problem to find a new common integral. With reference to such a system the connexion above referred to is as follows:—If we obtain a new integral K of the first equation of the system, and, associating this with the functions H, form with it a new equation of the same type as the former ones, so that, corresponding to the series of integrals of the first equation

$$H_1, H_2, .. H_m, K,$$

we have the series of partial differential equations

$$[H_1, P]=0, [H_2, P]=0, .. [H_m, P]=0, [K, P]=0,$$

and then if to the former series we apply JACOBI's process for the derivation of integrals, to the latter the process of derivation of partial differential equations of my last paper, carrying each to its fullest extent, the result will be that to each new partial differential equation arising from the one will correspond a new integral (of the first partial differential equation) arising from the other. The theory now to be developed is founded upon the inquiry whether it is possible to satisfy the completed system of partial differential equations by a function of the completed system of the Jacobian integrals, *i. e.* to determine a *common* integral of the completed series of equations as a function of the *completed* series of integrals of the first equation. The reader is reminded that by the completed series of integrals is meant, not all the integrals of the first partial differential equations that exist, but all that arise from a certain root integral by a certain process of derivation, together with the root integral itself. Now the answer here to be established to this inquiry is the following. The first of the partial differential equations necessarily *will*, and others *may*, be satisfied by the proposed function irrespectively of its *form*. If the number of equations of the completed system which is *not* thus satisfied be odd (this is the condition in question), the form of the function which will satisfy all is determinable by the solution of a single differential equation of the first order, capable of being made integrable by means of a factor.

I have entered into some details upon the history of the problem, partly because I believe the theory of simultaneous partial differential equations to be an important one, but partly also in order that I might render that just tribute to the great German mathematician which I was unable to pay before*. JACOBI certainly originated the theory of systems in which the condition

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)\mathrm{P}=0$$

is satisfied. I learn from his distinguished pupil Dr. BORCHARDT, that this subject was fully discussed by him in lectures delivered at Königsberg in 1842–43, my informant having been one of the auditors. The present memoir is but a contribution to that theory. And though it does not appear that JACOBI has discussed the theory of systems not satisfying the above conditions, it is just to observe that the more general theory is at least to this extent contained in the particular one, that the recognition of the above equation as the condition of a mode of integration naturally suggests the inquiry how that equation is to be interpreted when not satisfied as a *condition*; and in the answer to this question the general theory is contained.

PROPOSITION I.

The solution of any non-linear partial differential equation of the first order may be made to depend upon that of certain linear systems of partial differential equations of the first order.

Although what is contained in this proposition is already known, it is presented here for the sake of unity and to avoid inconvenient references.

First, the solution of any partial differential equation is reducible to that of one in which the dependent variable does not explicitly appear.

For let z be the dependent variable, $x_1, x_2, .. x_n$ the independent ones, and $p_1, p_2, .. p_n$ the differential coefficients of z with respect to these. Represent the given equation by

$$f(x_1, x_2, .. x_n, z, p_1, p_2, .. p_n)=0, \quad \ldots \quad (1.)$$

and let

$$\phi(x_1, x_2, .. x_n, z)=0 \quad \ldots \quad (2.)$$

be any relation between the primitive variables which satisfies the given equation.

Differentiating with respect to $x_1, x_2, .. x_n$ respectively, and representing the first member of (2.) by ϕ, we have

$$\frac{d\phi}{dx_1}+p_1\frac{d\phi}{dz}=0, \; .. \; \frac{d\phi}{dx_n}+p_n\frac{d\phi}{dz}=0.$$

Hence determining $p_1, .. p_n$, and substituting in (1.), we have

$$f\left(x_1, x_2, .. x_n, z, \frac{-\frac{d\phi}{dx_1}}{\frac{d\phi}{dz}}, .. \frac{-\frac{d\phi}{dx_n}}{\frac{d\phi}{dz}}\right)=0, \quad \ldots \quad (3.)$$

a partial differential of the first order in which ϕ is the dependent variable, and $x_1, x_2, .. x_n, z$ the independent ones. Here ϕ does not explicitly appear.

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* The results of my former researches were communicated to Dr. SALMON on February 4, 1862. At that time I had not seen JACOBI's researches, which indeed could only just have been published. A note on the connexion of the two which accompanied my paper was cancelled in the proof sheet in the prospect of that fuller explanation which I then hoped to be able to give, and now give.

It will suffice, therefore, to develope the theory of the solution of partial equations not explicitly involving the dependent variable. The general form of such an equation is

$$f(x_1, x_2 \dots x_n, p_1, p_2, \dots p_n)=0.$$

Now, to solve this equation, we must find values of $p_1, p_2, \dots p_n$ as functions of $x_1, x_2 \dots x_n$, which, while satisfying it, will make the equation

$$dz-p_1dx_1-p_2dx_2 \dots -p_ndx_n=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

admit of a single integral containing n arbitrary constants. This integral will constitute a complete primitive, a form of solution from which all other forms can be derived.

Let $H_1=0$ represent the given equation, and

$$H_1=0,\ H_2=c_2,\ \dots\ H_n=c_n$$

the system of equations from which $p_1, p_2, \dots p_n$ are to be thus found. Their values will contain the $n-1$ arbitrary constants $c_2, c_3, \dots c_n$; and the remaining constant will be introduced by the integration of (4.).

Let U and V represent any two of the functions $H_1, H_2, \dots H_n$. Then differentiating the corresponding equations with respect to any one of the independent variables x_i, explicitly as it appears, and implicitly as involved in $p_1, p_2, \dots p_n$, we have

$$\frac{dU}{dx_i}+\Sigma_j \frac{dU}{dp_j}\, \frac{dp_j}{dx_i}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

$$\frac{dV}{dx_i}+\Sigma_j \frac{dV}{dp_j}\, \frac{dp_j}{dx_i}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

the summations extending from $j=1$ to $j=n$.

Multiply the first of these equations by $\frac{dV}{dp_i}$, and sum the result from $i=1$ to $i=n$. Then

$$\Sigma_i \frac{dU}{dx_i}\, \frac{dV}{dp_i}+\Sigma_i\Sigma_j \frac{dU}{dp_j}\, \frac{dV}{dp_i}\, \frac{dp_j}{dx_i}=0;$$

or, since $\frac{dp_j}{dx_i}=\frac{dp_i}{dx_j}$,

$$\Sigma_i \frac{dU}{dx_i}\, \frac{dV}{dp_i}+\Sigma_i\Sigma_j \frac{dU}{dp_j}\, \frac{dV}{dp_i}\, \frac{dp_i}{dx_j}=0.$$

Interchange in the second term i and j, since the limits of summation are the same with respect to both, then

$$\Sigma_i \frac{dU}{dx_i}\, \frac{dV}{dp_i}+\Sigma_j\Sigma_i \frac{dU}{dp_i}\, \frac{dV}{dp_j}\, \frac{dp_j}{dx_i}=0,$$

or

$$\Sigma_i \frac{dU}{dx_i}\, \frac{dV}{dp_i}+\Sigma_i \frac{dU}{dp_i}\, \Sigma_j \frac{dV}{dp_j}\, \frac{dp_j}{dx_i}=0,$$

whence, reducing the second term by (6.),

$$\Sigma_i \left(\frac{dU}{dx_i}\, \frac{dV}{dp_i}-\frac{dU}{dp_i}\, \frac{dV}{dx_i}\right)=0.$$

Representing the first member of this equation in the form [U, V], it appears that the functions H_1, H_2, .. H_n must satisfy mutually the $\frac{n(n-1)}{2}$ equations of which the type is

$$[H_i, H_j]=0;$$

and by these conditions H_2, H_3, .. H_n must be deduced from H_1, which is known*.

Now all these conditions will be satisfied if we determine H_2 to satisfy the single linear partial differential equation

$$[H_1, H_2]=0,$$

then H_3 to satisfy the binary system

$$[H_1, H_3]=0,\ [H_2, H_3]=0,$$

then H_4 to satisfy the ternary system

$$[H_1, H_4]=0,\ [H_2, H_4]=0,\ [H_3, H_4]=0,$$

and finally, H_n to satisfy the system of $n-1$ equations

$$[H_1, H_n]=0,\ [H_2, H_n]=0, \ldots [H_{n-1}, H_n]=0.$$

All these are cases of the general problem of determining a function P to satisfy the simultaneous equations

$$[H_1, P]=0,\ [H_2, P]=0, \ldots [H_m, P]=0, \quad \ldots\ldots \quad (7.)$$

H_1, H_2, .. H_m being known functions mutually satisfying the conditions

$$[H_i, H_j]=0,$$

for each P thus determined gives the succeeding function H_{m+1}, and so on till all are found.

This is the problem with which we are concerned. But before proceeding to its solution we must notice certain properties of the symbolic combination [U, V].

Properties of [U, V].

1st. It is evident from the definition that

$$[U, V]=-[V, U], \quad \ldots\ldots \quad (8.)$$

$$[U, U]=0. \quad \ldots\ldots \quad (9.)$$

2nd. The case sometimes arises in which one of the functions under the symbol [ ] is itself a function of several other functions of the independent variables. Suppose V to be a function of v_1, v_2, .. v_q, then it may be shown that

$$[U, V]=[U, v_1]\frac{dV}{dv_1}+[U, v_2]\frac{dV}{dv_2}\ldots+[U, v_q]\frac{dV}{dv_q}, \quad \ldots\ldots \quad (10.)$$

a theorem of especial use in transformations.

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* On the sufficiency of these conditions see Professor DONKIN's excellent memoir "On a Class of Differential Equations, including those of Dynamics," Philosophical Transactions, 1854.

For

$$[\mathrm{U}, \mathrm{V}]=\Sigma_i\left(\frac{d\mathrm{U}}{dx_i}\frac{d\mathrm{V}}{dp_i}-\frac{d\mathrm{U}}{dp_i}\frac{d\mathrm{V}}{dx_i}\right)$$

$$=\Sigma_i\left\{\frac{d\mathrm{U}}{dx_i}\left(\frac{d\mathrm{V}}{dv_1}\frac{dv_1}{dp_i}+\frac{d\mathrm{V}}{dv_2}\frac{dv_2}{dp_i}\cdots+\frac{d\mathrm{V}}{dv_q}\frac{dv_q}{dp_i}\right)-\frac{d\mathrm{U}}{dp_i}\left(\frac{d\mathrm{V}}{dv_1}\frac{dv_1}{dx_i}+\frac{d\mathrm{V}}{dv_2}\frac{dv_2}{dx_i}\cdots+\frac{d\mathrm{V}}{dv_q}\frac{dv_q}{dx_i}\right)\right\}$$

$$=\frac{d\mathrm{V}}{dv_1}[\mathrm{U}, v_1]+\frac{d\mathrm{V}}{dv_2}[\mathrm{U}, v_2]\cdots+\frac{d\mathrm{V}}{dv_q}[\mathrm{U}, v_q].$$

In like manner, if U be a function of $u_1, u_2, \ldots u_q$, which are themselves functions of the independent variables, then

$$[\mathrm{U}, \mathrm{V}]=[u_1, \mathrm{V}]\frac{d\mathrm{U}}{du_1}+[u_2, \mathrm{V}]\frac{d\mathrm{U}}{du_2}\cdots+[u_q, \mathrm{V}]\frac{d\mathrm{U}}{du_q}. \quad . \quad . \quad . \quad . \quad (11.)$$

3rd. The important theorem

$$[\mathrm{U}, [\mathrm{V}, \mathrm{W}]]+[\mathrm{V}, [\mathrm{W}, \mathrm{U}]]+[\mathrm{W}, [\mathrm{U}, \mathrm{V}]]=0 \quad . \quad . \quad . \quad . \quad . \quad (12.)$$

is implicitly contained in the results of the following Proposition. All these are known relations.

PROPOSITION II.

To determine the result of the application of the general theorem of derivation to any system of partial differential equations of the form

$$[u_1, \mathrm{P}]=0,\ [u_2, \mathrm{P}]=0, \ldots [u_m, \mathrm{P}]=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$u_1, u_2, \ldots u_m$ *being given functions of the independent variables* $x_1, x_2, \ldots x_n, p_1, p_2, \ldots p_n$.

We adopt in the expression of this proposition $u_1, u_2, \ldots u_m$ in the place of $\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_m$, because we suppose the functions given to be unrestrained by connecting conditions.

If we represent any two equations of the system by

$$[\mathrm{U}, \mathrm{P}]=0, \quad [\mathrm{V}, \mathrm{P}]=0,$$

and then give to these the symbolical forms

$$\Delta_i\mathrm{P}=0, \quad \Delta_j\mathrm{P}=0,$$

we shall have

$$\Delta_i=\Sigma_r\left(\frac{d\mathrm{U}}{dx_r}\frac{d}{dp_r}-\frac{d\mathrm{U}}{dp_r}\frac{d}{dx_r}\right),$$

$$\Delta_j=\Sigma_s\left(\frac{d\mathrm{V}}{dx_s}\frac{d}{dp_s}-\frac{d\mathrm{V}}{dp_s}\frac{d}{dx_s}\right),$$

the summations with respect to r and s extending in each case from 1 to n inclusive.

Hence collecting into separate groups the terms which contain differential coefficients of P with respect to p_s, p_r, and with respect to x_s, x_r, we have

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)\mathrm{P}=\Sigma_r\Sigma_s\left\{\frac{d\mathrm{U}}{dx_r}\frac{d^2\mathrm{V}}{dp_rdx_s}\frac{d\mathrm{P}}{dp_s}-\frac{d\mathrm{U}}{dp_r}\frac{d^2\mathrm{V}}{dx_rdx_s}\frac{d\mathrm{P}}{dp_s}-\frac{d\mathrm{V}}{dx_s}\frac{d^2\mathrm{U}}{dp_sdx_r}\frac{d\mathrm{P}}{dp_r}+\frac{d\mathrm{V}}{dp_r}\frac{d^2\mathrm{U}}{dx_sdx_r}\frac{d\mathrm{P}}{dp_r}\right\}$$

$$-\Sigma_r\Sigma_s\left\{\frac{d\mathrm{U}}{dx_r}\frac{d^2\mathrm{V}}{dp_rdp_s}\frac{d\mathrm{P}}{dx_s}-\frac{d\mathrm{U}}{dp_r}\frac{d^2\mathrm{V}}{dx_rdp_s}\frac{d\mathrm{P}}{dx_s}-\frac{d\mathrm{V}}{dx_s}\frac{d^2\mathrm{U}}{dp_sdp_r}\frac{d\mathrm{P}}{dx_r}+\frac{d\mathrm{V}}{dp_s}\frac{d^2\mathrm{U}}{dx_sdp_r}\frac{d\mathrm{P}}{dx_r}\right\}.$$

Now this expression will not be affected if in any of its terms we interchange s and r. If we do this in the terms involving $\frac{dP}{dp_s}$, the first aggregate will become

$$\Sigma_r\Sigma_s\left\{\frac{dU}{dx_s}\frac{d^2V}{dp_sdx_r}\frac{dP}{dp_r}-\frac{dU}{dp_s}\frac{d^2V}{dx_sdx_r}\frac{dP}{dp_r}-\frac{dV}{dx_s}\frac{d^2U}{dp_sdx_r}\frac{dP}{dp_r}+\frac{dV}{dp_s}\frac{d^2U}{dx_sdx_r}\frac{dP}{dp_r}\right\}$$

$$=\Sigma_r\frac{dP}{dp_r}\frac{d}{dx_r}\Sigma_s\left(\frac{dU}{dx_s}\frac{dV}{dp_s}-\frac{dU}{dp_s}\frac{dV}{dx_s}\right)=\Sigma_r\frac{dP}{dp_r}\frac{d[U,V]}{dx_r}.$$

In a similar way the second aggregate of terms will reduce to

$$-\Sigma_r\frac{dP}{dx_r}\frac{d[U,V]}{dp_r}.$$

Hence

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=\Sigma_r\left(\frac{d[U,V]}{dx_r}\frac{dP}{dp_r}-\frac{d[U,V]}{dp_r}\frac{dP}{dx_r}\right).$$

Therefore

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=[[U,V],P]\text{[*]}. \qquad (2.)$$

Thus the theorem of derivation applied to the two equations

$$[U,P]=0,\ [V,P]=0,$$

gives

$$[[U,V],P]=0,$$

an equation of the same general form as the equations from which it was derived.

Apply this to the separate pairs of equations in the given system

$$[u_1,P]=0,\ [u_2,P]=0,\ ..\,[u_m,P]=0,$$

and to the equations thus generated, and so on in succession till no new equations arise, and the result will be a system of the form

$$[u_1,P]=0,\ [u_2,P]=0,\ ..\,[u_q,P]=0,\ \ldots\ q ⁊ m. \qquad (3.)$$

This constitutes the *completed* system, and it possesses, in accordance with the doctrine of my former paper, the property that, if we form the equation

$$\frac{dP}{dx_1}dx_1\,..+\frac{dP}{dx_n}dx_n+\frac{dP}{dp_1}dp_1\,..+\frac{dP}{dp_n}dp_n=0,$$

eliminate thence q of the differential coefficients of P, and equate to 0 the coefficients of the $2n-q$ which remain, we shall obtain a system of $2n-q$ ordinary differential equations susceptible of reduction to the *exact* form, and yielding by integration the common integrals of the system given.

The completed system is one of independent equations in which P is brought into successive relations with a series of functions u_1, u_2, .. u_q. It is important to show that the independence of the equations involves the independence of the functions, as also that if the equations were dependent the functions would be so too.

1st. The independence of the equations involves the independence of the functions.

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[*] This is not a new theorem. It is but another form of the theorem (12.) Prop. I. It has also been explicitly given by JACOBI and CLEBSCH.

For suppose the equations independent and the functions not independent, so that one of them, u_q, could be expressed as a function of the others, $u_1, u_2, \ldots u_{q-1}$. Then by (11.), Prop. 1,

$$[u_q, \mathrm{P}]=[u_1, \mathrm{P}]\frac{du_q}{du_1}+[u_2, \mathrm{P}]\frac{du_q}{du_2}\ldots+[u_{q-1}, \mathrm{P}]\frac{du_q}{du_{q-1}},$$

which would imply that the equation

$$[u_q, \mathrm{P}]=0$$

was not independent of the other equations of the system, as by hypothesis it is. Hence the functions are independent.

2ndly. If the equations of a system of the form (3.) are algebraically dependent, then are the functions $u_1, u_2, \ldots u_q$ dependent.

The algebraic dependence of the equations implies, in consequence of the linearity of their developed forms, the existence of at least one relation of the form

$$[u_q, \mathrm{P}]=\lambda_1[u_1, \mathrm{P}]+\lambda_2[u_2, \mathrm{P}]\ldots+\lambda_{q-1}[u_{q-1}, \mathrm{P}], \quad \ldots\ldots \quad (4.)$$

$\lambda_1, \lambda_2, \ldots \lambda_{q-1}$ being, on the most general supposition, functions of the independent variables.

Now the functions $u_1, u_2, \ldots u_{q-1}$ are either dependent or independent. If dependent, the proposition is granted; if independent, then equating the coefficients of $\frac{d\mathrm{P}}{dx_i}$ in the developed members of (4.), we have

$$\frac{du_q}{dp_i}=\lambda_1\frac{du_1}{dp_i}+\lambda_2\frac{du_2}{dp_i}\ldots+\lambda_{q-1}\frac{du_{q-1}}{dp_i};$$

and equating the coefficients of $\frac{d\mathrm{P}}{dp_i}$ in the same equation, we have

$$\frac{du_q}{dx_i}=\lambda_1\frac{du_1}{dx_i}+\lambda_2\frac{du_2}{dx_i}\ldots+\lambda_{q-1}\frac{du_{q-1}}{dx_i}.$$

Hence if we represent the series of variables $x_1, \ldots x_n, p_1, \ldots p_n$ taken in any order by $y_1, y_2, \ldots y_{2n}$, both the above equations will be included in the general one,

$$\frac{du_q}{dy_i}=\lambda_1\frac{du_1}{dy_i}+\lambda_2\frac{du_2}{dy_i}\ldots+\lambda_{q-1}\frac{du_{q-1}}{dy_i}.$$

Now, $u_1, u_2, \ldots u_{q-1}$ being by hypothesis independent, u_q will be expressible as, at most, a function of $u_1, u_2, \ldots u_{q-1}$ and $2n-q+1$ of the original variables. Regard then u_q as a function of $u_1, u_2, \ldots u_{q-1}, y_q, y_{q+1}, \ldots y_{2n}$. Of course $u_1, u_2, \ldots u_{q-1}$ will be functionally independent with respect to the quantities $y_1, y_2, \ldots y_{q-1}$ which they replace.

Then for all values of i, from 1 to $q-1$ inclusive, the last equation becomes

$$\frac{du_q}{du_1}\frac{du_1}{dy_i}\ldots+\frac{du}{du_{q-1}}\frac{du_{q-1}}{dy_i}=\lambda_1\frac{du_1}{dy_i}\ldots+\lambda_{q-1}\frac{du_{q-1}}{dy_i},$$

or

$$\frac{du_1}{dy_i}\left(\frac{du_q}{du_1}-\lambda_1\right)+\frac{du_2}{dy_i}\left(\frac{du_q}{du_2}-\lambda_2\right)\ldots+\frac{du_{q-1}}{dy_i}\left(\frac{du_q}{du_{q-1}}-\lambda_{q-1}\right)=0, \quad \ldots\ldots \quad (5.)$$

while for all values of i greater than $q-1$ it becomes

$$\frac{du_1}{dy_i}\left(\frac{du_q}{du_1}-\lambda_1\right)+\frac{du_2}{dy_i}\left(\frac{du_q}{du_2}-\lambda_2\right)\,..+\frac{du_{q-1}}{dy_i}\left(\frac{du_q}{du_{q-1}}-\lambda_{q-1}\right)+\frac{du_q}{dy_i}=0. \quad . \quad . \quad . \qquad (6.)$$

From the system of $q-1$ linear equations of the first type (5.), we find

$$\frac{du_q}{du_1}-\lambda_1=0,\ \frac{du_q}{du_2}-\lambda_2=0,\ \ldots\ \frac{du_q}{du_{q-1}}-\lambda_{q-1}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7.)$$

unless the determinant

$$\begin{vmatrix} \frac{du_1}{dy_1}, & \frac{du_2}{dy_1}, & \ldots & \frac{du_{q-1}}{dy_1} \\ \frac{du_1}{dy_2}, & \frac{du_2}{dy_2}, & \ldots & \frac{du_{q-1}}{dy_2} \\ . & . & . & . \\ \frac{du_1}{dy_{q-1}}, & \frac{du_2}{dy_{q-1}}, & \ldots & \frac{du_{q-1}}{dy_{q-1}} \end{vmatrix}$$

vanish identically. But this would, by the known property of determinants, imply that $u_1, u_2, \ldots u_{q-1}$ are not as functions of $y_1, y_2, \ldots y_{q-1}$ independent, which we have seen that they are. Hence the system (7.) is true. Reducing by it the system represented by (6.), we have

$$\frac{du_q}{dy_q}=0,\ \frac{du_q}{dy_{q+1}}=0,\ \frac{du_q}{dy_{2n}}=0.$$

It results therefore that u_q will be simply a function of $u_1, u_2, \ldots u_{q-1}$.

From these conclusions united we see that, if from any system of equations of the form

$$[u_1, \mathrm{P}]=0,\ [u_2, \mathrm{P}]=0,\ \ldots\ [u_m, \mathrm{P}]=0$$

we separate the functions $u_1, u_2, \ldots u_m$, derive from these all possible independent functions of the form $[u_i, u_j]$, and representing these by u_{m+1}, u_{m+2}, &c. continue with the aid of these the process of derivation until no new functions can be formed, then if we represent the completed series of functions by $u_1, u_2, \ldots u_q$, the corresponding system of equations

$$[u_1, \mathrm{P}]=0,\ [u_2, \mathrm{P}]=0, \ldots [u_q, \mathrm{P}]=0$$

will be precisely that system to which the theorem of derivation of my former paper, applied to the given system of equations, would lead.

PROPOSITION III.

To integrate the system of simultaneous partial differential equations

$$[\mathrm{H}_1, \mathrm{P}]=0,\ [\mathrm{H}_2, \mathrm{P}]=0, \ldots [\mathrm{H}_m, \mathrm{P}]=0,$$

it being given as a condition that $\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_m$ *satisfy mutually all relations of the form*

$$[\mathrm{H}_i, \mathrm{H}_j]=0.$$

This we have seen to be the general problem upon the solution of which the integration of any non-linear partial differential equation depends.

We learn by the last proposition that the system of equations given is already a complete system. For, applying the theorem of derivation to any two equations contained in it, we have a result of the form

$$[[H_i, H_j]P]=0;$$

but this is identically satisfied by virtue of the connected condition.

Instead, however, of solving the equation as a complete system, let us, with JACOBI, deduce a new integral of the first partial differential equation of the system, *i. e.* an integral distinct from H_1, H_2, ... H_m, which, in virtue of the condition given, are already integrals of that equation—in fact, common integrals of the system. For if we make $P=H_i$ in any of the equations, that equation will be identically satisfied.

Represent by

$$u=c$$

this new integral. It may happen that it proves on trial to satisfy all the other equations. In that case a common integral is found and the problem is solved. Suppose, however, the new integral of the first equation not to be a common integral, and, constructing the equation $[u, P]=0$, incorporate it with the given system of equations so as to form the larger system

$$[H_1, P]=0,\ [H_2, P]=0, \ldots [H_m, P]=0,\ [u, P]=0. \quad . \quad . \quad . \quad . \quad (1.)$$

Any common integral of this system will be also a common integral of the given system which is contained in this one.

Such common integral, if it exist, we propose to seek.

First let us complete the system just formed by the last proposition. The completed system will be of the form

$$[u_1, P]=0,\ [u_2, P]=0, \ldots [u_q, P]=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

in which u_1, u_2, ... u_{m+1} are for symmetry employed to represent H_1, H_2, ... H_m, u, and u_{m+2}, u_{m+3}, ... u_q are new functions.

Now all the functions u_1, u_2, ... u_q are integrals of the first partial differential equation of the system; for, the system (2.) being a formal consequence of the system (1.), if we substitute H_1 for P, the system

$$[u_1, H_1]=0,\ [u_2, H_1]=0, \ldots [u_q, H_1]=0$$

will be seen to be a consequence of the system

$$[H_1, H_1]=0,\ [H_2, H_1]=0, \ldots [u, H_1]=0.$$

But the latter system is true, therefore the former; therefore, since $[u_i, H_1]=-[H_1, u_i]$, the system

$$[H_1, u_1]=0,\ [H_1, u_2]=0,\ [H_1, u_q]=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

is true; therefore u_1, u_2, ... u_q are integrals of the first equation of the system. They are independent, Prop. II. And as the process by which such of them as are new have been formed is identical in character with that which Jacobi makes use of, differing only in the extent to which it is here applied, I shall speak of them as Jacobian integrals, or Jacobian functions.

Let us inquire whether it is possible to satisfy the completed system of equations (2.) by a function of the completed series of Jacobian integrals of the first equation of the system.

Suppose, then, P a function of $u_1, u_2, \dots u_q$. The equation $[u_i, P]=0$, which is the type of the equations of the system to be integrated, assumes, by (10.) Prop. I., the forms

$$[u_i, u_1]\frac{dP}{du_1}+[u_i, u_2]\frac{dP}{du_2}\cdots+[u_i, u_q]\frac{dP}{du_q}=0. \quad . \; . \; . \; . \; . \; . \quad (4.)$$

Hence since by (3.)

$$[H_1, u_i]=[u_1, u_i]=0,$$

the first equation of the system is identically satisfied, as it ought to be. Let us, to make the problem more definite, suppose for the present that none of the other equations are identically satisfied. Then, giving to i the successive values 2, 3, ... q, and observing that always

$$[u_i, u_i]=0, \quad [u_i, u_1]=0,$$

we see that the system of equations represented by (4.) will be

$$\left.\begin{array}{lllll}
* & +[u_2, u_3]\frac{dP}{du_3} & +[u_2, u_4]\frac{dP}{du_4}\cdots & +[u_2, u_q]\frac{dP}{du_q} & =0, \\
[u_3, u_2]\frac{dP}{du_2} & * & +[u_3, u_4]\frac{dP}{du_4}\cdots & +[u_3, u_q]\frac{dP}{du_q} & =0, \\
[u_4, u_2]\frac{dP}{du_2} & +[u_4, u_3]\frac{dP}{du_3} & * \quad \cdots & +[u_4, u_q]\frac{dP}{du_q} & =0, \\
\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\
[u_q, u_2]\frac{dP}{du_2} & +[u_q, u_3]\frac{dP}{du_3} & +[u_q, u_4]\frac{dP}{du_4}\cdots & * & =0.
\end{array}\right\} \quad . \; . \; . \; . \; . \quad (5.)$$

Now the coefficients $[u_i, u_j]$ are on the most general supposition functions of $u_1, u_2, \dots u_q$, since the system (2.) is complete. Hence, if from any two equations $[u_i, P]=0$, $[u_j, P]=0$, we derive an equation

$$[[u_i, u_j]P]=0,$$

that equation will be algebraically dependent on the equations of the system, and therefore, by Prop. II., $[u_i, u_j]$ will be functionally dependent on $u_1, u_2, \dots u_q$.

The system (5.) is then one of partial differential equations, in which $u_1, u_2, \dots u_q$ are the sole independent variables. If we express that system symbolically in the form

$$\Delta_2P=0, \; \Delta_3P=0, \dots \Delta_qP=0,$$

all equations of the form

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$$

derived from it will be simply what the corresponding equations derived from (2.), when $x_1 \dots x_n$, $p_1 \dots p_n$ are the independent variables, would become under the limitation that P is a function of $u_1, u_2, \dots u_q$. They are therefore not new equations, but combinations of the old ones.

It follows, therefore, that if from the equation

$$\frac{dP}{du_1}du_1+\frac{dP}{du_2}du_2\dots+\frac{dP}{du_q}du_q=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

we eliminate, by means of the system (5.), as many as possible of the differential coefficients of P, and equate to 0 the coefficients of the remaining ones, we shall obtain a system of differential equations of the first order, reducible to the exact form, and giving on integration all the common integrals of the original system which are expressible as functions of $u_1, u_2, \dots u_q$.

First let us suppose the number of equations in the system (5.) to be odd. Then are those equations not independent. For the determinant of the system is

$$\begin{vmatrix} * & [u_2,u_3] & [u_2,u_4] & .. & [u_2,u_q] \\ [u_3,u_2] & * & [u_3,u_4] & .. & [u_3,u_q] \\ [u_4,u_2] & [u_4,u_3] & * & .. & [u_4,u_q] \\ . & . & . & . & . \\ [u_q,u_2] & [u_q,u_3] & [u_q,u_4] & .. & * \end{vmatrix}$$

and this, since $[u_i,u_j]=-[u_j,u_i]$, belongs to the class of symmetrical skew determinants, of which it is a known property that when the number of rows or columns is odd the determinant vanishes; and this, by another known theorem, indicates that one of the corresponding linear equations is dependent. The system (5.) is therefore in general equivalent to a system of $q-2$ independent equations determining the ratios of the $q-1$ differential coefficients $\frac{dP}{du_2}, \frac{dP}{du_3}, \cdots \frac{dP}{du_q}$ in the form

$$\frac{\frac{dP}{du_2}}{U_2}=\frac{\frac{dP}{du_3}}{U_3}\dots=\frac{\frac{dP}{du_q}}{U_q},$$

$U_2, U_3, \dots U_q$ being known functions of $u_1, u_2, \dots u_q$. Eliminating by these $q-2$ equations the $q-2$ differential coefficients $\frac{dP}{du_3}, \frac{dP}{du_4}, \dots \frac{dP}{du_q}$ from (6.), we have

$$\frac{dP}{du_1}du_1+\frac{dP}{du_2}\times\frac{1}{U_2}(U_2du_2+U_3du_3\dots+U_qdu_q)=0,$$

whence, equating to 0 the coefficients of $\frac{dP}{du_1}$ and $\frac{dP}{du_2}$, we have

$$du_1=0,$$

$$U_2du_2+U_3du_3\dots+U_qdu_q=0.$$

The first of these gives

$$u_1=c,$$

a common integral already known. The second equation being reducible by means of the first to the exact form, it follows that, if in that equation we regard u_1 as constant, it will admit of an integral of the form

$$f(u_1, u_2, \ldots u_q)=\text{const.};$$

and this is the new common integral sought.

We supposed that only the first of the differential equations of the system (2.) was satisfied independently of the *form* of P as a function of u_1, u_2, .. u_q. Suppose, however, that any number of equations

$$[u_a, \mathrm{P}]=0, \quad [u_b, \mathrm{P}]=0, \ \ldots$$

are identically satisfied, independently of the form of P. Then

$$u_a=c, \quad u_b=c', \ldots$$

are common integrals of the system; and if any one of these be not contained in the series of known common integrals,

$$u_1=c_1, \ u_2=c_2, \ldots u_m=c_m,$$

it will be a new common integral, and the problem is solved. But whether such is or is not the case, if the number of equations

$$[u_\alpha \mathrm{P}]=0, \quad [u_\beta \mathrm{P}]=0, \ldots$$

not identically satisfied be odd, we shall, proceeding as before, arrive at an equation

$$\frac{d\mathrm{P}}{du_a}\,du_a+\frac{d\mathrm{P}}{du_b}\,du_b\ldots+\frac{d\mathrm{P}}{du_\alpha}\times\frac{1}{\mathrm{U}_\alpha}\,(\mathrm{U}_\alpha du_\alpha+\mathrm{U}_\beta du_\beta\ldots)=0,$$

resolvable therefore into

$$du_a=0, \ du_b=0, \ldots$$

$$\mathrm{U}_\alpha du_\alpha+\mathrm{U}_\beta du_\beta, \ldots=0.$$

Then obtaining from the first line of this system the already known integrals

$$u_\alpha=c, \quad u_\beta=c',$$

we shall be able to reduce by these the equation of the second line to an integrable form, and thence obtain the common integral sought.

Generally, then, when the number of equations of the completed system

$$[u_1, \mathrm{P}]=0, \ [u_2, \mathrm{P}]=0, \ldots [u_q, \mathrm{P}]=0$$

which, on the supposition that P is a function of u_1, u_2, .. u_q, is not satisfied independently of the *form* of that function is odd, the form which will satisfy all is determinable by the solution of a single differential equation of the first order*.

June 27, 1863. Secondly, suppose the number of equations of the above system not satisfied independently of the *form* of P as a function of u_1, u_2, .. u_q to be even.

In this case no form can generally be assigned to that function that will cause all the equations of the system to be satisfied.

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* Communicated to the British Association in October 1862,—without demonstration.

But neither is it required that all should be satisfied. It is only necessary to satisfy the original system,

$$[u_1, \mathrm{P}]=0,\ [u_2, \mathrm{P}]=0, \ldots [u_m, \mathrm{P}]=0,$$

which is also a complete system.

If in this system we make P a function of $u_1, u_2, \ldots u_q$, we get

$$\begin{array}{llll}
* & +[u_2, u_3]\dfrac{d\mathrm{P}}{du_3}+[u_2, u_4]\dfrac{d\mathrm{P}}{du_4}\ldots+[u_2, u_q]\dfrac{d\mathrm{P}}{du_q}=0, \\
(u_3, u_2)\dfrac{d\mathrm{P}}{du_2} & \quad * \quad +[u_3, u_4]\dfrac{d\mathrm{P}}{du_4}\ldots+[u_3, u_q]\dfrac{d\mathrm{P}}{du_q}=0, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
[u_m, u_2]\dfrac{d\mathrm{P}}{du_2} & \quad \ldots \ * \ \ldots +[u_m, u_q]\dfrac{d\mathrm{P}}{du_q}=0,
\end{array}$$

in which the coefficients $[u_i, u_j]$ are all functions of $u_1, u_2, \ldots u_q$, and in the integration of which u_1 is to be regarded as a constant. I shall prove that to this system of $m-1$ partial differential equations, the same treatment may be applied as to the original system of m partial differential equations.

Let v be any new integral of the first equation of the above system. As such, v will be a function of $u_1, u_2, \ldots u_q$, and will therefore be also an integral of the prior equation

$$[u_1, \mathrm{P}]=0.$$

Forming the equation $[v_1, \mathrm{P}]=0$, incorporate it with the original system, so as to form the larger system

$$[\mathrm{H}_1, \mathrm{P}]=0,\ [\mathrm{H}_2, \mathrm{P}]=0, \ldots [\mathrm{H}_m, \mathrm{P}]=0,\ [v_1, \mathrm{P}]=0,$$

and, expressing $\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_m, v_1$ as functions of the original variables, complete the system by the theorem of derivation. Let the result be

$$[v_1, \mathrm{P}]=0,\ [v_2, \mathrm{P}]=0, \ldots [v_r, \mathrm{P}]=0,$$

in which $v_1, v_2, \ldots v_{m+1}$ are identical with $\mathrm{H}_1, \mathrm{H}_2, \ldots \mathrm{H}_m, v_1$, and let it be sought to satisfy this system by regarding P as a function $v_1, v_2, \ldots v_r$. The two first equations will be satisfied identically; the others will take the form

$$\begin{array}{llll}
* & +[v_3, v_4]\dfrac{d\mathrm{P}}{dv_4}\ldots+[v_3, v_r]\dfrac{d\mathrm{P}}{dv_r}=0, \\
[v_4, v_3]\dfrac{d\mathrm{P}}{dv_3} & \quad * \ldots+[v_4, v_r]\dfrac{d\mathrm{P}}{dv_r}=0, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
[v_r, v_3]\dfrac{d\mathrm{P}}{dv_3}+[v_r, v_4]\dfrac{d\mathrm{P}}{dv_4} & \ldots\ldots\ldots \quad * =0;
\end{array}$$

and if the number of these equations, or more generally the number of these equations not identically satisfied, be odd, a common integral will be found by the solution of a single differential equation of the first order, of the form

$$\mathrm{V}_3dv_3+\mathrm{V}_4dv_4\ldots+\mathrm{V}_rdv_r=0.$$

In solving that equation v_1, v_2 are to be regarded as constant. If the number be even, we must proceed as before, and we shall thus reduce the original system to a system of the same character, but possessing only $m-2$ equations. In the most unfavourable case, the emerging systems being always odd, the common integral will ultimately be found by the solution of a single final partial differential equation.

I have supposed that, of the symmetrical equations which arise from the introduction of the Jacobian integrals as independent variables, only one is dependent when the number of equations is odd, and none when even. But exceptions may conceivably arise from the splitting of the determinant into component factors each of the skew-symmetrical form, and the corresponding resolution of the system of equations into partial systems each complete in itself. To such partial systems, and not to the general system, the law is to be applied. The connexion of the common integral with the odd skew determinant does not suffer exception even in those cases in which the integral is obtained without a final integration, or is primarily given. Thus for each of the common integrals u_1, u_2, .. u_m the determinant reduces to a single vanishing term on the diagonal. The possibility of cases of real exception seems to be a subject well worthy of inquiry.

POSTSCRIPT.—September 24, 1863.

Since communicating the above I have discovered that the number of independent equations of the final symmetrical system (5.) is necessarily even. This confirms the foregoing observations. It follows that whether we take all or some of the Jacobian functions u_1, u_2, .. u_q, if there exist one common integral of the system expressible by means of those functions, the determinant of the system will either be or will contain as a component factor an odd symmetrical skew determinant.

XXIII. *On the Nature of the Sun's Magnetic Action upon the Earth.*

By CHARLES CHAMBERS. *Communicated by General* SABINE, *P.R.S.*

Received April 30,—Read May 21, 1863.

1. IN attempting to frame a theory which shall account for the relations which have been shown to exist between the variations of terrestrial magnetism and the position of the sun with respect to the place of observation on the earth's surface, the following question presents itself for consideration at an early stage of the inquiry, "Are the magnetic effects produced on the earth such as could be explained by the simple supposition that the sun is a great magnet, or not?" The solution of this question will, to a certain extent, limit the range of probable sources from which to seek the true cause of magnetic variations, and is therefore worthy of attention.

2. In the first place, let us endeavour to find the *law of the diurnal variations* of the Declination, Horizontal Force, and Vertical Force at a given place on the earth's surface, on the supposition that these variations arise from the varying relations, as to position, of the sun acting as a magnet upon the earth.

3. The sun would affect the magnets used for showing the earth's changes of force in two ways—first directly, and secondly by inducing magnetism in the soft iron and other inductive matter forming part of the body of the earth, this induced magnetism reacting upon the observed magnets.

4. Now this subject has been discussed mathematically by POISSON in the case of masses of soft iron having any possible arrangement, and he has given expressions for the combined effect of the direct and induced forces upon the magnets, making only this restriction, which is allowable in the case under consideration, that the length of the magnets must be infinitesimally small in comparison with their distance from the nearest particle of iron. The expressions are as follows:—

$$X'=(1+a)X+bY+cZ, \quad \ldots \quad (a.)$$

$$Y'=dX+(1+e)Y+fZ, \quad \ldots \quad (b.)$$

$$Z'=gX+hY+(1+k)Z, \quad \ldots \quad (c.)$$

where, assuming the direction and intensity of the magnetic force exerted directly by the sun upon the centre of the earth to be approximate values of the same elements throughout the body of the earth, X represents the resolved part of that force along the perpendicular to the earth's axis from the place of observation, being reckoned positive when the resolved force acting alone would make the north end of a magnet at the station point towards the earth's axis; Y is the resolved part of the same force perpendicular to the meridian plane, the positive direction being to eastward; Z represents the part of the force resolved along the earth's axis, plus to northward, and a, b, c, d, e, f, g, h, and k are constants which depend on the amount and arrangement of the soft iron of the earth. X′, Y′, and Z′ are the combined effects of the direct force of the sun, and of the reaction of the magnetism which it induces in the earth, upon the magnets at the place of observation, in the directions of X, Y, and Z respectively.

POISSON's equations* may be derived from the following considerations. The inducing force X produces a certain magnetic state in the mass of iron, and the magnetism thus induced (including that which is caused by the inducing action of the particles of iron upon each other) will exert an attraction at the place of observation in the direction of each of the coordinate axes, which attraction is assumed to be proportional to the inducing force X; the forces due to X in the directions of X, Y, and Z respectively will therefore be aX, dX, and gX, and those due to Y, bY, eY, hY; those due to Z, cZ, fZ, kZ. Now, adding the forces in the same directions and including the forces X, Y, and Z, which act directly upon the magnets at the place of observation, we get

$$X'=(1+a)X+bY+cZ,$$
$$Y'=dX+(1+e)Y+fZ,$$
$$Z'=gX+hY+(1+k)Z.$$

5. In order that POISSON'S formulæ should be strictly applicable to the case under consideration, we must assume that the whole effect of an inducing force acting upon the earth requires only a few minutes to be acquired, and that it is lost with equal facility on the removal of the inducing force. It is not necessary that the effects of gain and loss should be momentary.

6. We must now transform the coordinates of POISSON'S equations to those directions in which the variations of the earth's magnetism are always observed, viz. towards the *magnetic* east (x), towards the *magnetic* north (y), and vertically downwards (z), the forces being reckoned positive when, acting alone, they would make the north end of a magnet point in the directions named, and we get a new set of equations,

$$x=AX+BY+CZ, \quad \ldots\ldots \quad (d.)$$
$$y=DX+EY+FZ, \quad \ldots\ldots \quad (e.)$$
$$z=GX+HY+KZ, \quad \ldots\ldots \quad (f.)$$

in which A, B, C, ... K are constant numbers, being functions of a, b, c, ... k, and of the magnetic declination and latitude of the place of observation.

7. Let us observe that, during the time in which the earth is making a single revolution upon its axis, the variations in direction and amount of Z and of R (the resultant of X and Y) may be disregarded, while the direction and value of X and Y vary with the hour-angle (h) of the sun, and may be expressed as follows:—

$$X=R\cos(h+\alpha), \quad \ldots\ldots \quad (g.)$$
$$Y=R\sin(h+\alpha), \quad \ldots\ldots \quad (h.)$$

$$\text{where } \tan\alpha=\frac{Y_0}{X_0}, \quad \ldots \quad (i.) \qquad \text{and } R^2=X_0^2+Y_0^2, \quad \ldots \quad (k.)$$

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* This mode of obtaining POISSON'S formulæ was suggested by Mr. ARCHIBALD SMITH, F.R.S.

X_0 and Y_0 being the values of X and Y *at noon.* It is evident that, Z being considered constant, the last terms of (*d.*), (*e.*), (*f.*) form no part of the diurnal variations, and may be neglected; omitting them therefore and substituting the values of X and Y given by (*g.*) and (*h.*), the equations (*d.*), (*e.*), and (*f.*) become

$$\left.\begin{aligned} x'_h &= \mathrm{R}\{\mathrm{A}\cos(h+\alpha)+\mathrm{B}\sin(h+\alpha)\} \\ &= \mathrm{R}\sqrt{\mathrm{A}^2+\mathrm{B}^2}\sin\{(h+\alpha)+\beta\}, \\ &\quad \text{where } \tan\beta=\frac{\mathrm{A}}{\mathrm{B}}; \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (l.)$$

$$\left.\begin{aligned} y'_h &= \mathrm{R}\{\mathrm{D}\cos(h+\alpha)+\mathrm{E}\sin(h+\alpha)\} \\ &= \mathrm{R}\sqrt{\mathrm{D}^2+\mathrm{E}^2}\sin\{(h+\alpha)+\beta_{\prime}\}, \\ &\quad \text{where } \tan\beta_{\prime}=\frac{\mathrm{D}}{\mathrm{E}}; \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (l'.)$$

$$\left.\begin{aligned} z'_h &= \mathrm{R}\{\mathrm{G}\cos(h+\alpha)+\mathrm{H}\sin(h+\alpha)\} \\ &= \mathrm{R}\sqrt{\mathrm{G}^2+\mathrm{H}^2}\sin\{(h+\alpha)+\beta_{\prime\prime}\}, \\ &\quad \text{where } \tan\beta_{\prime\prime}=\frac{\mathrm{G}}{\mathrm{H}}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (l''.)$$

As the angles α, β, $\beta_{\prime}$, and $\beta_{\prime\prime}$ may be regarded as constant for a single day, the equations (*l.*), (*l'.*), (*l''.*) show that the deviations of the magnets from their normal positions at any two hours whose interval is half a day are equal in amount but opposite in direction for each element of the earth's force; but this is quite at variance with the real character of the diurnal variations, the excursions of the magnets being invariably extensive on both sides of their normal positions during about eight hours of the day, which include the sun's upper transit, while deviations of comparatively small extent occur during the remaining hours.

This discordance will be more conclusively shown if we express the observed variations in the form of the series (*y*) described hereafter (17), when the coefficients of the second and following terms should be zero, while we find that those of the second and third terms have magnitudes not greatly inferior to the coefficient of the first term.

Again, it will be shown hereafter (19) that there should be no mean diurnal variation *for the whole year* due to direct action of the sun, while in fact there is such a variation of very considerable range, in middle and high magnetic latitudes, which is opposite in character for the north and south magnetic hemispheres.

With respect to the direct action of the sun upon the observed magnets, the first of the above-mentioned reasons was adduced by Dr. LLOYD as showing the inconsistency with observation in the law of the diurnal variations as derived from the theory in question, in a paper published in the 'Proceedings' of the Royal Irish Academy, February 22, 1858. The same conclusion is here extended by the aid of POISSON'S formula to the inducing action of the sun upon the soft iron of the earth. The hypothesis of the magnetic nature of the sun cannot, therefore, by itself be accepted as an explanation of the diurnal variations of the earth's magnetism.

9. It may be argued, however, that the diurnal variations are the combined effects of direct solar action and of other forces whose origin and nature are unknown; but if this be the case, I think we have reason for believing that the portion of the variations which proceeds from the former source is small in comparison with the part which is due to the other forces that are in operation.

10. Returning now to the equation (*l.*), if R_0 and α_0 be any particular values of R and α, and any other values be R_γ and $(\alpha_0+\gamma)$, we shall have in the latter case

$$\left.\begin{aligned} x'_h &= R_\gamma\sqrt{A^2+B^2}\sin(h+\alpha_0+\gamma+\beta),\\ \text{and if } \beta'=\alpha_0+\beta,\quad &= R_\gamma\sqrt{A^2+B^2}\sin(h+\gamma+\beta'). \end{aligned}\right\} \quad \dots\dots\dots \quad (p.)$$

Hence it appears that when R_0 and α_0 are changed to R_γ and $(\alpha_0+\gamma)$, the range of variation is altered in the proportion of R_γ to R_0, and the hour at which the maximum deviation occurs is advanced by the equivalent in time to the angle γ.

As we are only seeking the law of variation and not absolute quantities, we may reject the constant factor $\sqrt{A^2+B^2}$, when (*p.*) becomes

$$x''_h = R'_\gamma \sin(h+\gamma+\beta'). \quad \dots\dots\dots\dots \quad (t.)$$

12. Now let us observe that the period embraced by a large mass of hourly observations, extending over several years, comprises many revolutions of the sun on its axis, the sun's period of rotation being about twenty-five days, and that it is from such extended series of observations that the determinations of the regular diurnal variations, which we shall use for comparison with the results of our hypothesis, are derived.

13. As the distance of the sun from the earth is great in relation to the diameter of the sun, we may assume, in the calculation of the magnetic force exerted by the sun upon the earth, that the sun's free magnetism is accumulated in the extremities of a straight line whose middle point passes through the sun's centre; and this supposed magnet may be resolved into two others, one having its poles in the sun's axis of rotation, the other with its poles in the plane of the sun's equator. Now as regards the latter, if we notice its effect upon the earth when its north end points towards the earth, it is evident that when the sun has made somewhat more than half a revolution upon its axis, the south end will point towards the earth and the opposite effect be produced, and similarly, for every position of this magnet there is another position, when the sun has advanced about 180° in rotation, in which the effect neutralizes that of the former position.

14. It is necessary therefore to consider only the magnetism of the sun parallel to its axis of rotation as affecting the mean diurnal variations on the earth's surface. Now we have no means of finding the absolute magnetic force of the sun, neither do we require it; what we want is the *law of change* of direction and intensity of the force exerted by the sun upon the centre of the earth as the earth revolves in its orbit; and this we can easily calculate from the known elements of position of the sun's axis, which by

Mr. Carrington's latest determinations are—

Inclination of sun's axis to plane of ecliptic $(i)=82° 45'$,

Longitude of projection of northern half of sun's axis on plane of ecliptic

at epoch 1850·0 $(\lambda')=343° 40'$.

at epoch 1845·0 $(\lambda')=343° 35'$.

(See Monthly Notices of the Royal Astronomical Society, vol. xxii. page 300.)

15. Let I be the magnetic moment of the sun in the direction of its axis of rotation, supposed positive if the north end of the sun's axis is a north pole;

D the radius vector of the earth;
ω the obliquity of the ecliptic;
θ the right ascension of the sun;
λ the longitude of the sun;
δ the declination of the sun, positive when north;

and δ' and θ' the declination and right ascension which the sun had when his longitude was $(\lambda-90°)$.

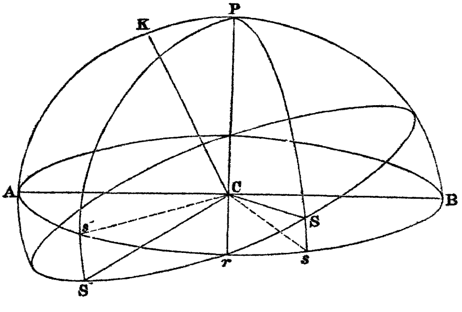
Now, if we suppose the magnetism of the sun along its axis to be resolved perpendicular to the plane of the ecliptic (+ to northward) $=\mathrm{I}\sin i$, and in that plane, and the latter part to be again resolved in the directions of the line joining the centres of the sun and earth (+ towards the earth) $=-\mathrm{I}\cos i\cos(\lambda-\lambda')$, and at right angles to that line (+ to the right hand of an observer who looks towards the earth, his head being supposed at the north pole and feet at south pole of the sun) $=-\mathrm{I}\cos i\sin(\lambda-\lambda')$, then the forces which these exert on the centre of the earth in the same directions respectively, are

$$\mathrm{L}=-\frac{\mathrm{I}}{\mathrm{D}^3}\sin i,$$

$$\mathrm{M}=-2\frac{\mathrm{I}}{\mathrm{D}^3}\cos i\cos(\lambda-\lambda'),$$

$$\mathrm{N}=+\frac{\mathrm{I}}{\mathrm{D}^3}\cos i\sin(\lambda-\lambda').$$

Now let us change the coordinates to those of X_0, Y_0, and Z; this will be facilitated by a reference to the accompanying figure, where C represents the centre of the earth, ArB the equator, and P its north pole, S'rS the ecliptic, and K its pole, and S the sun; then



$$\begin{aligned} r\mathrm{CS} &=\lambda, \\ r\mathrm{C}s &=\theta, \\ \mathrm{SCS'} &=90°, \\ s\mathrm{C}s' &=(\theta-\theta'), \\ s\mathrm{CS} &=\delta, \\ s'\mathrm{CS'} &=\delta', \\ \mathrm{PCK} &=\omega. \end{aligned}$$

The resolved part of L along $Z = -\frac{I}{D^3}\sin i\cos\omega$,

The resolved part of M along $Z = +2\frac{I}{D^3}\cos i\cos(\lambda-\lambda')\sin\delta$,

The resolved part of N along $Z = -\frac{I}{D^3}\cos i\sin(\lambda-\lambda')\sin\delta'$,

and the whole force

$$Z = -\frac{I}{D^3}\left\{\sin i\cos\omega - 2\cos i\cos(\lambda-\lambda')\sin\delta + \cos i\sin(\lambda-\lambda')\sin\delta'\right\}.$$

The resolved part of L along $X_0 = -\frac{I}{D^3}\sin i\sin\omega\sin\theta$,

The resolved part of M along $X_0 = -2\frac{I}{D^3}\cos i\cos(\lambda-\lambda')\cos\delta$,

The resolved part of N along $X_0 = +\frac{I}{D^3}\cos i\sin(\lambda-\lambda')\cos\delta'\cos(\theta-\theta')$,

and the whole force

$$X_0 = -\frac{I}{D^3}\left\{\sin i\sin\omega\sin\theta + 2\cos i\cos(\lambda-\lambda')\cos\delta - \cos i\sin(\lambda-\lambda')\cos\delta'\cos(\theta-\theta')\right\} \quad (u.)$$

The resolved part of L along $Y_0 = +\frac{I}{D^3}\sin i\sin\omega\cos\theta$,

The resolved part of M along $Y_0 = 0$,

The resolved part of N along $Y_0 = +\frac{I}{D^3}\cos i\sin(\lambda-\lambda')\cos\delta'\sin(\theta-\theta')$,

and the whole force $Y_0 = +\frac{I}{D^3}\left\{\sin i\sin\omega\cos\theta + \cos i\sin(\lambda-\lambda')\cos\delta'\sin(\theta-\theta')\right\}$. (*v.*)

16. Taking 1845 as a convenient year, being about the middle year of the observations which we shall make use of, and calculating by means of the formulæ (*u.*), (*v.*), and data taken from the Nautical Almanac, the values of X_0 and Y_0 for the middle day of each month of the year, we obtain the following results, which it may be remarked would be altered only in a trifling degree if any year between 1835 and 1855 were taken instead of 1845.

TABLE I.

	X_0.	Y_0.
	I ×	I ×
January	+ ·20618	+ ·08202
February	− ·01627	+ ·29425
March	− ·21637	+ ·41678
April	− ·33852	+ ·44028
May	− ·33349	+ ·35721
June	− ·34793	+ ·16582
July	− ·21580	− ·06992
August	− ·00104	− ·28116
September.........	+ ·20327	− ·41070
October............	+ ·32934	− ·44304
November	+ ·37919	− ·36388
December	+ ·34916	− ·17132

The constant I multiplies all these numbers, and therefore has no influence upon the law of change; hence it may be omitted, and we shall then call the numbers $X'_0\left(=\frac{X_0}{I}\right)$ and $Y'_0\left(=\frac{Y_0}{I}\right)$, the resultant of X'_0 and Y'_0 being called R_{γ}.

By means of the above Table and equations (*g.*), (*h.*), and (*i.*), we are enabled to construct a table of values of R'_{γ} and α, when, calling α_0 the January value of α, γ is found by subtracting α_0 successively from the α of each month.

TABLE II.

	R'_{γ}.	α.	γ.
January	·22190	21° 42′	0° 0′
February	·29479	93 10	71 28
March	·46960	117 26	95 44
April	·55537	127 33	105 51
May	·48869	133 2	111 20
June	·38543	154 31	132 49
July	·22686	197 58	176 16
August	·28116	269 47	248 5
September	·45824	296 20	274 38
October	·55204	306 38	284 56
November	·52555	316 11	294 29
December	·38892	333 52	312 10

If in equation (*t.*) x''_h become x'''_h when R_{γ} becomes R'_{γ}, we obtain, by inserting the above values of R'_{γ} and γ in

$$x'''_h = R'_{\gamma} \sin\{(h+\gamma)+\beta'\}, \quad \dots\dots\dots \quad (w.)$$

the following equations showing the diurnal variations in the different months:

$$\left.\begin{array}{lll} \text{January} & \dots & x'''_h = {\cdot}22190 \sin\ (h+\beta'), \\ \text{February} & \dots & x'''_h = {\cdot}29479 \sin\{(h+71^\circ\ 28')+\beta'\}, \\ \text{March} & \dots & x'''_h = {\cdot}46960 \sin\{(h+95^\circ\ 44')+\beta'\}, \\ \quad \&c.\ \&c. & & \end{array}\right\} \quad \dots\dots \quad (x.)$$

It will be noticed that the angle β' is undetermined, depending as it does upon the distribution of soft iron in the earth as well as upon the angle α_0; but it is not required in the application that we are to make of the formulæ.

17. Let us now turn our attention to the results of observation as to the diurnal variations. General SABINE has discussed this subject very fully for the observations made at the British Colonial Observatories of Toronto, St. Helena, Hobarton, &c. After a careful separation of disturbed observations, he has given hourly mean values of the deviations of the observed magnets from their normal positions, or their equivalents expressed in terms of the earth's force, for every month of the year, the means being deduced from several years' observations.

Now, if from the mean hourly deviations for a given month we determine by the method of least squares the constants in a series of the form

$$\Delta_h = B\sin(h+\delta) + C\sin 2(h+\epsilon) + D\sin 3(h+\iota) + \&c., \quad . \quad . \quad . \quad . \quad (y.)$$

where Δ_h is the deviation at any hour (h) of the day, it is easy to show that the values of the constants in any one term are independent of the values of those in any other term, if the data are complete for all the twenty-four hours, and if the series be not carried beyond a term $W\sin 23(h+\zeta)$.

18. We have proved (16) that the variation due to direct and inducing action of the sun is of the form

$$x_h''' = R'_\gamma \sin\{(h+\gamma)+\beta'\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\omega.)$$

and therefore we see that the first term of the series ($y.$) includes the whole of this action, the remaining terms being unaffected by it. Moreover the equations ($x.$) show us that the part of the term $B\sin(h+\delta)$ which is due to direct solar action varies as R'_γ, and has its maximum advanced as γ increases. Thus we are led to expect changes in B and δ from month to month which shall accord with the corresponding variations of R'_γ and γ, and the degree of accordance which is found to exist will serve to indicate the extent to which the direct and inducing action of the sun bears a part in the production of the regular diurnal variations.

19. We see from Table II. that the values of R'_γ for January and July are nearly equal, and that the angle γ for the former month differs by about 180° from its value for the latter month: consequently there should be no *mean* diurnal variation (or a very small one) for these two months, inasmuch as the hour of maximum deviation in the one is the same as the hour of minimum in the other, the magnitude of the deviations being nearly alike in the two months: the same observation holds with regard to any two months separated by an interval of half a year, and thus we perceive that the *mean* diurnal variation *for the whole year*, due to direct action of the sun, should be extremely small, and that, without sensible error, we may omit the consideration of it.

We may therefore separate from the quantity $B\sin(h+\delta)$ for the different months, its normal value for the year, calling the remaining quantities $B'\sin(h+\delta')$; and we shall then have to consider only the latter numbers for comparison with R'_γ and γ, for the reason just stated.

Now it is probable (from the simplicity of the instrument used and its independence of temperature corrections) that no periodical magnetic variations are so well determined as the diurnal variations of declination. We shall therefore confine ourselves to the examination of the variations of that element, for which the following Table gives the monthly values of B' and δ' at Toronto and St. Helena, these being derived from tables of variations given by General SABINE in his discussions of the observations made at the observatories of Toronto and St. Helena, published for the British Government by LONGMAN and Co., London. Positive values of $B'\sin(h+\delta')$ indicate easterly deviations of the north end of the declination magnet.

TABLE III.

	Toronto.		St. Helena.	
	B'.	δ'.	B'.	δ'.
January	1·208	24° 20′	·627	349° 48′
February	·883	35 44	·916	347 8
March	·467	188 22	·570	353 46
April	·664	228 21	·203	196 1
May	·884	204 14	·640	192 0
June	1·053	179 44	·858	174 35
July	1·006	177 6	·922	175 24
August	·963	207 23	·803	183 21
September	·569	261 12	·372	174 8
October	·444	17 49	·642	50 11
November	1·095	14 49	·688	4 40
December	1·472	17 26	·603	348 8
	B.	δ.	B.	δ.
Year	2·625	32 37	·126	292 29

20. As a diagram conveys to the mind a more distinct conception of a variation than a table of numbers, the curve (Plate XXV. fig. 1) is constructed to represent the successive values of R'_γ and γ: in this figure the lines $r\,1$, $r\,2$, . . . $r\,12$ are proportional to R'_γ, and the angles $A\,r\,1$, $A\,r\,2$, . . . $A r\,12$ are equal to the angle γ, for January, February, . . . December respectively. The curves of figs. 2 & 3, which are formed in a similar manner from the numbers in the preceding Table, are intended to show the variations of B' and δ'. Drawing lines $r\,G'$, $r\,H'$ at right angles to $r\,G$ and $r\,H$ respectively, it is easy to see, from the form of the expression $B'\sin(h+\delta')$, that the angles $G'r\,1$, $G'r\,2$, . . . $G'r\,12$ (reckoned by a right-handed revolution from $G'r$) represent the hour-angles of the sun at the time of the occurrence of the maximum deviation in the respective months from January to December, and that the extent of that deviation is represented by the lines $r\,1$, $r\,2$, . . . $r\,12$ respectively; and similarly for fig. 3.

21. Now we see that direct action of the sun is not the sole cause of the variations of the term $B\sin(h+\delta)$, because in that case fig. 1 (which represents the variations of the cause) would be similar to figs. 2 & 3 (which represent the variations of the effect), but would not be similarly situated unless the angle β' happened to be equal to $G\,r\,1$ or $H\,r\,1$; and we find but little appearance of similarity displayed by the curves; the extent of the likeness will be exhibited more distinctly in what follows. We may here state that if the quantity I were negative, it would only have the effect of increasing β' by 180°.

22. The lines C D, E F in figs. 2 & 3 have that direction which gives the sum of the squares of the perpendiculars let fall upon them from the points 1, 2, . . . 12 a minimum, and it is very noticeable that the points are arranged in closer proximity to these lines than to lines at right angles to C D and E F, the points 10, 11, 12, 1, 2 being

generally towards one extremity of the lines, while the remaining points are towards the opposite extremity.

The expression $B' \sin (h+\delta')$ may be written

$$B' \cos (\delta'-\sigma) \sin (h+\sigma) + B' \sin (\delta'-\sigma) \cos (h+\sigma),$$

where, σ being the angle $G r D$ or $H r F$, the former term represents that part of the variation of the term $B \sin (h+\delta)$ which gives a maximum of easterly declination when $h=90^\circ-\sigma$, its coefficient being the resolved part of the lines $r 1, r 2, \ldots r 12$ along CD or EF; and the latter term represents the part of the variation of $B \sin (h+\delta)$ which gives a maximum of declination when $h=-\sigma$, and its coefficient is the resolved part of $r 1, r 2, \ldots r 12$ at right angles to CD or EF. The following Table shows the values of $B' \cos (\delta'-\sigma)$ and $B' \sin (\delta'-\sigma)$ for the different months:—

TABLE IV.

	Toronto.		St. Helena.	
	$B' \cos (\delta'-\sigma)$.	$B' \sin (\delta'-\sigma)$.	$B' \cos (\delta'-\sigma)$.	$B' \sin (\delta'-\sigma)$.
January	+1·196	+·172	+·620	−·094
February	+ ·832	+·296	+·898	−·179
March	− ·463	+·063	+·569	−·046
April	− ·562	−·354	−·193	−·061
May	− ·875	−·124	−·622	−·150
June	−1·010	+·298	−·856	+·057
July	− ·951	+·328	−·921	+·048
August	− ·944	−·188	−·800	−·069
September	− ·240	−·516	−·371	+·028
October	+ ·444	+·013	+·397	+·504
November	+1·094	−·045	+·684	+·075
December	+1·471	+·033	+·593	−·107

In figs. 5, 6, 7, & 8 (Plate XXVI.) the monthly variation of these numbers is indicated by curves whose vertical ordinates are proportional to the numbers, positive values being reckoned upwards; and the abscissæ are divided into twelve equal parts to represent the successive months of the year.

23. Remembering what was said in art. 19, we see that if a straight line be drawn *in any direction* through the point r of fig. 1, it divides the curve into two sets of six consecutive months; and that if perpendiculars be drawn from the points 1, 2, 3, . . . 12 to that line, their lengths are least in the extreme months on each side of the line, and increase to maxima and then diminish as the numbers increase, there being only one maximum on each side. Now as this is true of any line, these conditions must be fulfilled (whatever be the value of the angle β') by the ordinates of figs. 5, 6, 7, & 8 if the variations which the curves represent are due to direct action of the sun: and if we confine our attention to figs. 5 & 7, these conditions are satisfied in a remarkable manner; but as we have found that they are to be demanded alike from figs. 6 & 8

before we can admit the truth of the hypothesis, let us see whether these curves also exhibit the required form.

24. In the first place, we observe that the range of these curves is much smaller than that of figs. 5 & 7, and still smaller than the whole diurnal range of declination at the two stations, and therefore, even if the curves had the required form, we should still have reason to believe that the direct effect of the sun was less than that of other forces in operation. For the ratio of the mean ordinate (disregarding signs) of fig. 6 is to that of fig. 5 as 1 to 4·2, and the corresponding ratio for figs. 8 & 7 is as 1 to 5·3, while the least possible ratio of the mean ordinates upon rectangular diameters of fig. 1 is as 1 to 2·7; and therefore, under the most favourable supposition as to the value of the angle β', the mean ordinates of figs. 5 & 7 should be considerably less, or those of figs. 6 & 8 greater, in order that the results of observation should accord with the hypothesis. But figs. 6 & 8 are far from fulfilling the conditions named as regards the form of the curves; for we see that neither of them has but a single maximum or a single minimum ordinate, nor are six consecutive months above and six below the horizontal line deviating most from that line in the middle months of each group of six.

The general appearance of figs. 6 & 8 conveys the impression that the numbers which they represent are possibly errors in the determinations of the respective values of B$'$ and δ', arising perhaps from the uneliminated part of the disturbances; but if it were possible to decompose them into two parts, one obeying the required law of variation and another following some other law, it is probable that the mean ordinate corresponding to the former part would be much less than the mean of the combined ordinates: hence it is probable that there is but little trace of direct solar action to be found in the regular diurnal variations of declination at Toronto and St. Helena.

25. It may be objected, however, that the agreement of figs. 5 & 7 with the requirements of the hypothesis does not appear as well in figs. 6 & 8, because the other variable forces in operation act in partial opposition to the sun, and so mask the character which figs. 6 & 8 would have if those opposing forces could be separated. To this objection it will be sufficient to reply that the variations of the quantities corresponding to $B' \cos(\delta'-\sigma)$ for the second and third terms of the series (y) have a similar character to those of the first term, and it appears reasonable therefore to infer that the same variable force is the cause of the variations of each of the three terms; but the second and third terms have been proved (18) to be independent of direct action of the sun; hence it is most probable that the variations of $B' \sin(\delta'-\sigma)$, though they accord with the hypothesis of direct solar action, are due to some different cause. The similarity of the variations of $B' \cos(\delta'-\sigma)$ is shown by the curves of figs. 9 & 11 (Plate XXVII.) in continuous black, red, and interrupted black lines, which have reference to the first, second, and third terms of (y) respectively. Figs. 10 & 12 are similar representations of the variations of $B' \sin(\delta'-\sigma)$ for the first three terms of (y); and it is observable that these are quite different one from another. In Table V. are given the numbers from which the curves of figs. 9, 10, 11, & 12 are constructed.

TABLE V.—Showing the variations of the quantities corresponding to $B' \cos(\delta'-\sigma)$ and $B' \sin(\delta'-\sigma)$ for the second and third terms of the series (y).

	Toronto.				St. Helena.			
	For second term.		For third term.		For second term.		For third term.	
	$B' \cos(\delta'-\sigma)$.	$B' \sin(\delta'-\sigma)$.	$B' \cos(\delta'-\sigma)$.	$B' \sin(\delta'-\sigma)$.	$B' \cos(\delta'-\sigma)$.	$B' \sin(\delta'-\sigma)$.	$B' \cos(\delta'-\sigma)$.	$B' \sin(\delta'-\sigma)$.
January ...	+ ·919	−·020	+·497	+·026	+·409	+·196	+·345	+·103
February...	+ ·868	−·096	+·429	−·195	+·503	−·604	+·235	−·461
March......	+ ·399	+·446	+·120	+·371	+·417	−·488	+·315	−·340
April	− ·184	+·145	−·014	+·138	+·050	−·069	+·232	−·041
May	− ·856	−·126	−·371	−·134	−·560	−·081	−·360	+·089
June	− ·875	+·281	−·295	−·070	−·543	+·098	−·533	+·135
July	− ·574	+·375	−·370	+·012	−·602	+·074	−·614	+·125
August ...	−1·212	+·061	−·751	+·018	−·825	+·012	−·690	+·191
September	− ·697	−·714	−·459	−·049	−·479	−·045	−·307	−·010
October ...	+ ·499	−·314	+·139	−·007	+·624	+·207	+·506	−·082
November	+ ·643	−·029	+·374	−·092	+·568	+·298	+·442	+·094
December	+1·075	−·006	+·694	−·017	+·440	+·398	+·457	+·116

An important inference that may be drawn from the curves of figs. 9, 10, 11, & 12, is that, as those of figs. 9 & 11 exhibit a remarkable likeness of character, while those of figs. 10 & 12 present little appearance of similarity, the former are probably representatives of true changes, while the latter may perhaps be regarded as errors in the determinations, when we obtain the following approximate expression for the variable part of the diurnal variations in any month of the year,

$$A\{B \sin(h+\alpha)+C \sin 2(h+\beta)+D \sin 3(h+\gamma)+\&c.\},$$

where A alone varies from month to month, being sometimes positive and sometimes negative. If we might regard this as a true statement, it would follow, as the law of variation remains constant throughout the year, the range only altering, that these variations do not result from those of different forces, but from those of a single force, or of a group of connected forces.

26. Having seen that it is probable that if the sun be a magnetic body its direct magnetic influence upon the earth, as evinced by the regular diurnal variations, is very small, I think we may safely infer that terrestrial magnetic disturbances are not caused by variations in the magnetism of the sun; for if the whole direct action of the sun (at ordinary times) be insensible, much more will the changes which take place in the sun's magnetic state be insensible in their effects upon the earth, as it would be difficult to conceive that the changes in question were ever of such magnitude as to approach the effect of reversing the sun's ordinary magnetic condition.

27. I conclude, therefore, that the mode in which forces originating in the sun influence the magnetic condition of the earth is not analogous to the action of a magnet upon a mass of soft iron placed at a great distance from it, but that these forces proceed from the sun in a form different from that of magnetic force, and are converted into the latter form of force probably by their action upon the matter of the earth or its atmosphere.

On the existence of a variation of diurnal range of Declination at St. Helena depending on the Sun's altitude.

When we express the diurnal variations of declination at St. Helena for the different months of the year in the form (*y*),

$$\Delta_h = \mathrm{B}\sin(h+\delta)+\mathrm{C}\sin 2(h+\varepsilon)+\mathrm{D}\sin 3(h+\iota),$$

the values of the coefficients B, C, D are seen to be larger about the time when at noon the sun is overhead at St. Helena, which occurs in the first weeks of February and November, than they are immediately before or after. This fact indicates the existence of a secondary variation of the diurnal range of declination which has maximum values at the times mentioned, and which is additional to the variation of range caused by the semiannual inequality discovered by General Sabine. The monthly values of B, C, and D are shown in the following Table:—

	B.	C.	D.
January	′·703	′·726	′·622
February	·994	1·043	·968
March	·640	·904	·904
April	·226	·375	·619
May	·629	·292	·352
June	·808	·243	·428
July	·872	·299	·499
August	·771	·524	·534
September	·331	·204	·427
October	·594	·939	·857
November	·737	·903	·713
December	·682	·817	·663

Had this been a feature of only one or even two of the three coefficients, we might have supposed it to be an accidental circumstance which would perhaps be reversed on a repetition of the observations; but as the three coefficients agree in giving the same testimony, I think we must allow it to be not merely an apparent but a real variation.

Before concluding, I must acknowledge the important help that I have derived from Mr. Archibald Smith's 'Mathematical Theory of the Errors of Ship's Compasses,' of which what has preceded is little more than an application to another purpose. I am also indebted to Mr. Thomas W. Baker, of the Kew Observatory, for the very efficient assistance which he has afforded me in the rather laborious computations involved in the discussion.

Note on §§ 26 *and* 27. *By Professor* W. Thomson, *M.A.*, *LL.D.*, *F.R.S.*

Received October 28, 1863.

If the sun were a magnet as intense on the average as the earth, the magnetic force it would exert at a distance equal to the earth's, or, let us suppose, 200 radii, would be only $\frac{1}{8,000,000}$ of the earth's surface magnetic force in the corresponding position relatively to its magnetic axis. Considering, therefore (according to the principles explained by Mr. CHAMBERS), the sun as a magnet having its magnetic axis nearly perpendicular to the ecliptic, we see that, with an average intensity of magnetization equal to the earth's, the effect of reversing the sun's magnetization would be to introduce, in a direction perpendicular to the ecliptic, a disturbing force equal to about $\frac{1}{8,000,000}$ of the earth's average polar force, and would therefore be absolutely insensible to the most delicate terrestrial magnetic observation yet practised. A disturbing force of this amount, acting perpendicularly to the direction of the terrestrial magnetic force about the equator, would produce a disturbance in declination of only half a second; and the sun's magnetization would therefore need to be 120 times as intense as the earth's to produce a disturbance of 1′ in declination even by a *complete reversal* in the most favourable circumstances. These estimates appear to me to give strong evidence in support of the conclusion at which Mr. CHAMBERS has arrived by a careful examination of the disturbances actually observed, that no effect of the sun's action as a magnet is sensible at the earth.

The same estimates are applicable to the moon, her apparent diameter being the same as the sun's. It is of course most probable that the moon is a magnet; but she must be a magnet thousands or millions of times more intense than the earth to produce any sensible effect of the character of any of the observed terrestrial magnetic disturbances.

XXIV. *On the Calculus of Symbols.*—Third Memoir. *By* W. H. L. Russell, *Esq., A.B.* *Communicated by* A. Cayley, *F.R.S.*

Received May 15,—Read May 21, 1863.

In my second Memoir "On the Calculus of Symbols," I worked out the general case of multiplication according to one of the two systems of combination of non-commutative symbols previously given. In the present paper I propose to investigate the general case of multiplication according to the other system. I commence with the Binomial Theorem, to which the second system gives rise. In my previous researches I obtained the general term of the binomial theorem when the symbols combine according to the first system by equating symbolical coefficients; here, on the other hand, I consider the nature of the combinations which arise from the symbolical multiplication, and obtain the general term by summation. I next proceed to the multiplication of binomial factors. Here the general term is obtained by considering the alteration of weight undergone by certain symbols in the process of multiplication. The multinomial theorem according to the second system is next considered and its general term calculated. I conclude the memoir with some applications of the calculus of symbols to successive differentiation. This paper completes the investigation of symbolical multiplication and division according to the two systems of combination, the general case of division having been worked out by Mr. Spottiswoode in a very beautiful memoir recently published in the Transactions of this Society.

I shall commence with expanding the binomial $(\pi+\theta(\varrho))^n$ in terms of π.

I shall put δ in place of the symbol $\varrho\frac{d}{d\varrho}$, and write a_{a_1} for $a, \frac{a-1}{2}, \frac{a-2}{3}, \ldots.. \frac{a-a_1+1}{a_1}$.

Now
$$(\pi+\theta\varrho)^2=\pi^2+\theta\varrho\,.\,\pi+\pi\theta\varrho+\theta(\varrho)^2,$$
$$(\pi+\theta\varrho)^3=\pi^3+(\pi^2\theta\varrho+\pi\theta\varrho\pi+\theta\varrho\pi^2)+(\theta\varrho^2\pi+\theta\varrho\pi\theta\varrho+\pi\theta\varrho^2)+\theta\varrho^3.$$

Hence we evidently have
$$(\pi+\theta\varrho)^n=\pi^n+\Sigma(\pi^{n-1}\theta\varrho)+\Sigma(\pi^{n-2}\theta\varrho^2)+\Sigma(\pi^{n-3}\theta\varrho^3)+\ldots..,$$
where $\Sigma(\pi^m\theta\varrho^r)$ denotes the sum of all the factors in which the symbol (π) occurs (m) times and the symbol $(\theta\varrho)$ (r) times irrespective of position.

Now if $a+b=m$,
$$\pi^b\theta\varrho\pi^a=\theta\varrho\pi^m+b_1\delta\theta\varrho\pi^{m-1}+b_2\delta^2\theta\varrho\pi^{m-2}+b_3\delta^3\theta\varrho\pi^{m-3}+\ldots$$

Again, if $a+b+c=m$,
$$\begin{array}{rlllll}\pi^c\theta\varrho\pi^b\theta\varrho\pi^a= & \theta\varrho^2\pi^m & +b_1\theta\varrho\delta\theta\varrho\pi^{m-1} & +b_2\theta\varrho\delta^2\theta\varrho\pi^{m-2} & +b_3\theta\varrho\delta^3\theta\varrho\pi^{m-3} & +\ldots \\ & & +c_1\delta\theta\varrho^2\pi^{m-1} & +b_1c_1\delta\theta\varrho\delta\theta\varrho\pi^{m-2} & +b_2c_1\delta\theta\varrho\delta^2\theta\varrho\pi^{m-3} & +\ldots \\ & & & +c_2\quad\delta^2\theta\varrho^2\pi^{m-2} & +b_1c_2\delta^2\theta\varrho\delta\theta\varrho\pi^{m-3} & +\ldots \\ & & & & +\quad c_3\delta^3\theta\varrho^2\pi^{m-3} & +\ldots;\end{array}$$

and if $a+b+c+e=m$, we have

$$\begin{array}{rllll}
\pi^e\theta\varrho\pi^c\theta\varrho\pi^b\theta\varrho\pi^a=(\theta\varrho)^3\pi^m & +b_1\theta\varrho^2\delta\theta\varrho\pi^{m-1} & +b_2\theta\varrho^2\delta^2\theta\varrho\pi^{m-2} & +b_3\theta\varrho^2\delta^3\theta\varrho\pi^{m-3} & +\ldots \\
 & +c_1\theta\varrho\delta\theta\varrho^2\pi^{m-1} & +b_1c_1\theta\varrho\delta\theta\varrho\delta\theta\varrho\pi^{m-2} & +b_2c_1\theta\varrho\delta\theta\varrho\delta^2\theta\varrho\pi^{m-3} & +\ldots \\
 & & +c_2\theta\varrho\delta^2\theta\varrho^2\pi^{m-2} & +b_1c_2\theta\varrho\delta^2\theta\varrho\delta\theta\varrho\pi^{m-3} & +\ldots \\
 & & & +\ c_3\theta\varrho\delta^3\theta\varrho^2\pi^{m-3} & +\ldots \\
 & +e_1\delta\theta\varrho^3\pi^{m-1} & +b_1e_1\delta\theta\varrho^2\delta\theta\varrho\pi^{m-2} & +b_2e_1\delta\theta\varrho^2\delta^2\theta\varrho\pi^{m-3} & +\ldots \\
 & & +c_1e_1\delta\theta\varrho\delta\theta\varrho^2\pi^{m-2} & +b_1c_1e_1\delta\theta\varrho\delta\theta\varrho\delta\theta\varrho\pi^{m-3} & +\ldots \\
 & & & +c_2e_1\delta\theta\varrho\delta^2\theta\varrho^2\pi^{m-3} & +\ldots \\
 & & +e_2\delta^2\theta\varrho^3\pi^{m-2} & +e_2b_1\delta^2\theta\varrho^2\delta\theta\varrho\pi^{m-3} & +\ldots \\
 & & & +e_2c_1\delta^2\theta\varrho\delta\theta\varrho^2\pi^{m-3} & +\ldots \\
 & & & +e_3\delta^3\theta\varrho^3\pi^{m-3} & +\ldots
\end{array}$$

Hence we see that if $a+b+c+e\ldots\ldots+l=n$, the general term of $\pi^l\theta\varrho\ldots\ldots\pi^e\theta\varrho\pi^c\theta\varrho\pi^b\theta\varrho\pi^a$ will be

$$\Sigma b_{b_0}c_{c_0}\ldots l_{l_0}\delta^{l_0}\theta\varrho\ldots\delta^{c_0}\theta\varrho\delta^{b_0}\theta\varrho\,.\,\pi^{m-s},$$

where the sum Σ extends to all values of $a_0b_0c_0\ldots l_0$ included in the equation

$$b_0+c_0+\ldots\ldots+l_0=s;$$

and if (r) be the number of the factors $\theta\varrho$, we shall have

$$\begin{aligned}
\pi^l\theta\varrho\ldots\ldots\pi^e\theta\varrho\pi^c\theta\varrho\pi^b\theta\varrho\pi^a=(\theta\varrho)^r\pi^m&+\{b_1(\theta\varrho)^{r-1}\delta\theta\varrho+c_1(\theta\varrho)^{r-2}\delta(\theta\varrho)^2+e_1(\theta\varrho)^{r-3}\delta(\theta\varrho)^3+\ldots\}\pi^{m-1} \\
&+\{b_2(\theta\varrho)^{r-1}\delta^2\theta\varrho+c_2(\theta\varrho)^{r-2}\delta^2(\theta\varrho)^2+e_2(\theta\varrho)^{r-3}\delta^2(\theta\varrho)^3+\ldots+b_1c_1(\theta\varrho)^{r-2}\delta\theta\varrho\delta\theta\varrho \\
&+b_1e_1(\theta\varrho)^{r-3}\delta(\theta\varrho)^2\delta\theta\varrho+\ldots\ldots\}\pi^{m-2}+\&\text{c.}+\Sigma b_{b_0}c_{c_0}\ldots l_{l_0}\delta^{l_0}\theta\varrho\ldots\delta^{c_0}\theta\varrho\delta^{b_0}\theta\varrho\pi^{m-s}+\ldots,
\end{aligned}$$

and hence we shall have

$$\begin{aligned}
\Sigma\pi^m(\theta\varrho)^r=(\theta\varrho)^r\pi^m&+\{(b_1'+b_1''+\ldots)(\theta\varrho)^{r-1}\delta\theta\varrho+(c_1'+c_1''+\ldots)(\theta\varrho)^{r-2}\delta(\theta\varrho)^2+(e_1'+e_1'+\ldots)(\theta\varrho)^{r-3}\delta(\theta\varrho)^3+\ldots\}\pi \\
&+\{b_2'+b_2''+\ldots)(\theta\varrho)^{r-1}\delta^2\theta\varrho+(c_2'+c_2''+\ldots)(\theta\varrho)^{r-2}\delta^2(\theta\varrho)^2+\ldots+(b_1'c_1'+b_1''c_1''+\ldots)\delta\theta\varrho\delta\theta\varrho+\ldots\}\pi^{m-2} \\
&+\ldots+\Sigma_m\Sigma_s b_{b_0}c_{c_0}\ldots l_{l_0}\delta^{l_0}\theta\varrho\ldots\delta^{c_0}\theta\varrho\delta^{b_0}\theta\varrho\pi^{m-s}+\ldots\ldots,
\end{aligned}$$

where Σ_s extends to all values of $b_0c_0\ldots$ included in the equation

$$b_0+c_0+e_0+\ldots+l_0=s,$$

and Σ_m to all values of abc included in the equation

$$a+b+c+\ldots\ldots+l=m.$$

Now the general term of the binomial $(\pi+\theta\varrho)^n$ will be the term involving π^μ in the sum of

$$\Sigma(\pi^{n-1}\theta\varrho),\quad \Sigma(\pi^{n-2}\theta\varrho^2),\ \&\text{c.}$$

Hence we easily see that the coefficient of π^{μ} in the binomial is

$$\Sigma_{n-1}\Sigma_{n-\mu-1}b_{b_0}\delta^{b_0}\theta\varrho+\Sigma_{n-2}\Sigma_{n-\mu-2}b_{b_0}c_{c_0}\delta^{c_0}\theta\varrho\delta^{b_0}\theta\varrho+\Sigma_{n-3}\Sigma_{n-\mu-3}b_{b_0}c_{c_0}e_{e_0}\delta^{e_0}\theta\varrho\delta^{c_0}\theta\varrho\delta^{b_0}\theta\varrho+\ldots$$
$$+\Sigma_{n-\nu}\Sigma_{n-\mu-\nu}b_{b_0}c_{c_0}\ldots\ldots l_{l_0}\delta^{l_0}\theta\varrho\ldots\delta^{c_0}\theta\varrho\delta^{b_0}\theta\varrho+\ldots,$$

where

$$a+b+c+e+\ldots\ldots l=n-\nu$$
$$b_0+c_0+\ldots\ldots+l_0=n-\mu-\nu.$$

I shall next investigate the general term of the symbolical product,

$$(\pi+\theta_1\varrho)(\pi+\theta_2\varrho)(\pi+\theta_3\varrho)\ldots\ldots(\pi+\theta_n\varrho).$$

Now

$$(\pi+\theta_1\varrho)(\pi+\theta_2\varrho)=\pi^2+\pi\theta_1\varrho+\theta_2\varrho\pi+\theta_1\varrho\theta_2\varrho;$$
$$(\pi+\theta_1\varrho)(\pi+\theta_2\varrho)(\pi+\theta_3\varrho)=\pi^3+(\theta_1\varrho\pi^2+\pi\theta_2\varrho\pi+\pi^2\theta_3\varrho)$$
$$+(\theta_1\varrho\theta_2\varrho\pi+\theta_1\varrho\pi\theta_3\varrho+\pi\theta_2\varrho\theta_3\varrho)+\theta_1\varrho\theta_2\varrho\theta_3\varrho;$$
$$(\pi+\theta_1\varrho)(\pi+\theta_2\varrho)(\pi+\theta_3\varrho)(\pi+\theta_4\varrho)=\pi^4+(\theta_1\varrho\pi^3+\pi\theta_2\varrho\pi^2+\pi^2\theta_3\varrho\pi+\pi^3\theta_4\varrho)$$
$$+(\theta_1\varrho\theta_2\varrho\pi^2+\theta_1\varrho\pi\theta_3\varrho\pi+\pi\theta_2\varrho\theta_3\varrho\pi+\theta_1\varrho\pi^2\theta_4\varrho+\pi\theta_2\varrho\pi\theta_4\varrho+\pi^2\theta_3\varrho\theta_4\varrho)$$
$$+(\theta_1\varrho\theta_2\varrho\theta_3\varrho\pi+\theta_1\varrho\theta_2\varrho\pi\theta_4\varrho+\theta_1\varrho\pi\theta_3\varrho\theta_4\varrho+\pi\theta_2\varrho\theta_3\varrho\theta_4\varrho)+\theta_1\varrho\theta_2\varrho\theta_3\varrho\theta_4\varrho.$$

Hence we deduce the following law of multiplication in this case.

Consider the symbolical expression

$$\pi^a\theta_{a_1}(\varrho)\pi^b\theta_{b_2}(\varrho)\pi^c\theta_{c_3}(\varrho)\ldots\ldots\pi^l\theta_{l_r}(\varrho)\pi^s,$$

where

$$a+b+c+\ldots\ldots+s=n-r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$
$$a_1 \ = \ a+1,$$
$$b_2 \ = \ a+b+2,$$
$$c_3 \ = \ a+b+c+3,$$
$$\&c. = \ \&c.,$$
$$l_r \ = \ a+b+c+\ldots\ldots+l+r,$$

and let $a, b, c, \ldots s$ have all the values which can be assigned consistently with equation (1.); the sum of the symbolical terms thus formed will be that group of the symbolical product in which the factor π occurs $(n-r)$ times. And by giving (r) the successive values 0, 1, 2, 3 ... n, we obtain the entire product, and may write

$$(\pi+\theta_1\varrho)(\pi+\theta_2\varrho)(\pi+\theta_3\varrho)\ldots\ldots(\pi+\theta_n\varrho)=\pi^n+\Sigma\theta_1\varrho\pi^{n-1}+\Sigma\theta_1\varrho\theta_2\varrho\pi^{n-2}+\Sigma\theta_1\varrho\theta_2\varrho\theta_3\varrho\pi^{n-3}+\ldots$$
$$+\Sigma\theta_1\varrho\theta_2\varrho\theta_3\varrho\ldots\ldots\theta_r\varrho\pi^{n-r}+\ldots,$$

where $\Sigma\theta_1\varrho\pi^{n-1}$, &c. are the groups we have just considered.

Now if ψ, ϕ, θ are any functions of (ϱ), and $a+b+c+e=m$, we have

$$\begin{array}{rll}\pi^{e}\psi\pi^{c}\phi\pi^{b}\theta\pi^{a}=\psi\phi\theta\pi^{m}+b_1\psi\phi\delta\theta\pi^{m-1} & +b_2\psi\phi\delta^2\theta\pi^{m-2} & +\dots \\ +c_1\psi\delta\phi\theta\pi^{m-1} & +b_1c_1\psi\delta\phi\delta\theta\pi^{m-2} & +\dots \\ & +c_2\psi\delta^2\phi\theta\pi^{m-2} & +\dots \\ +e_1\delta\psi\phi\theta\pi^{m-1} & +b_1e_1\delta\psi\phi\delta\theta\pi^{m-2} & +\dots \\ & +c_1e_1\delta\psi\delta\phi\theta\pi^{m-2} & +\dots \\ & +e_2\delta^2\psi\phi\theta\pi^{m-2} & +\dots\end{array}$$

where b_1, c_1, &c. have their original significations.

Hence we see that the general term of $\pi^{l}\theta_1\varrho\dots\dots\pi^{e}\theta_{r-2}(\varrho)\pi^{c}\theta_{r-1}\varrho\pi^{b}\theta_r\varrho\,.\,\pi^{a}$ is

$$\Sigma b_{b_0}c_{c_0}\dots\dots l_{l_0}\delta^{l_0}\theta_1\varrho\dots\delta^{c_0}_{r-1}\theta_{r-1}(\varrho)\delta^{b_0}_{r}\theta_r(\varrho)\pi^{m-s},$$

where

$$a+b+c+e+\dots+l=m,$$
$$b_0+c_0+e_0+\dots+l_0=s;$$

hence the general term of $\Sigma\theta_1\varrho\theta_2\varrho\theta_3\varrho\dots\theta_r\varrho\pi^{m}$ is

$$\Sigma_m\Sigma_s b_{b_0}c_{c_0}\dots l_{l_0}\delta^{l_0}\theta_{1+l}\varrho\dots\delta^{e_0}\theta_{r+e+\dots+l-2}(\varrho)\,.\,\delta^{c_0}\theta_{r+c+e+\dots l-1}(\varrho)\delta^{b_0}\theta_{r+b+c+e+\dots+l}(\varrho)\pi^{m-s};$$

and we find the symbolical coefficient of π^{μ} in $(\pi+\theta_1\varrho)(\pi+\theta_2\varrho)(\pi+\theta_3\varrho)\dots(\pi+\theta_n\varrho)$ to be

$$\Sigma_{n-1}\Sigma_{n-\mu-1}b_{b_0}\delta^{b_0}\theta_{1+b}(\varrho)+\Sigma_{n-2}\Sigma_{n-\mu-2}b_0c_0\delta^{c_0}_{1+c}\theta\varrho\delta^{b_0}\theta_{2+b+c}(\varrho)+\Sigma_{n-3}\Sigma_{n-\mu-3}b_{b_0}c_{c_0}e_{e_0}\delta^{e_0}_{1+e}\theta(\varrho)\delta^{c_0}\theta_{2+c+e}(\varrho)\delta^{b_0}\theta_{3+b+c+e}(\varrho)$$
$$+\&c.+\Sigma_{n-\nu}\Sigma_{n-\mu-\nu}b_{b_0}c_{c_0}e_{e_0}\dots l_{l_0}\delta^{l_0}\theta_{1+l}(\varrho)\dots\delta^{c_1}\theta_{\nu+e+\dots l-2}(\varrho)\delta^{c_0}\theta_{\nu+c+e+\dots l-1}(\varrho)\delta^{b_0}\theta_{\nu+b+c+a+\dots+l}(\varrho)+\&c.$$

where

$$a+b+c+\dots+l=n-\nu,$$
$$b_0+c_0+\dots+l_0=n-\mu-\nu.$$

Let us now expand the binomial $(\pi^2+\theta(\varrho)\pi)^{r}$. We might of course do this by putting

$$\theta_1\varrho=\theta_3\varrho=\theta_5\varrho=\dots$$
$$\theta_2\varrho=\theta_4\varrho=\theta_6\varrho=0$$

in the previous investigation, but we shall proceed as follows:—

$$(\pi^2+\theta\varrho\,.\,\pi)^{n}=\pi^{2n}+\Sigma(\pi^2)^{n-1}(\theta\varrho\,.\,\pi)+\Sigma(\pi^2)^{n-2}(\theta\varrho\,.\,\pi)^2+\&c.+\Sigma(\pi^2)^{n-r}(\theta\varrho\,.\,\pi)^{r}+\dots,$$

and the general term of $\Sigma(\pi^2)^{m}(\theta\varrho\,.\,\pi)^{r}$ will be

$$\Sigma_m\Sigma_s(2b+1)_{b_1}(2c+1)_{c_1}\dots(2l)_{l_1}\delta^{l_1}\theta\varrho\dots\delta^{c_1}\theta\varrho\delta^{l_1}\theta\varrho\pi^{2m+r-s},$$

where

$$a+b+c+\dots+l=m,$$
$$b_0+c_0+\dots+l_0=s,$$

and the symbolical coefficient of π^{μ} in the binomial will be

$$\Sigma_{n-1}\Sigma_{2n-\mu-1}(2b)_{b_1}\delta^{b_1}\theta(\varrho)+\Sigma_{n-2}\Sigma_{2n-\mu-2}(2b+1)_{b_0}(2c)_{c_0}\delta^{c_0}\theta(\varrho)\delta^{b_0}\theta(\varrho)+\dots$$
$$+\Sigma_{n-\nu}\Sigma_{2n-\mu-\nu}(2b+1)_{b_0}(2c+1)_{c_0}\dots(2l)_{l_0}\delta^{l_0}\theta\varrho\dots\delta^{b_0}\theta\varrho,$$

where

$$a+b+c+\ldots+l=n-\nu,$$
$$b_0+c_0+\ldots+l_0=2n-\mu-\nu.$$

To determine the general term in the expansion of the multinomial expression

$$(\pi^n+\theta_1\varrho\pi^{n-1}+\theta_2\varrho\pi^{n-2}+\theta_3\varrho\pi^{n-3}+\ldots)^m.$$

This reduces itself to finding the symbolical coefficient of π^μ in the expression

$$\Sigma(\pi^n)^\alpha(\theta_1\varrho\pi^{n-1})^\beta(\theta_2\varrho\pi^{n-2})^\gamma\ldots.,$$

where $\alpha+\beta+\gamma+\ldots.=m$, and π^n, $\theta_1\varrho\pi^{n-1}$, &c. are combined in every possible way so that π^n should occur (α) times, $\theta_1\varrho\pi^{n-1}$ (β) times, $\theta_2\varrho\pi^{n-2}$ (γ) times, &c. irrespective of position,—or, in other words, to determining the symbolical coefficient of π^μ in the expression

$$\pi^{n\alpha_1}(\theta_1\varrho\pi^{n-1})^{\beta_1}(\theta_2\varrho\pi^{n-2})^{\gamma_1}(\theta_3\varrho\pi^{n-3})^{\zeta_1}\ldots\pi^{n\alpha_2}(\theta_1\varrho\pi^{n-1})^{\beta_2}(\theta_2\varrho\pi^{n-2})^{\gamma_2}(\theta_3\varrho\pi^{n-3})^{\zeta_2}\ldots\pi^{n\alpha_3}(\theta_1\varrho\pi^{n-1})^{\beta_3}(\theta_2\varrho\pi^{n-2})^{\gamma_3}(\theta_3\varrho\pi^{n-3})^{\zeta_3}\ldots,$$

where

$$\alpha_1+\alpha_2+\alpha_3+\ldots.=\alpha,$$
$$\beta_1+\beta_2+\beta_3+\ldots.=\beta,$$
$$\gamma_1+\gamma_2+\gamma_3+\ldots.=\gamma, \text{ \&c.}$$

Now the symbolical coefficient of

$$nm-(\beta+2\gamma+3\zeta+\ldots.)-s$$

in the preceding expression will be as follows:—

$$\begin{aligned}\Sigma\ldots(n-3)_{\zeta'_3}(n-3)_{\zeta''_3}\ldots(\zeta_3\text{ factors})(n-2)_{\gamma'_3}(n-2)_{\gamma''_3}\ldots(\gamma_3\text{ factors})(n-1)_{\beta'_3}(n-1)_{\beta''_3}\ldots(\beta_3\text{ factors})(n\alpha_3)_{\alpha'_3}\ldots\\ (n-3)_{\zeta'_2}(n-3)_{\zeta''_2}\ldots(\zeta_2\text{ factors})(n-2)_{\gamma'_2}(n-2)_{\gamma''_2}\ldots(\gamma_2\text{ factors})(n-1)_{\beta'_2}(n-1)_{\beta''_2}\ldots(\beta_2\text{ factors})(n\alpha_2)_{\alpha'_2}\ldots\\ (n-3)_{\zeta'_1}(n-3)_{\zeta''_1}\ldots(\zeta_1\text{ factors})(n-2)_{\gamma'_1}(n-2)_{\gamma''_1}\ldots(\gamma_1\text{ factors})(n-1)_{\beta'_1}(n-1)_{\beta''_1}\ldots(\beta_1\text{ factors})(n\alpha_1)_{\alpha'_1}\end{aligned}$$

$$\begin{aligned}&\delta^{\alpha'_1}\theta_1\varrho\delta^{\beta'_1}\theta_1\varrho\delta^{\beta''_1}\theta_1\varrho\ldots\delta^{\gamma'_1}\theta_2\varrho\delta^{\gamma''_1}\theta_2\varrho\ldots\delta^{\zeta'_1}\theta_3\varrho\ldots\\ &\delta^{\alpha'_2}\theta_1\varrho\delta^{\beta'_2}\theta_1\varrho\delta^{\beta''_2}\theta_1\varrho\ldots\delta^{\gamma'_2}\theta_2\varrho\delta^{\gamma''_2}\theta_2\varrho\ldots\delta^{\zeta'_2}\theta_3\varrho\ldots\\ &\delta^{\alpha'_3}\theta_1\varrho\delta^{\beta'_3}\theta_1\varrho\delta^{\beta''_3}\theta_1\varrho\ldots\delta^{\gamma'_3}\theta_2\varrho\delta^{\gamma''_3}\theta_2\varrho\ldots\delta^{\zeta'_3}\theta_3\varrho,\ \ldots\text{ \&c.,}\end{aligned}$$

where the sum of the indices of δ must equal (s).

From this expression, by putting

$$nm-(\beta+2\gamma+3\zeta+\ldots.)-s=\mu,$$

the symbolical coefficient of π^μ in the expression of the given multinomial may be immediately deduced.

There are certain methods of expressing the differential coefficients of implicit functions by means of symbolical notation which I notice in this place, as the method of summation employed for that purpose is similar to some of the symbolical summations we have already considered in this paper.

Let $F(x, y)$ be a function of x and y, y being a function of (x), and $y_1, y_2, y_3, \ldots$ the successive differential coefficients of (y). Then

$$\frac{dF}{dx}=\frac{dF}{dx}+\frac{dF}{dy}y_1,$$

$$\frac{d^2F}{dx^2}=\frac{d^2F}{dx^2}+2\frac{d^2F}{dxdy}y_1+\frac{d^2F}{dy^2}y_1^2+\frac{dF}{dy}y_2,$$

$$\frac{d^3F}{dx^3}=\frac{d^3F}{dx^3}+3\frac{d^3F}{dx^2dy}y_1+3\frac{d^3F}{dxdy^2}y_1^2+\frac{d^3F}{dy^3}y_1^3+3\frac{d^2F}{dxdy}y_2+3\frac{d^2F}{dy^2}y_1y_2+\frac{dF}{dy}y_3,$$

$$\frac{d^4F}{dx^4}=\frac{d^4F}{dx^4}+4\frac{d^4F}{dx^3dy}y_1+6\frac{d^4F}{dxdy^2}y_1^2+4\frac{d^4F}{dx^2dy^3}y_1^3+\frac{d^4F}{dy^4}y_1^4$$

$$+6y_2\left\{\frac{d^3F}{dx^2dy}+2\frac{d^3F}{dxdy^2}y_2+\frac{d^3F}{dy^3}\right\}+4\frac{d^2F}{dxdy}y_3+\frac{d^2F}{dy^2}(4y_1y_3+3y_2^2)+\frac{dF}{dy}y_4.$$

Now let $U^r_{m,n}$ be the coefficient of $\frac{d^{m+n}}{dx^m dy^n}F$ in the expansion of $\frac{d^rF}{dx^r}$. Then

$$\frac{d^rF}{dx^r}=\frac{d^rF}{dx^r}+r\frac{d^rF}{dx^{r-1}dy}y_1+r\cdot\frac{r-1}{2}\frac{d^rF}{dx^{r-2}dy^2}y_2+r\cdot\frac{r-1}{2}\cdot\frac{r-2}{3}\cdot\frac{d^rF}{dx^{r-3}dy^3}y_3$$

$$+r\cdot\frac{r-1}{2}y_2\left\{\frac{d^{r-1}F}{dx^{\ -2}dy}+(r-2)\frac{d^{r-1}F}{dx^{r-3}dy^2}y_1+\frac{(r-2)(r-3)}{1\,.\,2}\frac{d^{r-1}F}{dx^{r-4}dy^3}y_1^2+\text{\&c.}\right\}$$

$$+\frac{d^{r-2}F}{dx^{r-3}dy}U^{r-2}_{r-3,1}+\frac{d^{r-2}F}{dx^{r-4}dy^2}U^{r-2}_{r-4,2}+\frac{d^{r-2}F}{dx^{r-5}dy^3}U^{r-2}_{r-5,3}+\ldots$$

$$+\frac{d^{r-3}F}{dx^{r-2}dy}U^{r-3}_{r-4,1}+\frac{d^{r-3}F}{dx^{r-5}dy^2}U^{r-3}_{r-5,2}$$

$$+\frac{d^{r-4}F}{dx^{r-5}dy}U^{r-4}_{r-5,1}+\text{\&c.}$$

We easily obtain this law of coefficients,

$$U^r_{m,n}=\frac{d}{dx}U^{(r-1)}_{m,n}+y_1U^{(r-1)}_{m,n-1}+U^{r-1}_{m-1,n}$$

whence

$$U^r_{0,n}=\frac{d}{dx}U^{r-1}_{0,n}+y_1U^{r-1}_{0,n-1}$$

$$=\frac{d^2}{dx^2}U^{r-2}_{0,n}+\left(\frac{d}{dx}y_1+y_1\frac{d}{dx}\right)U^{r-2}_{0,n-1}+y_1^2U^{r-2}_{0,n-2}$$

$$=\frac{d^3}{dx^3}U^{r-3}_{0,n}+\Sigma\left(\frac{d^2}{dx^2}y_1\right)U^{r-3}_{0,n-1}+\Sigma\left(\frac{d}{dx}y_1^2\right)U^{r-3}_{0,n-2}+y_1^3U^{r-3}_{0,n-3}$$

$$=\frac{d^{r-1}}{dx^{r-1}}U^1_{0,n}+\Sigma\left(\frac{d^{r-2}}{dx^{r-2}}y_1\right)U^1_{0,n-1}$$

$$+\Sigma\left(\frac{d^{r-3}}{dx^{r-3}}y_1^2\right)U^1_{0,n-2}+\ldots+\Sigma\left(\frac{d^{r-\nu-1}}{dx^{r-\nu-1}}y_1^\nu\right)U^1_{0,n-\nu}+\text{\&c.}$$

Let $n-\nu=1$, then $U^1_{0,n-\nu}=U^1_{0,1}=y_1$, and $U^1_{0,n}=U^1_{0,n-1}=\&c.=0$. Hence

$$U^r_{0,n}=\Sigma\left(\frac{d^{r-n}}{dx^{r-n}}y_1^{n-1}\right)y,$$

where $\Sigma\left(\frac{d^{r-n}}{dx^{r-n}}y_1^{n-1}\right)$ denotes the sum of all the symbolical products in which $\frac{d}{dx}$ occurs $r-n$ times, and y_1 $(n-1)$ times.

We have

$$U^r_{1,n}=\frac{d}{dx}U^{r-1}_{1,n}+y_1U^{(r-1)}_{1,n-1}+U^{r-1}_{0,n},$$

from which, substituting for $U^{r-1}_{0,n}$ the value just obtained, $U^r_{1,n}$ may be found in a similar manner, and so we may proceed. I shall not, however, enter upon these higher coefficients, my object being principally to call attention to the use of symbolical expressions in expansions of this nature.

The subject of implicit differentiation has been treated by Mr. GEORGE SCOTT of Trinity College, Dublin, in a very elegant paper in the Quarterly Journal of Mathematics, vol. iv. p. 77. His results have great generality, but do not appear to include the above.

XXV. *Numerical Elements of Indian Meteorology.* By Hermann de Schlagintweit, *Ph.D., LL.D., Corr. Memb. Acad. Munich, Madrid, Lisbon, &c.* *Communicated by* Major-General Sabine, *P.R.S.*

Received May 4,—Read May 21, 1863.

First Series.—*Temperatures of the Atmosphere, and Isothermal Lines of India.*

I. Materials collected: calculation of the Daily Mean.
II. Tables of 207 Stations of Mean Temperature—Months, Seasons, and Years.
III. Decrease of Temperature with Height in the Tropics.
IV. Thermal Types of the Year and the Seasons.

I. *Materials collected: calculation of the Daily Mean.*

The numerical elements of the mean temperature[*] of the atmosphere for India and the Indian Archipelago here presented, I had occasion to collect during the years 1854–58. For judging of the value of the data I had obtained, and for working out the general results, it was very favourable that, for most of the stations, I had occasion personally to see the instruments employed and the mode of their being put up.

Already some years ago a considerable number of these stations had been published for the year 1851, by Dr. Lambe in the Journal of the Asiatic Society of Bengal, as well as by Colonel Sykes in the Report of the British Association for 1852; but as the materials sent in consisted, nearly exclusively, of results presented as means, which however were but the plain arithmetical mean of the respective hours of observation without any further modification, it was particularly welcome to me that the Indian Government, by the mediation of Dr. Macpherson, handed me over the original manuscripts, now forming thirty-nine volumes in folio.

A new calculation of the mean temperatures showed for many of these stations, particularly for the warmer period of the year, results lower by many degrees than the values formerly adopted; the difference would have been greater still and more frequent, if for many of the Indian stations the daily variation of temperature had not been included altogether within comparatively narrow limits.

The publication of Colonel Sykes[†] in 1850, the observations communicated being his own, or those of contemporaneous residents, contains throughout means based upon hours carefully selected.

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[*] All temperatures are Fahrenheit.

[†] "Discussion of Meteorological Observations taken in India," by Colonel W. H. Sykes, F.R.S., Philosophical Transactions, Part II. 1850.

Also the meteorological publications of Dove and Schmidt* contained important contributions for completing the number of the Indian stations, and for comparing them with the surrounding regions†.

The hours of observation at the various stations had been in general selected so as to include the minimum at the time of sunrise and the hours 10 A.M., 4 P.M., these two nearly coinciding with the barometrical extremes; also the maximum of the day a little after 2 P.M., and an evening observation is very frequently contained in these Tables; but, with few exceptions, the latest period was 6 P.M., or sunset. This circumstance excluded therefore the introduction of an evening hour more distant from the maximum, such as 9 P.M. or 10 P.M., into the calculation of the mean. A very favourable modification it was, however, that hourly observations existed for several stations, very accurately made, though situated in regions where the daily variation of temperature is not a very great one. These stations are Bombay, Calcutta, Madras, Trevandrum. Already Dove, so very careful in completing his collections of meteorological materials, has published several years for each of these stations‡. For calculating such Indian stations as show a more continental character in their variation of temperature, I could take advantage of the observations which we had occasion to make ourselves during our travels, a material which, I think, presented sufficient data for defining the mode of calculation, by their number as well as by their geographical distribution.

A combination of sunrise and sunset with either the maximum of the day or the observation at 4 P.M. showed very unfavourable results, even if variable coefficients were introduced for the different months, since, for the various geographical regions, the changes in the daily variation of temperature during the year are very great. Also the combination of the extremes with one morning hour, as I formerly had applied them to Alpine stations§, gave no satisfactory results, since in India the morning hours 9 A.M. or 10 A.M. had risen already considerably more above the mean of the day than is the case in the temperate zone.

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* Dove, "Tafel der mittleren Temperaturen verschiedener Orte in Réaumur'schen Graden," and "Ueber die nicht periodischen Aenderungen der Temperaturvertheilung," 6 parts.

† Amongst the 207 stations of the numerical Tables, pages 532–537, the following stations had to be taken over without recalculation, or without the addition of new material: from the publications of Colonel Sykes, Átăre Mállē, Ahmednắgar, Mahabaléshvar, Máhu, Phắltăn, Púna, Satára; from the series of the Medical Board Observations only Gughéra remained without the addition of new material; and from Dove's Tables I took over, with their values unchanged, Álor Gája, Áva, Bangkók, Chandernagúr, Chúsan, Kắlsi, Kándi, Kánton, Makáo, Manilla, Mozăfarpúr, Pondichéri, Trivándrum. Dove's Seringapatám is the year 1816 for the neighbouring fort, French Rocks, for which I was able to add 1814, 1853, and 1854. In the "Lehrbuch der Meteorologie," von Schmidt, 1860, I found in addition, for the Archipelago, Banjuvángi, Palanbáng, Lahút. For want of details about the decrease of temperature with height in these regions I excluded them, their height being 2138, 2119, and 2104 feet.

‡ On the Daily Variations of the Temperature of the Atmosphere, Abhandl. Berl. Akad. for 1846, pp. 104–6.

§ Schlagintweit, "Neue Unters. phys. Geogr. d. Alpen," page 325; I had obtained there the following coefficients for deducing the mean temperature from the extremes and 9 A.M.: for the minimum 0·5; for the maximum 0·4; for 9 A.M. 0·11.

The arithmetical mean of the extremes, where registering-instruments had been used, showed temperatures in general too warm throughout the year; but this very circumstance induced me to try the combination of 4 P.M. (which I had for all stations) with the observations at sunrise; the latter is nearly always identical (except at stations in very great heights) with the minimum obtained by registering-instruments, and four o'clock is cooler, though but little, than the true maximum; the result was a much more satisfactory one than I had expected.

The coincidence of the minimum of temperature with sunrise is particularly general in the tropics. It materially depends upon the rapid ascent of the sun above the horizon, whilst with us, especially in summer, the effect of *insolation* upon clouds and the higher strata of the atmosphere is partly felt already on the surface of the earth before the sun himself is visible above the horizon. In very great heights, again, chiefly if it be a peak in a very isolated position, the tropics show also modifications similar to those of the temperate zones. There I found, just as I formerly had seen, too, on the Vincent Hütte (southern slope of Monte Rosa), that the temperature frequently began to rise several hours before sunrise*.

As another characteristic modification of the morning period in the tropics, I may add here that very frequently the absolute minimum is followed by a second, though minor depression. This becomes best marked in the tropical seas; I found it greatest, when the sky was clear, five to ten minutes after sunrise, and it amounted not unfrequently to a full degree, but it never went lower than the absolute minimum preceding. I considered the cause of it to be the change in the relative humidity, which has attained its maximum nearly at the moment of sunrise. The appearance of the sun above the horizon coincides, too, with the heaviest precipitation of dew, and from this moment the relative humidity is rapidly decreasing whilst the temperature begins to rise. Not only is radiation now increased with the transparency of the atmosphere, but also the amount of heat becoming latent in consequence of the dissolving of vesicular vapours might participate in producing the second depression of temperature.

For presenting an immediate comparison of the value $\frac{\text{min.} + 4 \text{ P.M.}}{2}$ with the mean of the 24 hours, I have given the corrections to be applied in the following Tables (with "—" if the calculated value is too large, with "+" if it is too small), and have added the corresponding corrections for three other combinations. At Bombay and Calcutta, hourly observations are made every day, Sundays excepted; I took 1855 as the year the least distant from any other observations. For Tónglo, Falút, Islamabád, and Leh, the periods of observation are only months. For Ambála I had no quite regular series completely including the daily period, but the considerable number of observations from morning to night, combined (by the particular kindness of the observer, Dr. TRITTON) with very good extremes and isolated nocturnal observations, allowed me to define with sufficient precision the form of the monthly curves, and to deduce from these the hours still wanted.

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* Neue Unters. Geogr. Alpen, pp. 278–80.

I thought it not uninteresting to complete the comparison of my mode of calculating with those generally used, by adding the value of $\frac{\text{min.}+4\ \text{A.M.}}{2}$ also for some other stations, situated beyond India, and greatly differing in reference to their climatological character.

A. *From India, the Himálaya, and Tibet.*

Bombay in the Kónkan, lat. N. 18° 53′ 30″, long. E. Green. 72° 49′ 5″, height L.a.L.S.*

1855.	Mean.	$\frac{\text{S. R.}+\text{IV.}}{2}$.	$\frac{\text{Max.}+\text{Min.}}{2}$.	$\frac{\text{VI.}+\text{II.}+\text{X.}}{3}$.	$\frac{\text{VII.}+\text{II.}+2\,.\,\text{IX.}}{4}$.
January	74·7	−0·6	−0·9	+0·1	0
February ...	76·9	−0·5	−0·8	+0·1	0
March	79·3	0	−0·5	+0·5	+0·4
April	82·0	+0·3	−0·4	+0·6	+0·3
May	86·0	−0·3	−0·7	+0·4	+0·2
June	83·8	+0·1	−0·5	+0·2	+0·3
July	82·0	+0·1	−0·7	+0·1	+0·3
August	82·1	−0·5	−0·7	+0·4	+0·1
September ...	81·0	−0·2	−0·7	+0·1	+0·3
October	82·6	0	−0·7	+0·2	0
November ...	80·6	−0·7	−1·2	−0·1	−0·1
December ...	77·7	−0·7	−1·2	+0·1	−0·1
Mean ...	......	−0·12	−0·38	+0·11	+0·08

Calcutta in Bengal, lat. N. 22° 33′ 1″, long. E. Green. 88° 20′ 34′, height L.a.L.S.

1855.	Mean.	$\frac{\text{S. R.}+\text{IV.}}{2}$	$\frac{\text{Max.}+\text{Min.}}{2}$	$\frac{\text{VI.}+\text{II.}+\text{X.}}{3}$.	$\frac{\text{VII.}+\text{II.}+2\,.\,\text{IX.}}{4}$.
January	66·5	0	−0·9	0	0
February ...	72·1	−0·8	−1·1	−0·3	−0·3
March	79·3	−0·6	−0·8	+0·4	+0·5
April	82·3	0·	−0·3	+1·1	+1·3
May	85·9	−0·6	−1·1	+0·3	+0·7
June	85·6	+0·1	−0·6	+0·4	+0·3
July	82·3	+0·4	−0·5	+0·1	0
August	83·7	+0·2	−0·5	+0·3	+0·3
September ...	82·3	+0·3	−0·6	0	+0·1
October	81·2	+0·2	−0·4	+0·3	+0·2
November ...	74·4	+0·2	−0·9	0	+0·3
December ...	66·9	+0·1	−1·2	0	0
Mean ...	......	−0·02	−0·73	+0·11	+0·14

Ambála in the Pănjáb, lat. N. 30° 21′ 25″, long. E. Green. 76° 48′ 49″, height 1026 feet.

1855.	Mean.	$\frac{\text{S. R.}+\text{IV.}}{2}$.	$\frac{\text{Max.}+\text{Min.}}{2}$.	$\frac{\text{VI.}+\text{II.}+\text{X.}}{3}$.	$\frac{\text{VII.}+\text{II.}+2\,.\,\text{IX.}}{4}$.
January	50·1	−0·1	−0·6	+0·5	+0·5
February ...	59·5	−0·1	−0·7	+1·3	+1·0
March	56·5	−0·2	−0·3	+1·4	+0·4
April	76·0	+0·7	+0·2	+2·3	+0·2
May	92·1	+1·7	+1·1	+0·5	−1·1
June	95·4	+1·2	+0·9	+0·2	−1·0
July	83·8	+0·3	+0·2	+0·1	+1·3
August	87·9	+1·1	+0·5	−0·6	+2·0
September ...	82·4	+1·1	+0·9	+1·1	+0·4
October	73·4	+0·3	+0·1	+1·8	+0·7
November ...	60·2	−1·9	−2·2	−0·7	−1·7
December ...	55·9	+0·8	−0·2	−0·9	+0·8
Mean	......	+0·41	−0·01	+0·58	+0·22

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* This abbreviation is placed for "a little above the level of the sea." The feet are English.

Tónglo Peak in Síkkim, lat. N. 27° 1′ 50″, long. E. Green. 28° 3′ 55″, height 10,080 feet.

1855.	Mean.	$\frac{\text{S. R.}+\text{IV.}}{2}$.	$\frac{\text{Max.}+\text{Min.}}{2}$.	$\frac{\text{VI.}+\text{II.}+\text{X.}}{3}$.	$\frac{\text{VII.}+\text{II.}+2.\text{IX.}}{4}$.
May	48·1	+0·5	−1·5	−0·2	0

Falút Peak in Síkkim, lat. N. 27° 6′ 20″, long. E. Green. 87° 59′ 0″, height 12,042 feet.

1855.	Mean.	$\frac{\text{S. R.}+\text{IV.}}{2}$.	$\frac{\text{Max.}+\text{Min.}}{2}$.	$\frac{\text{VI.}+\text{II.}+\text{X.}}{3}$.	$\frac{\text{VII.}+\text{II.}+2.\text{IX.}}{4}$.
May	46·9	−0·1	−0·5	0	0

Islamabád in Kashmír, lat. N. 33° 44′, long. E. Green. 75° 8′, height 5160 feet.

1856.	Mean.	$\frac{\text{S. R.}+\text{IV.}}{2}$.	$\frac{\text{Max.}+\text{Min.}}{2}$.	$\frac{\text{VI.}+\text{II.}+\text{X.}}{3}$.	$\frac{\text{VII.}+\text{II.}+2.\text{IX.}}{4}$.
October	51·3	+0·7	+0·3	+1·3	−0·7

Leh in Ladák, lat. N. 24° 8′ 2″, long. E. Green. 77° 14′ 36″, height 11,527 feet.

1856.	Mean.	$\frac{\text{S. R.}+\text{IV.}}{2}$.	$\frac{\text{Max.}+\text{Min.}}{2}$.	$\frac{\text{VI.}+\text{II.}+\text{X.}}{3}$.	$\frac{\text{VII.}+\text{II.}+2.\text{IX.}}{4}$.
September ...	60·1	−0·1	−0·2	+0·7	−0·2

B.* *From the temperate zone in low elevations.*

Rome, lat. N. 41° 54′, long. E. Green. 12° 25′, height 170 feet.

	Mean.	$\frac{\text{S. R.}+\text{IV.}}{2}$.	$\frac{\text{Max.}+\text{Min.}}{2}$.	$\frac{\text{VI.}+\text{II.}+\text{X.}}{3}$.	$\frac{\text{VII.}+\text{II.}+2.\text{IX.}}{4}$.
January	49·95	−0·07	−1·15	−0·22	+0·09
July	75·47	+0·36	+0·20	+1·62	+0·97

Greenwich, lat. N. 51° 29′, long. E. Green. 0° 0′, height 156 feet.

	Mean.	$\frac{\text{S. R.}+\text{IV.}}{2}$.	$\frac{\text{Max.}+\text{Min.}}{2}$.	$\frac{\text{VI.}+\text{II.}+\text{X.}}{3}$.	$\frac{\text{VII.}+\text{II.}+2.\text{IX.}}{4}$.
January	35·45	−0·02	−0·40	−0·31	−0·22
July	59·65	+0·40	−0·34	+0·45	−0·13

St. Petersburgh, lat. N. 59° 36′, long. E. Green. 30° 18′, height L.a.L.S.

	Mean.	$\frac{\text{S. R.}+\text{IV.}}{2}$.	$\frac{\text{Max.}+\text{Min.}}{2}$.	$\frac{\text{VI.}+\text{II.}+\text{X.}}{3}$.	$\frac{\text{VII.}+\text{II.}+2.\text{IX.}}{4}$.
January	13·57	+0·16	−0·11	−0·29	−0·25
July	62·37	−0·12	−0·13	+0·47	−0·11

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* The date is not added, the means being taken from various series, all of several years' duration.

Toronto, lat. N. 43° 40′, long. W. Green. 79° 22′, height 340 feet.

	Mean.	$\frac{\text{S.R.}+\text{IV.}}{2}$.	$\frac{\text{Max.}+\text{Min.}}{2}$.	$\frac{\text{VI.}+\text{II.}+\text{X.}}{3}$.	$\frac{\text{VII.}+\text{II.}+2.\text{IX.}}{4}$.
January	26·37	+0·22	−0·36	−0·18	−0·40
July	65·60	−0·06	−0·07	+0·94	+0·20

C. *From the Alps.*

Geneva, lat. N. 40° 12′, long. E. Green. 6° 10′, height 1334 feet.

	Mean.	$\frac{\text{S. R.}+\text{IV.}}{2}$.	$\frac{\text{Max.}+\text{Min.}}{2}$.	$\frac{\text{VI.}+\text{II.}+\text{X.}}{3}$.	$\frac{\text{VII.}+\text{II.}+2.\text{IX.}}{4}$.
January	30·81	−0·13	−0·54	−0·18	−0·16
July	64·16	+0·59	+0·43	0	−0·81

St. Bernard Hospital, lat. N. 45° 50′, long. E. Green. 6° 6′, height 8108 feet.

	Mean.	$\frac{\text{S. R.}+\text{IV.}}{2}$.	$\frac{\text{Max.}+\text{Min.}}{2}$.	$\frac{\text{VI.}+\text{II.}+\text{X.}}{3}$.	$\frac{\text{VII.}+\text{II.}+2.\text{IX.}}{4}$.
January	13·41	+0·14	−0·31	+0·02	−0·02
July	42·84	+0·61	−0·18	0	−0·31

II. *Tables of Mean Temperature for the Month, Seasons, and the Year* (207 *Stations*).

Ten geographical groups are formed of the meteorological materials, and within these the stations are arranged alphabetically.

The number of stations is 207, and they are distributed as follows:—

1. Eastern India: 1, Assam; 2, Khássia Hills	12
2. Bengál and Bahár, and Delta of the Ganges and Brahmapútra	36
3. Hindostán, the upper Gangetic plain	27
4. Pănjáb, including the stations west of the Indus	24
5. Western India: Răjvára, Guzrát, Kăch, Sindh	10
6. Central India: Berár, Oríssa, Málva, Bandelkhănd	15
7. Southern India, hilly districts: 1, Dékhan and Maissúr; 2, Nílgiris	29
8. Southern India, coasts: Kónkan, Málabar, Karnátik	24
9. Ceylon	10
10. Indo-Chinese Peninsula, Archipelago, and China	20

The *transcription* of the geographical names is the same used and detailed by me in our ‘Results’*; the vowels are written as in Italian and German, the consonants as in English, with very few modifications, such as "th" being an aspirated "t," &c. Nasal modifications of the vowels are indicated by a circumflex. Every word has its principal accent marked by the usual sign. The sign ˘ above a vowel shows its imperfect phonetic formation, such as "e" in *herd*.

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* The full detail is contained in vol. iii. pp. 139–60.

The *latitude* is north, unless an S is written before the respective numbers.

The *longitude*, east of Greenwich, is referred to the Madrás observatory, its value being adopted =80° 13′ 56″. The sign * before the stations indicates that the latitude and longitude have been determined by the great Trigonometrical Survey of India; our own determinations are marked by the sign †. For the remaining stations the coordinates are taken from the most detailed maps.

The *height* is given in English feet; I took it from our "Hypsometry," vol. ii. of the 'Results.' Heights in round numbers, for which I had no detailed data, are put in brackets. To places very little elevated above the level of the sea L.a.L.S. is added.

The *seasons* are formed as it is usually done also for the stations of other latitudes; these groups coincide besides, for Central and Northern India, with the character of the climate in general. For the stations in lower latitudes, however, the type of the climate only allows of distinguishing a hot season, a rainy one, and a cool one.

The *numerical values** are corrected only for instrumental errors, or combinations of hours not sufficiently careful; but the influence of height, and in consequence the difference from the isothermal lines next to the respective stations, had to follow separately.

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* The stations where no decimals are seen (but fractions or full numbers only) are stations of somewhat minor accuracy.

1. EASTERN INDIA.

1. ASSÁM.

Station.	Lat.	Long.	Height.	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	D.J.F.	M.A.M.	J.J.A.	S.O.N.	Year.
			ft.																	
Bărpétah …………	26° 18′	91° 0′	(100)	62·2	66·2	71	75·7	78·5	80	82	83	82·5	88·5	71	65·2	64·5	75·1	81·7	80·7	75·5
Dibrugárh ………†	27 32	94 57·6	396	62·2	63·4	71·3	72·7	77·1	80·7	83·7	81·8	81	75·6	67·4	61	62·2	73·7	82·1	74·7	73·2
Goalpára …………*	26 11	90 36·6	(120)	63 2	67·8	73·7	78	80·2	79·8	81·4	81·6	81·1	78·5	70·6	64·9	65·3	77·3	80·9	76·7	75·1
Gohátti …………†	26 5·8	91 43·8	134	63·6	67·6	74·5	77·4	80·4	81·8	83	82·9	82·2	79·2	71·1	65·5	65·6	77·4	82·6	77·5	75·8
Golaghát …………	26 33	93 58	(350)	59·5	61·5	71·5	76·5	79·5	81·5	84·5	85	83	79·2	66	60	60·3	75·8	83·7	76·1	74
Lăkhimpúr ………	27 31	94 55	410	61	62·5	68·5	73·5	79·5	82	..	…	…	…	66	62·5	62	73·8	…	…	…
Măngaldái …………	26 24	92 1	155	66·5	69	70	77	80	79·5	82·5	82	81·5	78·5	71·5	64·5	66·7	75·7	81·3	77·2	75·2
Naugóng …………	26 21	92 49	(250)	61·3	66·4	72	76·4	81·1	83·4	84	84·4	83·2	78·6	69·8	63·5	63·7	76·5	83·9	77·2	75·3
Naziruaghát …… …	26 52	94 42	(400)	61·4	64·2	68	72	78·4	81·5	82·7	82	81·7	77·1	67·4	60·4	62	72·8	82·1	76·9	73
Sibságar ……………	27 2	94 39	(370)	60	64·1	69·3	73·8	78·05	82·4	83·6	83·5	83·1	78·3	69·4	62·4	62·2	73·7	83·2	76·9	74
Tézpur……………†	26 34·6	92 46·8	278	59·7	63·8	69·5	74·5	78·9	82	82·7	81·2	81·3	77	68·5	61·3	61·6	74·3	82	75·6	73·4

2. KHÁSSIA HILLS.

Station.	Lat.	Long.	Height.	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	D.J.F.	M.A.M.	J.J.A.	S.O.N.	Year.
Cherrapúnji ……†	25 14·2	91 40·5	4125	51·8	54·7	61·5	63·5	67·2	67·1	68·5	68·1	67·8	65·8	58·8	55·1	53·9	64·1	67·9	64·1	62·5

2. BENGÁL AND BAHÁR,

AND DELTA OF THE GANGES AND BRAHMAPÚTRA.

Station.	Lat.	Long.	Height.	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	D.J.F.	M.A.M.	J.J.A.	S.O.N.	Year.
			ft.																	
Băkúra……………*	23° 14′·8	87° 3′·1	L. a. L. S.	63	72·7	79·7	85·5	87·9	87·2	84·2	84·3	81·9	74·4	70·5	63·2	66·3	84·4	85·2	75·6	78
Bărdván …………*	23 13·2	87 48·9	93	66·6	73·7	81·2	86	90	86·5	85·2	85·4	85·3	82	73·1	69·3	69·9	85·7	85·7	80·1	80·4
Barisál……………*	22 35·7	90 13·6	L. a. L. S.	64·1	70	72·2	80·4	84·8	81·8	81·1	81·8	81·9	81·3	72·9	65·1	66·4	79·1	81·6	78·7	76·4
Barrakpúr…………	22 42·6	88 21·8	L. a. L. S.	67·6	69·4	79·5	86·1	83·7	85·1	83·3	83·3	83·1	80·2	73·7	69·7	69·6	83·1	83·9	79	78·7
Bhăgalpúr ………*	25 14·8	86 56·6	154	62 6	73·4	78·6	89·9	92·1	87·9	85·7	82·5	84	77·5	69·5	63·1	66·4	86·9	85·4	77	78·9
Birbhúm …………*	23 54·4	87 30·6	L. a. L. S.	67·1	72·9	81	86·6	89·3	86·6	84	83·7	83·8	80·2	73·7	68·5	69·5	85·6	84·8	79·2	79·8
Bógra………………	24 50	89 22	L. a. L. S.	63·8	64·8	74	77·9	82·9	81·3	83·5	82·0	82·4	78·3	71·1	64·3	64·3	78·3	82·3	77·3	75·5
Calcutta …………*	22 33	88 20·6	L. a. L. S.	65·60	71·06	77·99	83·37	85·37	84·18	82·69	83·05	83·06	81·35	74·68	67·70	68·12	82·24	83·31	79·70	78·34
Chaiabásso ………*	22 31·7	85 42·8	L. a. L. S.	67	70·9	79·6	86	88·6	86·4	82·8	81·5	81·3	78	71·5	65	67·6	84·7	83·6	76·9	78·2
Chandernagúr ……	22 50	88 23	46	60·8	68·8	77·3	88·2	93·7	94·2	86 8	87·2	83·7	79·7	72·7	63·7	64·4	86·4	89·4	78·7	79·7
Chápra…… ………*	26 45	85 25·8	250	69·7	67	75·4	86·4	89·6	86·1	85·1	84·2	78·8	79·1	68·5	60	62·2	83·8	85·1	75·5	76·6
Chittagóng ………*	22 20·5	81 44·1	191	63·6	69·4	76	80·5	83	79·7	80·2	80·7	81·1	79·7	72·2	66·2	66·4	79·8	80·2	77·7	76·0
Chunár……………*	25 7·5	82 51·6	(300)	68	62½	77¼	84	89	88	90	85½	82¼	79¾	69½	63¼	61·2	83·5	87·8	77·2	77·4
Dáinajpur ………*	25 36·6	83 36·8	180	63	68½	80	87	90	80½	83½	81½	81½	80	72	66	66	86	81·8	77·8	77·8
Dămdắm…………*	22 37·9	88 21·2	L. a. L. S.	66·2	72·7	79	85·7	86	85·1	82·7	82·8	82·9	81·2	74·7	68·7	69·2	83·6	83·5	79·6	79
Dháka …. …. ….†	23 42·7	90 20·3	72	65·6	69·6	80·1	82·8	85·1	83·5	82·7	83·8	83·8	81·6	74·1	67·7	67·6	82·7	83·3	79·8	78·4
Fărídpur …………*	23 36·5	89 48·9	L. a. L. S.	65·8	70·2	78·9	81·9	78·5	82·2	82·4	82·1	82·1	80·1	73·8	67·2	67·7	79·8	82·2	78·7	77·1
Gáya ……………	24 49	85 0	280	63·4	70·2	78·2	86·5	91·9	89·5	84·5	85·2	84·6	81·1	71·9	65·5	66·4	85·5	86·4	79·2	79·4
Hazaribágh ………*	24 0	85 20·9	1750	60	65·2	73·5	80·5	84·9	80·6	80·6	77·5	76·0	72·9	66·8	62	62·4	79·6	79·6	71·9	73·4
Húgli ……………*	22 53·4	88 23·1	L. a. L. S.	63·4	68	80	85	87	83·2	80·7	80·9	82·5	80	72·2	64·6	65·3	84	81·6	78·2	77·3
Jessór ……………*	23 9	89 7·1	L. a. L. S.	62	71·2	78·6	84·4	84·8	83·35	83·7	83·6	83·85	85·2	75·1	64·85	66	82·6	83·5	81·4	78·4

Kachár ……………*	24 48·7	92 43·9	L. a. L. S.	62·95	66·65	73·45	76·8	80·95	82·2	82·4	82·9	82·45	79·65	71·6	65·6	65·1	77·1	82·5	77·9	75·6
Kishánpur …………	23 28	85 20	(200)	60·15	72·15	78·8	85·75	88·65	83·2	78·15	78·95	78·7	76·85	70·35	63	65·1	84·4	80·1	75·3	76·2
Maimănsingh ……*	24 44·8	90 20·9	L. a. L. S.	62·45	65·95	75·8	74·45	83·8	82·3	83·5	82·9	82	81·3	72·7	65·55	64·6	78·0	82·9	78·7	76
Mídnapur ………*	22 24·3	87 17·9	L. a. L. S.	…	71	77·7	…	…	81·6	80·4	78 1	78·2	…	…	…	…	..	…	…	…
Monghír …………*	25 27·4	86 40·2	200	65¼	68	76½	84½	92½	80	86	86½	85	80½	70½	…	…	84·5	87·5	78·7	…
Murshedabád ……*	24 11·8	88 9·9	L. a. L. S.	64	69·2	80·4	86	87	86·5	84·6	86·7	86·5	82·5	74	68	67·1	84·5	85·9	81	79·6
Noakólli …………*	22 45·5	90 57·8	L. a. L. S.	65·6	70·1	78·8	80·4	84·9	82·3	83·2	81·6	82·3	80·7	75	67·8	67·7	81·4	82·4	79·4	77·5
Pắbna………………	24 1	89 12	L. a. L. S.	65·2	72·8	78·9	83·9	85·3	83·7	85·3	84·1	84·2	79·5	71·8	66	68	82·7	84·4	78·5	78·4
Pắrnea ……………*	25 48	87 29·6	L. a. L. S.	63⅓	64⅓	70½	87	83½	84½	87½	83½	83·6	79·8	71·1	64	64	80·3	85·2	78·2	76·9
Pátna ……………*	25 37·2	85 7·5	170	60·8	66·1	76·1	83·9	88·8	87·3	84·5	84·7	83·9	80·4	71·1	62·2	63	82·9	85·5	78·5	77·5
Rámpur Bólea……*	24 21·8	88 34·3	56	64·8	69·7	77·2	82·5	85·8	83·5	83·8	83·7	83·8	81·1	71·5	66·8	67·1	81·8	83·7	78·8	77·8
Rắngpur …………*	25 42·8	89 11·4	72	61·5	67·5	75·1	81·8	82·3	81·3	83	83	83·4	80·3	74·8	69·2	66·1	79·7	82·4	79·5	76·8
Silhét ……………*	24 53	91 47·1	L. a. L. S.	65·5	69·9	75	77·4	81·1	80·6	82·2	81·8	82	79·1	71	68·5	67·6	77·8	81·5	77·4	76·2
Típpera …………*	23 27·5	91 2·3	L. a. L. S.	64·7	68·4	75·6	81·1	82	81·5	81·6	81·8	81·3	80·2	72·2	65·8	66·3	79·6	81·6	77·9	76·3
Tirhút ……………*	26 7·3	85 22·8	255	58·2	64·9	73·7	82·3	87·2	86·8	80·7	80·3	84	79·2	69·6	61	61·4	81·1	82·6	77·6	75·7

3. HINDOSTÁN,

THE UPPER GANGETIC PLAIN.

Station.	Lat.	Long.	Height.	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	D.J.F.	M.A.M.	J.J.A.	S.O.N.	Year.
Ágra ……………*	27° 10′·2	70° 1′·7	ft. 657	57·5	66·4	76	85·5	94·8	93·9	86	85·3	84	78·1	69·2	60·2	61·4	85·4	88·4	77·1	78·1
Aligărh……………†	25 53·8	78 39	750	58·6	63·3	75	83·9	91·4	93·6	87·2	86·3	84·4	78·4	68·3	60·6	60·8	83·4	89	77	77·5
Allahabád ………*	25 26	81 51·9	316	64·2	67·6	77·3	92·6	97·6	91·9	89·0	86·3	85·7	83·1	71·6	64·9	65·6	89·2	89·1	80·1	81
Azimgărh…………*	26 32	83 9·9	(550)	60	70·6	78·2	85·6	88·5	90	84·8	85·3	84·2	80·2	69·6	61·7	64·1	84·1	86·7	78	78·2
Baréli ……………*	28 22·2	79 23·2	693	58·9	65·8	71·6	77·8	86·8	89·7	84·3	86·5	81	78·4	71·2	60·2	61·6	78·7	86·8	76·9	76·0
Benáres ………… *	25 18·4	82 59·8	347	62·1	69·2	79·4	88·9	95·3	91	84·9	85·7	84·9	81·4	71·6	64·3	65·2	87·9	87·2	79·3	79·9
Bijnúz ……………	29 22	78 9	530	53	64	74½	81	87½	92	86	87⅓	86·4	76·8	65·9	57	58	81	88·5	76·4	76·0
Déhli ……………*	28 38·9	77 13·1	827	55	60·7	69·2	79·3	87	91·1	85·2	82·5	81·6	74·4	63	56·7	57·5	78·5	86·3	73	73·8
Étava ……………*	26 45·5	78 59·9	550	58·4	64·8	74·4	82·4	89·9	92·5	85·3	84·4	84	77·8	66·5	57·6	60·3	82·2	87·4	76·1	76·5
Fătigărh …………*	27 23·3	79 37	635	57 9	65·8	74·1	82·4	91·2	90·7	86·2	83·7	83·4	76·9	67·8	59	60·9	82·6	86·9	76	76·6
Gházipur …………*	25 33·6	83 31·8	351	63·6	67·5	79	87	93·7	90·1	88	85·4	86	83·5	70·6	64·9	65·3	86·6	87·8	80	80
Gorákhpur ………*	26 46·1	83 18·7	340	62·6	66·8	74·5	83·1	88 3	87·8	84·5	84·5	84·2	78·3	70·1	63·4	64·3	82	85·6	77·5	77·3
Javánpur…………*	25 43·8	82 40·7	(380)	57·5	63·4	74	82·3	86·2	87·2	…	82·5	83·2	80	67·4	59·3	60·1	80·8	…	76·9	…
Kắlsi ………………	30	77½	(1100)	58·3	61	62·6	77 7	81·1	86	83·7	80·2	77·2	70·8	63·2	59·7	59·6	73·8	83·3	70·3	71·8
Kánhpur …………*	26 28·3	80 20·3	525	58·3	66·4	75·7	86·3	93·8	93·4	85·2	85·9	84·8	78 1	69·8	61·7	62·1	85·3	88·2	77·6	78·3
Mainpúri …………	27 14	79 2	620	58	68	73	93	94	91	88	83	80	73	68	60	62	86·7	87·3	73·7	77·4
Mắthra……………*	27 30·2	77 40·3	655	60·4	67·5	74·4	84·1	89·5	93·4	86·8	86·5	86	80·1	72	62·7	63·5	82·7	88·9	79·4	78·6
Mírăth …………*	29 0·7	77 41·6	859	56·9	62·9	70·7	80·5	88·7	91·9	85·7	85·3	84	76·6	76·6	58·7	59·5	80	87·6	76·1	75·8
Mírzapur …………*	25 9·3	82 33·9	362	61½	64	75	83	90	90	87	86	85	80	70	64	63	83	88	78	78
Mozăfarpúr ………	26 7	83 21	(300)	56·8	64·6	73·6	77·1	86·9	86·4	84·4	83	82·6	78·1	67·5	60·1	60·8	80·7	84·7	76·1	75·6
Muradabád ………	28 49	78 56	673	58	60·9	72·6	80 6	86·75	88·75	84·1	84	83·5	74·9	66·1	58·75	58·9	80	85·6	74·7	74·8
Panipắt ……………	29 23	76 59	936	56·1	74·4	..	80·4	86	91·6	88·6	86·3	…	…	…	…	…	…	88·8	…	…
Saháranpur ………*	29 57·2	77 28·8	1002	55·5	58·9	70·5	78·5	88·3	91·8	87	86·5	83·1	77·4	67	60·7	58·4	79·1	88·4	75·8	75·4
Sărauli ……………	28 30	79 10	(700)	50·7	57·1	66·4	75·6	81	85·7	85·1	83·3	77·9	70·7	58·3	56·6	54·8	74·3	84·7	69	70·7
Shahjehắnpur ……*	28 1·6	79 31 8	(1200)	56·1	59·5	67·5	78·3	86·3	91·9	85·7	86·5	82·4	77·3	67 6	61·8	59·1	77·4	88	75·8	75·1
Sítapur ……………	27 35	80 44	(450)	…	…	…	76·95	85·9	87·6	86·95	83·05	82·7	74·5	…	..	…	…	85·9	..	…
Sultánpur…………*	26 15·6	82 33	(450)	60·7	68·3	76	87·1	…	…	85 4	…	…	…	…	58·9	62·6	…	..	…	…

4. PĂNJÁB,

INCLUDING THE STATIONS WEST OF THE INDUS.

Station.	Lat.	Long.	Height.	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	D.J.F.	M.A.M.	J.J.A.	S.O.N.	Year.
			ft.																	
Ambála ....†	30° 21′·4	76° 48′·8	1026	54·5	61	69·9	78·4	86·5	91·1	85·8	85·7	84·3	75	63·2	56·2	57·2	78·3	87·5	74·2	74·3
Ásni	29 12	70 7	(410)	52	61	70½	79	86	93	90	86½	86	72½	...	...	...	78·5	89·8	...	...
Bắnnu	32 40	70 30	(1800)	49·5	55	64	74	82	93	91	88·5	86·3	75	64·8	55·1	53·2	73·3	90·8	75·4	73·2
Dera Gházi Khan ...	30	70 54	(430)	51·4	59·7	68·9	78·6	85·2	93·8	91·3	88·3	86·2	71·7	64·5	55·5	55·5	77·6	91·1	74·1	74·6
Dĕra Ismáel Khan †	31 39·6	70 56·5	478	49	58·6	70	79	86·5	94·5	95·6	91·5	90	81·2	67	52·7	53·4	78·5	93·9	79·4	76·3
Firózpur ....*	30 57·1	74 38·4	1120	51	58·5	66·9	77·1	83	91·1	88·2	86·4	85·4	73·4	62·2	53·7	54·4	75·7	88·6	73·7	73·1
Govindgárh	31 40	74 45	(900)	51	56	64·7	73·2	81·2	85·3	85 5	84	83·5	76·5	63·2	55·5	54·2	73	84·9	74·4	71·6
Gugéra	30 51	73	(600)	50	59	69½	77½	90	95½	...	89·5	82	...	58	...	...	79	...	...	...
Hánsi .. ..*	29 6·1	75 57·1	(1000)	55·5	62	71·5	88·1	95·4	96·9	86	82	79·8	72	65	57·5	58·3	85	88·3	72·3	76
Hoshiárpur ....*	31 32·2	75 53·9	1066	53·9	59	66·6	80·5	85·2	92·1	86·2	84·7	84·1	78·4	63	56·5	56·5	77·4	87·7	75·2	74·2
Jălhándar ....*	31 19·5	75 33·3	(900)	53·5	59·6	66·9	76·4	81·5	87·4	86·9	84·4	83·9	75·9	65·6	57·9	57	74·9	86·2	75·1	73·3
Jhílum ....*	32 55·2	73 42	1620	50·45	58	65 2	73·3	83·4	91·5	87·25	85·2	84 95	74·5	64·6	49·05	52 5	74	88	74·7	72·3
Kartárpur ....*	31 26·7	75 29·1	(800)	57	61·4	67·5	76·6	86·4	91	87·5	87	88·6	79·6	69	60	59·5	76·8	88·5	79·1	76
Kohắt ....*	33 32·5	71 22·9	1725	51·8	59·4	68·7	75·9	86·7	91·3	90	87·6	86·7	78	62·2	57·6	56 3	77·1	89·6	75·6	74·7
Lahór ....†	31 31·1	74 14·6	839	52·3	60·7	68·3	78·6	87·1	92·2	87·6	86·4	85·7	77	66·4	57·7	56·9	78	88·7	76·4	75
Láya	30 59	70 57	(450)	49·5	59·3	66·6	76·5	83·2	91·9	88·6	85·9	83·8	71·6	58·8	50·5	53·1	75·4	88·8	71·4	72·2
Ludhiána..... ......*	30 55·4	75 50·2	893	51·5	61·8	67·8	77·4	84·7	91	87·8	87·8	84·8	75·3	..	53·4	55·6	76·6	88·9	...	...
Multán.............†	30 10·2	71 34·6	480	56½	61	67	80	84·7	94	92	90	88	79	70·5	59½	59	77·2	92	79·2	76·8
Nakódar..............	31 7	75 27	(840)	53·9	...	69·4	78·6	85·05	91·8	89	84·5	85·7	77·5	68·2	57·8	...	77·6	88·4	77·1	...
Naushéra...........*	34 3·1	71 58·4	(1200)	51·1	51·1	61·4	74	81·9	95·2	91·2	89·7	85·9	74	57·6	56	51·5	72·4	92	72·5	72·1
Pesháur ....*	34 3·2	71 33·3	1280	52·3	57·1	62·5	73·2	81	90·1	91·8	87·6	86·1	73·7	63·2	55·9	55·1	72·2	89·8	74·3	72·9
Raulpíndi ....†	33 36·5	72 59·8	1737	51·4	56·4	60·4	72·8	81·5	90·1	86·1	83·3	81	76	63·1	54·3	54	71·6	86·5	73·4	71·4
Shắhpur ....†	32 14	72 32·5	681	53	54	62	80	86	95	91	93	87	76	65	58	55	76	93	76	75
Vazirabád...........*	32 26·3	74 6·4	(900)	52	61·9	67·8	77·8	86·1	93·3	89·1	88	87·3	77·7	66·1	57·3	57·1	77·2	90·1	77	75·4

5. WESTERN INDIA,

RAJVÁRA, GUJRẮT, KĂCH, SINDH.

Station.	Lat.	Long.	Height.	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	D.J.F.	M.A.M.	J.J.A.	S.O.N.	Year.
			ft.																	
Ábu	24° 45′	72° 46′	3850	...	...	74	76·7	78	78	74	...	...	...	...	...	...	76·2	...	...	...
Ajmír ....*	26 27·2	74 40·6	(1500)	61·7	69·3	78·3	87·4	94·3	92·5	85·6	82·7	83·5	82·2	73	65·1	65·3	86·6	86·9	79·5	79·6
Barṓda	22 16	73 14	L. a. L. S.	69 3	73·2	81·7	90·6	94·7	88·4	80 8	81·4	82·1	79·8	73·3	71·2	71·2	89	83·5	78·4	80·5
Beávr	26 6	74 21	(2000)	...	64·75	75·1	88·2	94·5	91·7	85·8	82·6	82·6	...	68·9	...	...	85 9	86·7	...	...
Bhū́j	23 17	69 40	281	65½	71½	79½	82	84	87	81	80	79	75	70	59	65·3	81·8	82·7	74·7	76·1
Erinpúra ....*	25 9·3	73 63	(1500)	48½	59	70 5	84·1	89·4	90·3	85	82·1	81·7	79·1	67·2	56	54·5	81·3	85·8	76	74·4
Kărráchi ....†	24 45·5	67 0·9	L. a. L. S.	67	67	77	79	83	87	84	88	86	80	72	63	66	80	86	79	77·7
Khervára	26 4	74 20	(2000)	60·6	68·6	79	87·4	91·8	87·4	80·4	78·9	79·6	77·2	70·6	61·3	63·5	86·1	82·2	75·6	76·9
Nazirabád	26 18	74 42	1487	59·2	62·7	71·8	82·6	90·4	90·1	85·7	82·7	82·5	78·8	68·5	58·2	60	81·6	86·2	76·6	76·1
Nímăch...............*	24 27·5	74 59	1356	63·9	70·8	78·6	86·8	90·3	86·6	79·5	78·1	78·5	78·7	73·5	65·2	66·6	85·2	81·4	76·9	77·5

6. CENTRAL INDIA,

BERÁR, ORÍSSA, MÁLVA, BĂNDELKHÁND.

Station.	Lat.	Long.	Height.	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	D.J.F.	M.A.M.	J.J.A.	S.O.N.	Year.
			ft.																	
Báitul ....*	21° 51′·2	77° 54′·8	(2000)	59·3	62·3	72·2	83·1	84·8	80·4	76·8	75·6	78·3	73·8	63·3	60·8	60·8	80	77·6	71·8	72·5
Hamírpur	25 58	80 12	645	64·7	74·2	76·7	87·6	93·8	93·4	85·3	85·7	85	82·1	69	63·7	67·5	86	88·1	78·7	80·1
Hushangabád	22 45	77 42	1050	...	...	80·1	90	94·7	90·5	83·9	81·6	80·8	80	70·7	...	...	88·3	85·3	77·2	...
Jáblpur ....*	23 9·7	79 56·3	1396	62·8	66·8	75	85·3	91·3	87·8	81	80·4	81·5	75·1	67·9	60·8	63·5	83·9	83·1	74·8	76·3
Jhánsi	25 28	78 35	745	63·6	70·5	78·9	88·4	93·7	92·6	84·1	82·4	82·2	80·7	73·4	64·7	66·3	87	85·4	78·8	79·6
Kokonáda	17 6	12 14	L. a. L. S.	74·5	77·9	81	85·1	87·8	86·3	83·3	82 3	82	79	77·3	75·4	75·9	84	84	79·4	81
Máhu (Mhow)	22 33	75 49	1862	...	...	...	...	...	...	...	..	...	...	...	...	...	...	...	...	74
Nágpur	21 10	79 7	9535	71·4	74·9	84·3	93	96·3	86	81	81·1	81·5	81·6	75·4	72·6	72·9	91·2	82·7	79·5	81·6
Nărsinghpur	22 57	79 8	·1305	60·8	64·4	74	83·1	84·8	86·7	81·1	79·8	79·5	77·7	70·5	62	62·4	80·6	82·5	75·9	75·4
Naugóng ....*	25 3·5	79 27·6	(570)	60·9	73·3	76·4	82·7	91·3	92·4	85·7	84·1	83·7	80·2	71·1	63·4	65·9	83·5	87·4	78·3	78·8
Orái	25 59	79 31	(1700)	65	75½	80	90	96½	95·2	90·2	87·2	86·2	82·5	68·2	66	68·8	88·8	90·9	79	81·9
Púri ....*	19 48·2	85 45 8	L. a. L. S.	69·05	73·55	80·6	85	86·5	85·85	84·2	84·7	83·95	81·7	76·5	72·1	71·5	84	84·9	80·7	80·3
Ságar ....*	23 50·2	78 43·4	1906	58	71½	73	86	91·5	92	86	81¼	76½	76½	65½	63½	64·5	83·5	86·4	72·8	76·8
Sehór	23 12	77 1	1620	60·8	66·5	75	83·9	90·4	86 4	80·4	78·4	78	74·5	68	65	64·1	83·1	81·7	73·8	75·7
Vizagapatám	17 41	83 21	L. a. L. S.	71·4	75·9	81·2	...	87·1	..	...	84·5	...	. .	...	...	...	...	...	...	...

7. SOUTHERN INDIA, HILLY DISTRICTS:

1. DÉKHAN AND MAISSÚR.

Station.	Lat.	Long.	Height.	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	D.J.F.	M.A.M.	J.J.A.	S.O.N.	Year.
			ft.																	
Ahmadnágar	19° 6′	74° 46′	2133	..	...	...	...	...	..	..	...	...	...	...	...	...	...	...	...	78
Bangalúr	12 57·6	77 33·5	2949	69·5	72·1	78 2	79·7	80·6	75·3	73 2	74·4	74·3	72·8	71·5	69	70·2	79·5	74·3	72·9	74·2
Belgáũ	15 50	74 32	2500	70	74	82	81·5	84	82	74	74·5	74	73·5	70·5	68	70·6	82·5	76·8	72·6	75·6
Bellári ....*	15 8·9	76 53·8	1538	73·8	78	84·8	87·5	86·4	83·3	80	80·1	78·4	78·8	77·3	73·5	75·1	86·2	81·1	78·2	80·2
Bíjapur	16 50	75 47	(1700)	78·7	75·3	84 8	88·4	88·5	85	81·2	78·7	78·2	76·6	76·7	79·2	77·7	87·2	81·6	77·2	81·7
Dharvár..............	15 27	75 1	2423	70·2	74 7	77·2	80·4	80·3	74·8	72·9	72·7	...	...	...	...	(70)	79·3	73·5	...	...
French Rocks	12 31	76 40	2620	72·8	77·6	81·4	83·5	83·9	79·2	75·4	74·3	76·9	77·1	75	73·7	74·7	82·9	76·3	76·3	77·6
Gŭti ....*	15 6·9	77 38·1	1115	...	79·7	85·1	90·5	...	84·2	...	...	...	...	...	...	...	...	...	...	...
Hărihár	14 31	75 51	1900	74	78·9	86·1	87·7	86·7	81·2	75·8	78·7	79	80·7	79·3	75	76	86·8	78·6	79·7	80·3
Jálna	19 51	75 54	1652	74·3	75·7	83·2	87	90	83	77·2	79	77	79·4	77	72·5	74·2	86·7	79·7	77·8	79·6
Kădapa ....*	14 28·8	78 48·4	364	76·8	83·1	87	90·9	91·4	89	86·7	83·8	83·2	82·3	77·8	74·3	78·1	89·8	86·5	81·1	83·9
Kărnúl..............*	15 49·9	78 2·1	(900)	78·9	80·9	86·8	91·3	90·4	87	82·4	82·5	82·3	81·5	78·9	76·6	78·8	89·5	84·0	80·9	83·3
Kírki ....*	18 33·5	73 50·2	1850	71	75	81	81·5	82·5	78·5	77	75	75	80	77	71	72·3	81·7	76·8	77·3	77
Mahabalĕshvar ...*	17 54·4	73 38·7	4300	64	66·3	71·5	74·5	72·4	66·3	63·2	63·2	63·9	66·6	64·4	63·2	64·5	72·8	64·2	65	66·6
Merkára..............	12 24	75 45	4506	69·4	72·4	76·8	75·9	74·2	69·1	67·3	67	67·8	69·6	70	69·3	70·4	75·6	67·8	71·5	71·3
Phăltan	17 59	74 26	(1700)	74·8	76·9	83·9	83·9	86·1	81·2	80·2	79·3	78·9	80	73·5	72·9	74·9	84·6	80·2	77·5	79·3
Púna ....*	18 30·4	73 52·1	1784	70·7	73·7	74·1	82	82·3	80·8	77·4	77·1	77·1	79·1	76·4	70·6	71·7	79·5	78 4	77·5	76·8
Purandăr...........*	18 16·6	73 57·3	3974	67·3	70·2	77·6	80·1	77·6	69·6	65·2	64·7	64·8	67·6	68·2	65·2	67·6	78·4	66·5	66·9	69·8
Satára................	17 41	74 2	2320	70	72·6	77·8	80·6	80·1	77	73·8	73	74	76·1	72	71·8	71·5	79·5	74·6	74·1	75
Shólapur	17 40	75 58	(1700)	74½	80	83	89	85	85	82	81	80	78	74	73	75·8	85·7	82·7	77·3	80·4
Sikăndarabád ......*	16 26·7	78 27	1830	68·2	70·1	79·9	84·7	87·5	81·9	78	77·8	76·6	76·5	75·5	71·2	70	84	79·2	76·2	77·4

2. NÍLGIRIS.

Station.	Lat.	Long.	Height.	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	D.J.F.	M.A.M.	J.J.A.	S.O.N.	Year.
			ft.																	
Átăre Mállē	8° 31′	77° 10′	(4500)	62·7	65	67·6	68·1	66·5	63·7	64·9	65·5	66·2	64·5	64·7	63·3	63·6	67·4	64·7	65·1	65·2
Dodabétta	11 23	76 44	8640	50·8	52·2	55·1	56·6	57·7	52·9	52·7	52·9	52·2	53	51·9	50·8	51·3	56·5	52·8	52·4	53·2
Jakunári	11 24	76 53	(5000)	57·1	55·2	60·8	62·9	63·5	62	64·3	62·5	58·3	61·8	57·1	59·7	57·3	62·4	62·9	59·1	60·4
Koimbatúr...........	11 1	76 58	1483	73·5	72·7	79·2	81·7	81	77	76·7	77·2	77·5	76	74·2	73·2	73·1	80·6	77	75·9	76·7
Koterghérri	11 26	76 57	6100	58·7	59·7	60·5	62	62·2	63·5	65	65·5	64	62	60·5	59	59·1	61·6	64·7	62·2	61·9
Manantvádi	11 48	76 1	2685	57	63¼	70¾	71½	72	69	67	67	67½	67¾	68	65¼	61·9	71·4	67·7	67·7	67·2
Sirlu	11 22	76 55	(3500)	68·5	69·5	73·6	...	76·1	...	...	...	...	...	...	...	...	...	...	...	...
Utakamánd........*	11 23·7	76 43·2	7490	51·5	52·8	57·3	60·1	60·8	57·9	55·8	56·1	56·4	55·9	53·9	51·9	52·1	59·4	56·6	55·4	55·9

8. SOUTHERN INDIA, COASTS,

KÓNKAN, MÁLABAR, KARNÁTIK.

Station.	Lat.	Long.	Height.	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	D.J.F.	M.A.M.	J.J.A.	S.O.N.	Year.
			ft.																	
Anjarakándi	11° 40′	75° 40′	L. a. L. S.	79·7	81·9	83·5	85·6	83·7	79·8	78·4	79·1	79·6	80·7	80·6	80·2	80·6	84·3	79·1	80·3	81·1
Árkot ....*	12 54·3	79 19	599	74	76	82	87	85	88	87	85	83	81	78	73	74·3	84·6	86·6	80·6	81·5
Bombay ....*	18 53·5	72 49·1	L. a. L. S.	74·5	76·2	80	83·3	85·6	83·3	81·3	80·7	80·4	82·1	79·7	76·5	75·7	83	81·8	80·7	80·3
Chittúr.............*	13 11	79 6	1112	75·6	77·8	82·2	88·4	90	86·9	84·8	83·1	83	78·5	74	73·7	75·7	86·9	84·9	78·5	81·5
Gantúr..............*	16 17·7	80 25·6	L. a. L. S.	74½	78½	81½	84½	88	85½	84	83½	82½	82	81	78½	77·2	84·7	84·3	81·8	82
Kádalur ....*	11 43·6	79 45·7	L. a. L. S.	77·4	79·2	83·3	84·9	87·3	86·1	86	83·9	84·5	82·8	78·2	76·3	77·6	85·2	85·3	81·8	82·5
Kalikát..............*	11 15·2	75 45·4	L. a. L. S.	81·3	82·2	83·9	84·8	85·5	80	78·7	79·4	79·5	80·3	80·4	79·2	80·9	84·7	79·4	80·1	81·3
Kananúr ....*	11 51·2	75 21·3	L. a. L. S.	79·7	81·0	83·7	85	84·5	79·5	78·3	78·8	79·3	79·7	80·7	79·2	80	84·4	78·9	79·9	80·8
Karikál	11 5	79 56	L. a. L. S.	76·6	79·0	82	86·3	87·3	88·7	85·6	83·5	84·1	82	78·9	75·1	76·9	85·2	85·9	81·7	82·4
Kóchin..............*	9 58·1	76 15·6	L. a. L. S.	79·4	80·7	83·6	84·7	82·7	78·2	77·1	79·1	80·1	80·1	80·3	79·8	80	83·7	78·1	80·2	80·5
Madrás..............*	13 4·2	80 13·9	L. a. L. S.	75·9	77·8	81·1	84·3	87·2	87·6	85·6	84·4	83·9	81·7	78·4	76·3	76·7	84·2	85·9	81·3	82
Madúra ....*	9 55·3	78 6·3	600	79·3	82·5	86·2	88·7	90	88·6	87·2	86·2	86·9	84·7	80·1	79·1	80·3	88·3	87·3	83·9	84·9
Mangalúr...........*	12 51·7	74 49·2	L. a. L. S.	80·2	80·5	83·6	85·1	86·1	79·1	78·3	77·9	79·3	79·9	81	80·8	80·5	84·9	78·4	80·1	81
Masulipatám ......*	16 9	81 8·2	L. a. L. S.	78·2	79·5	83	87·6	90·4	92·6	84·0	87·1	83	81·8	79½	78½	78·7	87	87·9	81·4	83·8
Nellúr ....*	14 28	79 58·3	81	77·2	76·7	77·5	84·2	88·7	89·2	87·4	86·3	84·8	81·7	77·9	73·5	75·8	83·5	87·6	81·5	82·1
Pallamkótta.........*	8 43·5	77 43·3	209	79	79·8	84·1	85	87·8	86	85·1	84·9	86·4	82·6	80	78·5	79·1	85·6	85·3	83	83·3
Pondichéri ....*	11 56	79 49·1	L. a. L. S.	80·1	81·4	82·8	84·5	86·9	87	86·0	86·3	88·5	84·2	82	84·2	81·9	84·7	86·4	84·9	84·5
Rajamándri.........†	17 10·5	81 46·6	L. a. L. S.	72·9	76·8	82	87·3	89·1	90·5	82·2	84	79·9	85·1	82	77½	75·7	86·1	85·6	82·3	82·4
S. Thomas Mount...	13	80 8	314	79·9	80·5	81·8	86·9	90·8	90·3	89·5	88	87	83·7	80	79	79·8	86·5	89·3	83·6	84·8
Sálem ....*	11 39·2	78 8·4	907	75·9	77·4	81·3	85·4	85·7	83·4	81·7	81·4	82	80·2	76·7	76·6	76·6	84·1	82·2	79·6	80·6
Tinevélli ....*	8 43·8	77 40·4	120	81·2	81·7	84·5	88·6	88	88·8	85·4	85·1	86	84	80·6	80·9	81·3	87	86·4	83·5	84·6
Trichinápalli ......*	10 49·8	78 40·9	297	80·2	83·1	86·3	88·4	90·6	87·3	85·1	84·9	84·4	82·8	82·2	81·1	81·5	88·4	85·8	83·1	84·7
Trivándrum	8 29	76 56	L. a. L. S.	78·1	79·7	81·7	82·7	81·5	78·5	78	78·3	78·5	78·5	77·6	78·1	78·6	82	78·3	78·2	79·3
Vingórla ....*	15 51·2	73 35·9	L. a. L. S.	76	81	81	81½	...	...	...	...	...	...	...	78	...	...	...	...	...

9. CEYLON.

Station.	Lat.	Long.	Height.	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	D.J.F.	M.A.M.	J.J.A.	S.O.N.	Year.
			ft.																	
Bádula	6° 59′	81° 11′	2450	66⅔	68⅓	68⅓	71⅓	72⅓	71	69⅓	72	72⅓	72⅓	71	70⅓	68·5	70·7	70·8	71·9	70·5
Battikótta	9 36	80	L. a. L. S.	78·5	80·6	82·4	85·1	85·4	84·6	83·3	83·2	82·9	82·3	79·9	78·4	79·2	84·3	83·7	81·7	82·2
Gálle †	6 2·5	80 10·8	L. a. L S.	...	...	...	...	82·1	...	...	...	...	...	...	...	...	...	...	...	81
Kándi	7 17	80 49	1739	70·6	72·5	74	73·8	74·6	73·1	72·5	73·3	72·7	72·2	72·1	71·8	71·6	74·1	73	72·3	72·8
Kolómbo †	6 56·1	79 49·8	L. a. L. S.	78·7	79·5	81·2	82·4	81·8	81·3	80·7	80·5	80·3	79·1	78 8	78·1	78·6	81·2	80·8	78·4	80·2
Máteli	7 32	80 47	1187	70·6	72·6	75·2	78·5	79·7	79·7	78·9	77·6	77·2	75·6	74·5	72	71·7	77·8	78·7	75·8	77
Nurélia	7 13	81 52	6218	55·9	58	59·7	59	60·1	59·2	59·6	59·1	59	58⅓	58	57	57	59·6	59·3	58·5	58·6
Pátlam	8 2·8	79 53·6	L. a. L. S.	75·5	78·9	83	83·2	81·1	82·1	79·9	81·4	80·4	80·3	78·2	77·7	77·4	82·4	81·1	79·6	80·1
Peredénia	7 17	80 49	1650	73·4	74·9	76·5	78	77·7	75·7	76·1	75·5	75·9	74·4	74	73·1	73·8	77·4	75·8	74·8	75·4
Trinkonomalí †	8 33·5	81 13·2	213	77·8	78·6	81·3	83·9	84	81·5	84·4	83 1	82·3	80 8	79·8	78·4	78·3	83·1	83	81	81·4

10. INDO-CHINESE PENINSULA, ARCHIPELAGO, AND CHINA.

Station.	Lat.	Long.	Height	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	D.J.F.	M.A.M.	J.J.A.	S.O.N.	Year.
			ft.																	
Akyáb *	20° 8′	92° 52′·6	L. a. L. S.	70·8	73·4	78·8	83·5	85·6	81	81·8	81	81·7	81·9	77·2	72·8	72·3	82·6	81·3	80·3	79·1
Álor Gájah	2 11	102 17	L. a. L. S.	...	...	...	...	...	...	...	82·7	81·4	80 6	80·6	72·2	...	...	...	81·8	...
Áva	21 50	96 2	L. a. L. S.	64·7	73·5	75·9	86·2	84	85 5	82·8	82·5	82·7	80·6	74·2	68·3	68·8	82	83·6	79·2	78·4
Bangkók	14	101 30	L. a. L. S.	76·7	79	83·8	84·2	83·7	82·8	82	81·7	81·3	80·8	80·5	76·9	76·5	83·9	82·2	80·9	81·14
Batavia S	6 10	106 58	L. a. L. S.	79	79·3	80·4	80·1	80·6	79·2	78·9	79	79	77	77·6	79·4	79·2	80·4	78·7	77·9	79
Chúsan	30 25	121 44	L. a. L. S.	40½	38¾	...	...	...	...	...	...	77¾	68¼	57½	43½	40·9	...	...	67·8	...
Hong-Kong	22 11	114 7	(140)	68	63·2	65·5	73·8	80·7	84·4	86	83·1	83·5	78·5	71·9	65·4	65·5	73·3	84·5	78	75·3
Kánton	23 8	113 16	L. a. L. S.	52·5	55	62·5	70	77	81	83	82	80	73·3	65·2	57·1	54·8	69·8	82	72·8	69·9
Kyuk-phyú *	19 25·2	93 32·2	L. a. L. S.	74·15	73·3	80·7	87·35	87·4	80·75	79·7	77·6	80·7	83·3	77·9	74·05	73·8	85·1	79·3	80·6	79·7
Makáo	22 11	113 34	L. a. L. S.	62	54·9	64·3	71·3	78·4	82·7	82·9	83·1	81·6	76·1	67·4	61·2	59·4	71·3	82·9	75	72·2
Manilla	14 36	121 9	L. a. L. S.	77·1	77·5	79·5	81·5	81·7	81·3	80·2	78·9	78·8	79·9	79·7	78·3	77·6	80·9	80·1	79·5	79·5
Mérgui	12 27	98 35	L. a. L. S.	79·5	81·3	80·6	82·8	80	78·6	78·4	78·6	77·5	79·5	...	...	...	81·1	78·5	...	...
Pádang S	0 59	100 31	L. a. L. S.	81·7	...	...	...	...	...	...	...	...	80·5	80·8	81·3	...	...	...	...	...
Rangún	16 48	96 17	L. a. L. S.	73·2	79·3	86·5	87	80 2	79·7	80·2	78	80·3	82·3	81·1	76·7	76·4	84·6	79·3	81·2	80·4
Sámarang S	7	110 31	L. a. L. S.	80·1	80·8	80·5	82·6	82·5	81·1	80·5	82·1	83·1	84·2	83·6	81·3	80·7	81·9	81·2	83·6	81·9
Sarávak	1 34	110 29	L. a. L. S.	...	...	...	...	...	79·6	78·8	78·7	...	...	...	...	...	...	...	...	...
Sándove	18 25	94 30	L. a. L. S.	69·1	71·4	75·9	81·9	82·3	80·2	77·7	79	79·6	80·7	76·3	71·7	70·7	80	79	78·9	77·1
Shanghái	31 2	121 20	L. a. L. S.	40·2	39·5	46·7	56·9	65·7	69	81·9	81·6	74·4	67·3	55·8	45·1	41·6	56·4	77·5	65·8	60·3
Singapúr	1 18	103 53	L. a. L. S.	78·7	80·1	80·5	81·1	81·8	81·8	82	81·3	81	81	80·2	79·3	79·4	81·1	81·7	80·7	80·7
Tavái	14 7	98 18	L. a. L. S.	76·5	79·5	81	82·6	81·6	79·2	78·6	78·6	77·8	79·6	80·6	78·1	78	81·7	78·8	79·3	79·5
Áden	12 46	45 15	187	72·5	73·4	76·8	80·6	84·9	85·5	83·3	81·5	84·9	83·3	78·6	77·2	74·4	80·8	83·4	82·3	80·2

III. *Decrease of Temperature with Height in the Tropics.*

The decrease of temperature with height had to be taken into consideration, not only on account of its practical importance for the selection of stations and sanitaria, but also for comparing the different parts of India, independently of the accidental height of the observer's residency, and for drawing finally the general isothermal lines.

For the Dékhan and Central India, Púna, Purandár, and French Rocks could be compared with the coasts of the Kónkan and the Karnátik; for the South I had three stations in the Nílgiris and one in Ceylon, which could be referred to the shores of the Indian Ocean.

The following Table shows the results I had obtained for the year and the seasons:—

A. Dékhan and Central India.

Places of observation.	Height above the level of the sea.	Height in feet=decrease of 1° F.				
		Year.	Dec. to Feb.	March to May.	June to Aug.	Sept. to Nov.
Púna	1784	410	370	360	310	595
Purandắr	3974	435	450	660	230	390
French Rocks	2620	750	900	1200	340	600

B. Nílgiris and Ceylon.

Places of observation.	Height above the level of the sea.	Height in feet=decrease of 1° F.				
		Year.	Dec. to Feb.	March to May.	June to Aug.	Sept. to Nov.
NÍLGIRIS.						
Átare Mállē	4500	270	310	260	220	290
Utakamắnd	7490	280	300	270	260	290
Dodabétta	8640	310	350	310	265	300
CEYLON.						
Nurélia	6218	280	290	280	270	290

For the Dékhan and Central India we see that the decrease is very slow; for the Alps, for instance, I formerly obtained 320 English feet for 1° Fahr.* As the principal cause of the decrease being not more rapid, we may consider, I think, the circumstance that the elevation, though not very considerable, extends itself with great uniformity over a large surface.

In the second group the values are less different from those in the Alps and in High Asia; for both groups of the Indian stations it is characteristic that the rainy season shows by far the most rapid decrease.

For showing *simultaneously* the variations of the decrease with the locality and the seasons, I have drawn three topographical profiles (see Plates XXIX.–XXXII.), and have indicated for each of the single seasons the difference of the respective decrease from its annual mean value by drawing a dotted line in connexion with the topographical outline. The dotted line shows the contour which the topographical section ought to have for the actual temperature of the season, supposing the value of the decrease would have remained the same throughout the year; if, therefore, the decrease in the season is too slow, the new ideal position of the station will be below the real topographical outline, on account of the station having a temperature as if it were in a less elevated situation; if, *vice versâ*, the decrease is more rapid than the annual mean, as we see it particularly to be the case in the rainy season, the dotted line will show, from the same reason, a profile which is higher than the topographical contour. For Ceylon I further added the point of its highest peak, Péduru tálla gálle, 8305 feet, for the sake of completing the general topographical profile of the island, though I had no higher station for its mountainous regions than Nurélia. The decrease of temperature with height in the Himálaya, the Karakorúm and Kuënlün had not to be calculated in connexion with the construction of these maps. As I had direct data for the beginning of the isothermal lines along the western and the eastern margin of these mountainous regions, the form of the dotted lines which I now have drawn across them could be obtained directly by uniting the terminal points. This circumstance is very valuable, too, when I come later to examine the influence which is exercised by the topographical formation (including vast plateaux, ridges, and isolated lofty peaks), and by the extent of the snowy regions, upon the alterations of the decrease of temperature with height.

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* Neue Unters. phys. Geogr. d. Alpen, p. 584. The numbers I have given there are 540 French feet for 1° C.

IV. *Thermal Types of the Year and the Seasons.*

The considerations about the distribution of temperature over the *surface* of India in general may best be combined with the analysis of the isothermal curves on the maps annexed (Plates XXVIII.–XXXII.).

In reference to the geographical details, I have limited myself to the principal river-systems; and to avoid interfering with the distinctness of the isothermal lines, the names of the stations, as well as the mountain-systems, are left out; for the means of the year, the Indian Archipelago and countries to the north-east of it are also added on a smaller scale.

In drawing the lines, I made these distinctions: besides the lines being dotted where they pass the regions of High Asia, also the thermal equator is distinguished, its line being a broken one. In consequence of the great difference in latitude between the western and the eastern end of the Himalaya, the curves extend along the western margin of the Map from 5° to 35° of latitude; along its eastern margin only from 5° to 30°. For the period corresponding to our summer, the isothermal lines could be continued for Central Asia somewhat further to the north, in connexion with our personal stay in these regions.

The eastern part in the higher latitudes of the Map is throughout cooler than the western part, as shown in the following Table, where the numerical values are nearly the same, though the difference in latitude amounts to 5°.

		Warmest isothermal line.	Minimum in the N.W.	Minimum in the N.E.
	Year	84°	73°	73°
Seasons.	December, January, February	80	57	60
	March, April, May	90	72	73
	June, July, August	92	89	81
	September, October, November	82	75	74

The isothermal lines of the year very decidedly show the influence of the topographical form of the Indian peninsula on the increase of the mean temperature: in the southern parts they follow the contours of the shores, or obtain forms evidently in connexion with them; in the northern part these lines are raised to the extent of a difference of five degrees of latitude where they pass over the central axis of India. At the same time, southern India presents one of those insular regions of greatest heat which are connected with each other by the thermal equator; the Indian archipelago shows us the next of these regions which follows to the east.

When comparing the seasons, we are particularly surprised by the unusually great *variety* of the four types, whilst in many of the more western regions of the tropics we see that it is more the numerical value of the lines which is changed than the type of their forms. In India and the Indian archipelago the thermal equator runs still to the south of the geographical one for all the three months of the cool season; but in the season corresponding to our summer, from July to August, we see it has been raised up to the latitude of 32° N. This part of the year is, for the greater part of the map, the rainy season, though for the region in the north-west it is the very season of an absolute maximum of heat. These variations have the more importance, as the territory here represented has a surface considerably larger than might be expected, perhaps, from the extent of European empires. The distance from the Bay of Biscay to the Caspian Sea can be considered as about equal to the difference in longitude of the borders of this Map; whilst 30° of latitude, referred to European regions, might be compared with the distance from the southern shores of the Mediterranean to St. Petersburgh.

The cool season.—This period already shows traces of the increase of temperature in the interior of the land when compared to surrounding seas; but, as it must be expected, the influence of insolation is, comparatively speaking, but little felt during this season in the provinces at some distance to the north of the equator, on account of the southern position of the sun. In the regions beyond the tropics the hibernal influence of continents, compared to that of the seas, causes depression of temperature. In reference to the Pănjáb, it must be further added that we have here, comparatively speaking, a greater number of stations for which the actual temperature is still lower than the values represented by the isothermal lines, as the latter had to be reduced to the level of the sea. The general elevation of the ground, and, throughout the season, a sky unusually clear, so favourable to nocturnal radiation, may be mentioned as the principal causes. The decrease of temperature with latitude is by far the most rapid in the cool season.

The second period of the year (March, April, and May), which is generally called the hot season all over India, also in its north-western parts, shows a remarkable difference in the type of the curves when compared to the cool season; the influence of the topographical forms of the peninsula has become now considerably more apparent. The thermal equator enters the western border of the Map already at an elevation of 24° of latitude, passes through a central region of maximum temperature exceeding 90°, and descends from thence directly to the south, to the very southern end of India. Great dryness is combined in this period with the high temperature, and is an important element for making its difference from the other seasons still more apparent; but it would be erroneous to expect, as it might appear rather probable, that in consequence the heat is felt the heavier by the human organism. Though the central parts, compared to the shores of the sea, show a rapid increase of temperature with the progress towards the interior, I must add that, on account of the moisture being greater along the shores, not only the heat is felt there more close and more oppressive, but also its influence on the health, particularly of the Europeans, is decidedly still more unfavourable. For the coasts, and for the interior of India up to latitude 25° N., these months remain the period of the year which includes the highest means, and also the greatest heat of single days.

The third period (June, July, and August) is, for the greatest part of India, the rainy season; its setting in is connected, particularly in Central India, with the most rapid sinking of temperature. Nearer to the shores the difference is felt less beneficial; the humidity has increased, too, and makes, in the shade at least, the heat the more oppressive. The power of the insolation now being broken by a sky nearly permanently clouded, must be named as the particular cause why the beginning of this season in general is considered as a welcome period. For the state of health, however, it is less favourable; dyspeptic complaints and fevers are particularly frequent in the latter part of this season. In the Pănjáb, and partly already in the north-west provinces of Hindostan, this period has no more the character of a rainy season. The precipitation takes the form of our summer rains with thunder-storms, and also the amount of precipitation most rapidly decreases towards the north-west.

At the same time the meteorological observations showed for these very regions a maximum of temperature which was unexpected to me, not only on account of the number of stations formerly existing being not very great, but also since I heard from the inhabitants, the Europeans as well as the natives, no unusual complaints about the heat being much greater than in other parts of India. Nevertheless these provinces include a region for which the mean temperatures during the three months exceed 92°, which therefore must be considered as one of the very hottest regions of our globe; besides, we must take further into consideration that clear days are not unfrequent, during which the purity of the sky is not even limited, as it was in the period preceding, by dust suspended in the atmosphere. Therefore also the absolute maxima in the shade as well as in the sun are higher here than in any other region of India.

I may further draw attention to the fact that for this region also the non-periodic variations of temperature, the variations between different years, have become much greater than we find them to be in the more southern tropical part of the territory examined. The thermal equator enters the west of the Map at the latitude of 32°, and only leaves the Indian peninsula near Ceylon in an easterly direction.

The influence of height in the Pănjáb is not very considerable in this season, and the curves I have drawn remain for some stations even still a little below the respective means; but in the other regions, where the character of the "rainy season" prevails, the decrease of temperature with height is more rapid than during any other part of the year.

Autumn (September, October, November) is the only one of the tropical seasons which shows here a very regular form of its curves, and a very slow decrease of temperature with latitude; it is not less characteristic for this season that in most regions, particularly in those along the banks of the larger rivers, the drying up of vast surfaces formerly inundated is the cause of most deleterious miasmatic vapours; but in the Pănjáb, and in the hilly regions along the Brahmapútra and in Central India, where these dangerous modifications of the atmosphere are not to be feared, this season frequently approaches the mild and refreshing character of the regions of southern Europe.

A more descriptive detail, together with the personal data in reference to the observers' names and the duration of the different series, will be given by me, later, in the 4th volume of our 'Results'*. In the present memoir I considered it my particular object to lay down the materials officially entrusted to me, and to give them at the same time as critically worked over as my travels allowed me to attempt.

The temperature in shade, however, can but insufficiently define the climate as it is particularly seen in these parts of the tropics, where the power of solar radiation, rains, and storms differ in no less proportions from those in temperate zones. I will consider it a particular pleasure to be able to forward, in not too distant a time also for these elements, my numerical data, together with some remarks about the principal general results.

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* It also will contain some additional stations from the materials I obtained, with their usual liberality, from the Indian authorities during my recent visit to England. Amongst the official publications, those of Glaisher and Macpherson have particularly to be quoted. These materials could no more be added to the present Tables, as most of these too will have to be recalculated for being reduced to true daily means.

XXVI. *On the Structure and formation of the so-called Apolar, Unipolar, and Bipolar Nerve-cells of the Frog.* *By* LIONEL S. BEALE, *F.R.S., M.B., Fellow of the Royal College of Physicians; Professor of Physiology and of General and Morbid Anatomy in King's College, London; Physician to King's College Hospital, &c.*

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IT is an opinion now very generally entertained by anatomists that in vertebrate and in certain invertebrate animals, there are in connexion with the nervous system apolar, unipolar, and multipolar, including bipolar cells.

It is easy to demonstrate the presence of multipolar cells, but it is another matter altogether to prove that certain cells which seem to be *apolar* or *unipolar* are really of this nature. An observer is justified in asserting very positively the existence of that which he has himself distinctly seen and has shown to others; but it does not follow that he is correct in concluding that, because a fibre or other structure has not been seen by him in certain specimens, it therefore does not exist, for the actual existence of a structure and its demonstration are two very different questions in minute anatomical inquiry. The one is a question of fact, no matter how it may be explained, or how many different interpretations may be offered of it. The other is but an inference arrived at in the absence of evidence, and may result from imperfect means of observation, or want of due care on the part of the observer in preparing the specimen.

A new fact, if capable of being demonstrated to others, will be received at once as true, but negative assertions, although supported by the authority of the most experienced, ought in matters of observation to be received with the greatest caution.

It is one thing to assert that a cell *has* no fibre proceeding from it, or that a cell has but one fibre, and another to state that no fibre or but one fibre has *been demonstrated.* Although many general facts are opposed to the notion of the presence of apolar and unipolar cells, the existence of such cells is generally admitted and taught. If apolar and unipolar cells exist at all, they are certainly less numerous than the multipolar cells; and it is clear that the arrangement and action of the three classes of cells, *apolar*, *unipolar*, and *multipolar*, must be very different. Indeed it almost follows, if the three classes of cells do exist, that there must be as many distinct kinds of principles of action. Nerve-cells, which bear such different relations to nerve-fibres as must exist in the case of the cell unconnected with any fibre at all, the cell connected with one, and that connected with more than one fibre, cannot possibly influence the fibres in precisely the same manner. The supposition of such an anatomical arrangement involves the existence of different principles of action.

There are, however, authorities who consider it consistent with many facts they have themselves observed, or accepted as having been demonstrated by others, that the arrangement of different parts of the nervous system may be upon a totally different type—that, for example, in some central parts nerves may be in bodily connexion with the cells that influence them, while in other cases a fibre may be influenced by a cell with which it is not structurally connected, and that in some tissues nerves may terminate in distinct ends, while in others they form networks, and in others again terminate in "cells." A high authority, Professor Kölliker, only last year suggested to me that, although it might be true that nerve-fibres formed plexuses or networks in the muscles of the mouse, it did not *therefore* follow that they did not terminate in free ends in the muscles of the frog, although no essential difference in minute structure, or different principle of action, is known to exist in the muscular tissue of these animals.

Now I submit that such arguments are utterly untenable unless the supposed difference in the arrangement of the particular structure be associated with a fundamental difference in the tissues under its immediate influence or holding a fixed relation to it. I venture to assert that if the nerve-fibres terminate in ends upon the sarcolemma of the muscles of the frog, they terminate in ends upon the sarcolemma of the muscles of the mouse. That in one case there may be more ends than in another, that there may be all sorts of modes of branching, that fibres may pass off from the trunks at different angles, and that an infinite number of variations in detail may exist, is consistent with all that has been observed in nature; but I submit that there is no reason for supposing that any typical or fundamental difference exists with regard to the arrangement of nerve-fibres in corresponding tissues in different animals. So, admitting as we must that by far the majority of ganglion-cells in different nervous centres have more than one fibre in connexion with them, it seems more reasonable to conclude that those cells which *appear* to have but one, and those in connexion with which no fibres whatever have been demonstrated, have really two or more fibres connected with them—but too fine to be demonstrated by ordinary means,—than to accept the necessary inference, *that there are in the very same part or organ three distinct classes of nerve-cells exhibiting as many fundamental differences of relation to the fibres which they are supposed to*

I have been compelled to conclude, from anatomical observation, that although many ganglion-cells exist which appear to be destitute of fibres, none do really exist without them. I have studied this question in various parts of the nervous system and in different animals, from mammalia down to the annelida, and, with regard to the arrangement of the nerve-fibres in nerve-centres, I feel justified in drawing the following general conclusions:—

1. That in all cases the fibres are in bodily connexion with the cell or cells which influence them, and this from the earliest period of their formation.
2. That there are no apolar cells, and no unipolar cells, in any part of any nervous

3. That every nerve-cell, central or peripheral, has at least two fibres in connexion with it.

Now, in the ganglia connected with many of the nerves distributed to the internal organs of the frog, there are many apparently *unipolar* and *apolar* cells; but one object of this memoir is to prove that all these nerve-cells have at least two fibres proceeding from them.

Although the present inquiry will be limited to the particular cells connected with the ganglia in different parts of the body of the frog, it necessarily involves the consideration of many fundamental principles of the utmost interest with regard to the intimate structure of the highest tissue existing in living animals. Nor must it be supposed, because I only describe the structure of certain special cells, that my observations have been limited to these cells alone. I have studied the arrangement of nerve-cells and fibres, in the centres and also in peripheral parts, in many different animals.

Upon examining some specimens of the ganglia near the branches of the large arteries distributed to the viscera of the frog, more than two years ago, I was surprised to find that from many of the pear-shaped cells two fibres proceeded—not from either pole, as delineated by WAGNER in the nerves of the pike, but the fibres emerged apparently pretty close together, from the narrowest extremity of the cell, although they did not appear to originate from the same part of the cell.

My specimens have all been prepared in precisely the same manner, and my observations have been made with the aid of powers magnifying more highly than those generally employed in such researches. All the drawings illustrating the subject are magnified from 700 to 1800 diameters, and a few are magnified to the extent of 2800. Many of the preparations have been preserved for two years, and will probably retain their characters for a considerable time longer.

I have arranged the questions considered in this communication under the following heads:—

1. General description of the ganglion-cells connected with the sympathetic and other nerves of the frog.
2. On the formation of ganglion-cells.
    a. Ganglion-cells developed from a nucleated granular mass like that which forms the early condition of all tissues.
    b. Formation of ganglion-cells by the division or splitting up of a mass like a single ganglion-cell.
    c. Formation of ganglion-cells by changes occurring in what appears to be a nucleus of a nerve-fibre.
3. Further changes in the ganglion-cell after its formation.
4. Of the spiral fibres of the fully formed ganglion-cell, and of ganglion-cells at different ages.

5. Of the essential nature of the changes occurring during the formation of all nerve-cells, and of the formation of the spiral fibres.
6. The sense in which the term "nucleus" is employed in this paper.
7. Of the fibres in the nerve-trunks continuous with the straight and spiral fibres of the ganglion-cells.
8. Of the ganglion-cells of the heart.
9. Of the ganglion-cells and nerve-fibres of the arteries.
10. Of the connexion of the ganglion-cells with each other.
11. Of the capsule of the ganglion-cell, and of the connective tissue and its corpuscles.

1. *General description of the ganglion-cells connected with the Sympathetic and other nerves of the Frog.*

All the ganglion-cells from which nerve-fibres proceed to the vessels and other parts in the submucous areolar tissue of the palate, to the tongue, lungs, and neighbouring organs, those from which nerves distributed to the viscera are derived, as well as the cells connected with the pneumogastric, those forming the ganglia upon the posterior roots of the spinal nerves, and some others, exhibit the same general structure—although there are many special peculiarities, and great differences in size, observed in the cells of which some of these ganglia are composed.

The general form of these cells is oval or spherical; but upon closer examination it is found that the most perfectly formed ganglion-cell is more or less pear- or balloon-shaped in its general outline, and by its narrow extremity it is continuous with nerve-fibres which may be followed into nerve-trunks.

In Plate XXXIII. fig. 1, a well-formed ganglion-cell from a ganglion close to one of the large lumbar nerves of the little green tree-frog (*Hyla arborea*) is represented. The substance of the cell consists of a more or less granular material, which by the slow and prolonged action of acetic acid becomes decomposed, oil-globules being gradually set free. Near the fundus or rounded end is seen the very large circular nucleus with its nucleolus. In some of these cells, at about the central part or a little higher, are observed a number of small oval nuclei, fig. 20, *c*. These oval nuclei are arranged transversely to the long axis of the cell, and several follow each other in lines which pass in a direction more or less obliquely downwards towards the lower narrow part of the cell. In fig. 1 very few of these nuclei are seen, but in figs. 20 & 26 many exist. The matter of which the mass of the cell consists gradually diminishes in diameter, and contracts so as to form a fibre, in which a nucleus is often seen. At the circumference of the cell, about its centre, the material gradually seems to assume the form of fibres, which contain numerous nuclei, and these pass around the first fibre in a spiral manner. Thus a fibre comes from the centre of the cell (*straight fibre*), and one or many fibres (*spiral fibres*, figs. 1, 3, 4, & 22 to 28, 31 to 38, & fig. 42) proceed from its surface. Neither of the fibres can be traced to the large "nucleus" of the cell.

2. *On the formation of the ganglion-cell.*

I now proceed to describe the changes which I believe take place in the structure of these ganglion-cells during their development, growth, and decay. The drawings which illustrate these observations, although arranged so as to form as far as possible a connected series, are actual copies from nature, made from very many different specimens at intervals during the last two years and a half. The clearest as well as the least uninteresting manner of describing the points of anatomical detail will probably be to arrange the facts of observation so as to construct, as far as may be possible, a connected history of the structural alterations occurring during the life of these elaborate organs. But it is not pretended that the sketch can amount to more than a rough and imperfect outline of the process which actually occurs; for the nature of the inquiry is such that it is absolutely impossible to observe the same cell at different periods of its life, so that the history is made up of fragmentary details obtained at intervals during a long course of investigation.

The development of these, as of most other structures, may be studied *in the fully formed animal as well as in the embryo*, and the various changes occurring may be observed with greater precision and distinctness in the former than in the latter. I shall not enter into the question of the origin of the "nucleated blastema" or of the "granular matter" observed in such quantity in all developing structures. The word "cell" I shall use in a general sense, because it is shorter than "elementary part."

The nerve-cell does not at all ages possess the well-defined structure (cell-wall, cell-contents, and nucleus) usually accorded to it by anatomists, but always consists of *matter that is living*, and *matter that has lived.* The first is coloured by an ammoniacal solution of carmine, the last is not so coloured. A reference to any of my drawings will enable the reader to see at a glance what I understand by a cell or elementary part, and will render further description needless*. In the very young animals these ganglion-cells gradually form from "nuclei" which appear to be imbedded in very soft granular matter. The fibres extend from the collection in at least two directions, and exist as granular nucleated bands, the course of which cannot be followed for any great distance, partly in consequence of their extreme transparency and tenuity, and partly because they ramify amongst the "nuclei," of which all the different tissues at this early period are in great part composed. The fibres do not *grow out from the cells*, but are formed as two masses of germinal matter gradually separate from each other. But new ganglion-cells, nerve-fibres, and nuclei are being constantly produced, not only in fully developed young animals, and in the adult, but certainly for a considerable time after the animal has arrived at maturity, and I believe almost to the close of the ordinary period of life. A young cell, a fully formed cell, and an old cell are represented in figs. 2, 3, & 4, magnified 700 diameters.

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* This question has been fully considered in my lecture "On the Structure and Growth of the Simple Tissues," delivered at the Royal College of Physicians in 1860; and the general conclusions then arrived at, and since confirmed by other observations, will be found to be strongly supported by facts recounted in this paper.

In the fully developed frog, ganglia are formed,

1. From a nucleated granular mass like that seen in the embryo, but continuous with nerve-fibres.
2. By the division and splitting up of a mass like a ganglion-cell.
3. By changes occurring in what appears to be an ordinary(?) nucleus of a nerve-fibre.

a. *Ganglion-cells developed from a nucleated granular mass like that which forms the early condition of all structures.*

At first an elongated delicately granular mass is observed in connexion with nerve-fibres, more or less of an oval form, tapering into a tolerably broad granular band or fibre at either extremity. This band often appears to consist of but one broad fibre, but it may generally be resolved into several finer fibres. It cannot be said that nuclei as distinct and separate and *isolable* bodies are imbedded in this; but by the action of carmine numerous rounded portions of matter are darkly coloured, and these may be seen to pass gradually into the surrounding substance, which is only slightly tinted by the red solution. In the small ganglion represented in Plate XXXIV. fig. 5, several such nuclei gradually passing into the surrounding uncoloured matter may be observed. These nuclei divide and subdivide; and at a more advanced period of development, the whole mass having increased in size, indications of its division into several lobes or portions may be seen (*b*, *c*, *d*). These divisions become more and more distinct, until a number of masses, all exhibiting the same structure, but still intimately connected with each other, result. At *b* several of the cells distinctly separated from each other, and beginning to assume a pear shape, are seen. As the cells approach their fully developed form they *move away from the point where they were first developed*, as represented in fig. 6. In this figure the cells that project furthest and have the longest stems are the *oldest*. From the mode of formation, it is obvious that, at least for a certain time, the cells are continuous with each other (fig. 5, *a*, *b*, *c*, *d*, *e*), while it is also clear that the very matter of which at first the substance of a cell is formed may subsequently be drawn out so as to form a fibre. It appears, however, that as the ganglion-cells grow and separate from each other they change their relative positions, so that it is impossible to trace any one fibre for more than a comparatively short distance. The connexions of the ganglion-cells will be considered separately.

b. *Formation of ganglion-cells by the division or splitting up of a mass like a single ganglion-cell.*

One of the masses (cells) just described (*b*, *d*, fig. 5) having reached a considerable size may divide and subdivide, and thus new cells may be developed from it. The stem by which the cell is connected with the rest of the ganglion divides and subdivides into numerous fibres, and in this manner it seems that a number of separate ganglion-cells may be formed by the division of one. The fibres proceeding from each of the new cells are arranged in a bundle which corresponds to the stem of the original cell. The changes just described are well seen in figs. 7 & 8 from a sympathetic ganglion, and in figs. 11 & 12 from the intervertebral ganglia*. This mode of multiplication generally occurs before the cell has assumed its complete and perfect form (figs. 1, 3, & 4). Although I have examined hundreds of ganglia, I have only seen on very few occasions such a change occurring in a cell which exhibits a distinct spiral fibre, and I have not been able to study it. The greater part of the old spiral fibres would probably waste and disappear, and new ones would be developed from a new germinal mass, resulting from the growth and multiplication of the "nuclei" of the old fibres. The fibres connected with a mass undergoing development into several ganglion-cells often exist in considerable number, but sometimes are so fine as only to be just visible under a power of 3000, and I believe others exist which are too fine and delicate to be seen by means at present at our disposal. In fig. 6 several fully formed ganglion-cells are observed; and here and there are oval masses of about the same size, but containing numerous nuclei, and connected with the ganglion by very fine fibres (fig. 6, at *a* & *b*).

In fig. 7 one of these is represented much more highly magnified; and in fig. 8 another which has increased greatly in size, and contains numerous separate ganglion-masses is magnified 1800 linear. These cells are dividing and subdividing, and from the whole mass would have been formed a new *ganglion* composed of upwards of *thirty cells.*

c. *Formation of ganglion-cells by changes occurring in what appears to be a nucleus of a nerve-fibre.*

I have several specimens which exhibit a single ganglion-cell in connexion with a very fine nerve-fibre. Not that the fibre is actually single, for it consists of at least three or four very fine fibres; but it runs by itself at a distance from any other fibres or ganglia, and is imbedded in transparent cord-like connective tissue.

The changes occurring during the formation of a single ganglion-cell, in connexion with what is undoubtedly at first a single granular fibre, are of the utmost interest. I cannot be perfectly confident that the history I give of these changes is absolutely true in all points of detail; for, as may be easily imagined, to isolate fine fibres with nuclei exhibiting the different stages of change is a difficulty of no ordinary character, while hundreds of specimens may be examined without finding one in which such fibres are observed.

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* My friend Mr. LOCKHART CLARKE has not observed this division of the cells which enter into the formation of the ganglia on the posterior roots of the nerves of mammalia. He describes the gradual increase in size and the alteration in structure and form of the cells as development advances, but says nothing about their *increase in number*. It is true that Mr. CLARKE's observations were made upon mammalia, while the statements in my paper refer only to the frog; but it is almost certain—I think indeed that it is quite certain—that if the cells do not multiply by division in mammalia they do not do so in the frog. Upon this question of fact, therefore, I regret to say that Mr. CLARKE and myself are at issue. The specimens from which my figures 11 & 12, which show this division most distinctly, have been copied, may be examined by any one who desires to see them. This question shall be further investigated by me. Mr. LOCKHART CLARKE's observations will be found on page 927 of the Philosophical Transactions for 1862, Part II.

The drawings represented in figs. 9, 10, 13, 14, 15, 16, 17, & 18 show the points I have observed in connexion with this question.

A nucleus which cannot at first be distinguished from the ordinary nuclei in connexion with the nerve-fibres, grows somewhat larger than the rest (figs. 9 & 14). I am not prepared to say that *any* nucleus connected with the fibres could or could not undergo the changes about to be described; but the nuclei from which the ganglion-cells are formed exhibit at first nothing peculiar. Sometimes several in different parts of a fibre may enlarge to some extent; but for the most part only one in the course of a long distance will be developed into a ganglion-cell. Of course the opinion that a ganglion-cell may be developed from *any nucleus* up to a certain period of its life is quite tenable, and it may be that the actual change is determined by the presence or absence of certain conditions, but this speculation I will not pursue. The enlarged nucleus which is about to become developed into a ganglion-cell soon exhibits a transparent portion at its circumference.

From what has been stated, it is clear that every ganglion-cell developed in the course of a nerve-fibre must have a fibre proceeding, as it were, from either pole (bipolar cells); but the difficulty of defining the changes with the utmost certainty arises from the rarity with which single fibres emanating from either end of the cell, like those represented by authors, occur. The only cells exhibiting these characters in the frog that I have seen are, in my opinion, young and imperfectly formed cells. Such an *appearance* is not uncommon in fully formed cells in ganglia; but in all cases in which I have observed it in the frog, more careful investigation has satisfied me that it was fallacious, and depended upon a fibre really passing close to the cell, without being connected with it, although it appeared at first to emanate from the end opposite to that which certainly terminated in a straight fibre. In the frog I feel almost certain that fully formed cells, with a nerve-fibre coming from the opposite poles, as delineated by WAGNER and KÖLLIKER in the pike, do not exist; and I have failed to find examples of the cells represented by KÖLLIKER in three specimens of the pike which I have examined. My want of success, however, probably depends upon imperfect investigation, as such fibres undoubtedly exist in the skate. In the posterior roots of the spinal nerves of the common skate such cells are easily found; but there is a difference in the arrangement of the two fibres, and their relation to the mass of the cell corresponding to that already described in the frog, which deserves more attentive examination. I shall describe the peculiar structure of these cells in another communication.

The cells which are not fully developed, delineated in figs. 9, 14, 15, & 16, have a fibre coming from either extremity, but, as will presently be shown, there is reason to believe that as the cell grows the arrangement of these fibres soon becomes altered. The appearances represented in fig. 18, when examined by a low power, would be considered to be due to the presence of fibres emanating from either end of the cell, but under a higher power the true arrangement, which is very different, can be clearly demonstrated.

3. *Further changes in the ganglion-cell after its formation.*

The ganglion-cell having assumed such forms as represented in figs. 9 & 14, now undergoes further changes. It becomes separated more and more from the point where its formation commenced (figs. 32 & 33, Plate XXXIX.). The two opposite extremities of the cell are drawn down (fig. 34). The fibres increase in length and lie parallel to each other, and the form of the cell becomes much altered. If formed in connexion with one of a bundle composed of numerous fibres, the cell seems to grow away, as it were, from the bundle, and sometimes is found at some distance from it, as represented in fig. 41 *d*; but more commonly its long axis corresponds to the direction in which the fibres of the bundle run (figs. 17 & 21). This is constantly observed in the cells in connexion with bundles of fibres running close to the large arteries in the abdomen; and they may easily be mistaken for cells having a fibre springing from each end. But in this case also the ganglion-cell is some distance from the spot where its first formation commenced, and the two fibres which now extend from it may lie parallel to each other for the distance of four or five thousandths of an inch from the cell, and then take opposite courses in the bundle of nerve-fibres.

The ganglion-cell, when fully formed, may lie on the outside of a bundle of nerve-fibres, while the fibre or fibres with which it is connected may run in the central part of the bundle. The two fibres passing from the cell run amongst the bundle of nerve-fibres, to which their course is at first more or less at right angles. Sometimes the fibres from a ganglion-cell pass partly round the circumference of a bundle of nerve-fibres, and then run amongst them. Often the fibres appear to pass quite to the centre of the bundle. At this point, in fortunate specimens, the two fibres may be seen to alter their course and run with other fibres of the bundle, but in opposite directions (see figs. 32 to 37; also figs. 1, 20 *a* & *c*, and fig. 25). As these fibres are often less than the $\frac{1}{60,000}$th of an inch in diameter, it is very difficult to follow them in the bundle for any great distance; and in ordinary specimens, in endeavouring to unravel the bundle of nerve-fibres, they are inevitably broken. Often they break close to the ganglion-cell; in which case the cell itself, especially if examined in water, appears as a round *apolar cell*, while the fibres which were continuous with it might be easily mistaken for fibres of connective tissue running transversely around the bundle of nerve-fibres imbedded in the neurilemma. Indeed it is probable that many authorities will still maintain not only that the spiral fibre, but that many fine fibres I have described as nerve-fibres really consist of modified connective tissue. If I had only specimens from the common frog, I might have experienced some difficulty in demonstrating that the spiral fibre was a true nerve-fibre to the satisfaction of every one; but many specimens which I have from the green tree-frog settle the question beyond dispute. The spiral fibre is as large and thick as the straight fibre, and, like it, has been traced into a dark-bordered fibre (figs. 1 & 19). Figs. 17 to 28, 31, 38, and figs. 41 & 42 represent different forms of ganglion-cells.

4. *Of the spiral fibres of the fully formed ganglion-cell, and of the ganglion-cells of different ages.*

In all but very young ganglion-cells the remarkable spiral fibres already alluded to exist. In fig. 1, which represents a fully formed cell, the beautiful spiral arrangement of one of the fibres is observed, and in figs. 21, 22, 27, & 28 several fibres are seen to proceed from one ganglion-cell. Close to the cell in which there is a considerable extent of spiral, the course of the fibres is almost transverse. The fibres seem to be coiled around the lower thinner portion of the ganglion-cell (figs. 1, 3, 4, 20, 25, 31). Then the fibres pass spirally round the straight fibre away from the cell, and each turn becomes more oblique than the one above it, until at last the fibre (or fibres) lies parallel with the other fibre which leaves the cell. The spiral fibres are necessarily longer than the straight fibre. The spiral fibre eventually takes a course in the compound nerve-trunk to which the fibres may be traced, the very opposite of that taken by the straight fibre; so that, although the two fibres run parallel to each other for some distance from the ganglion-cell, as STANNIUS observed was the case in the bipolar cells of the calf, it is certain that, at least in many instances, they at length pass in opposite directions (fig. 1). KÖLLIKER thinks that "by far the greater part, perhaps all," of the fibres from the ganglion-cells of the spinal nerves pass in a "*peripheral direction;*" but I believe this statement is erroneous, both as regards the existence of but one fibre, and its course. I have never been able to find any ganglion, large or small, the fibres of which passed in but one direction. Even when a ganglion consists of only three or four cells, *compound fibres pass from it in different directions.* I doubt if a nerve-cell anywhere exists whose fibres pass to their destination in the same direction. In connexion with this question I may revert to the fact I have already stated [*], that whenever any compound nerve-trunk, large or small, composed of any form or number of nerve-fibres in any part of the organism of man or animals, passes into another trunk running at, or nearly at, right angles to it, its component fibres divide into branches which pass in *opposite directions.* This general fact is most important; and I have never seen anything to make me believe that the disposition is not universal, and therefore essential even to the simplest nervous apparatus. As the disposition exists in bundles of fibres which there is reason to believe are purely sensitive, as well as in the case of those which are purely motor, it looks as if fibres invariably passed in two directions, whether they be traced from their peripheral distribution or from their central origin. The arrangement spoken of is represented in fig. 40, which is from the submucous areolar tissue of the palate of the frog [†].

The spiral fibres can be shown to be continuous with the material of which the body of the cell is composed as well as the straight fibre; but the former are connected with its surface, while the latter proceeds from the more central part; so that, in the most perfect of these cells, the straight fibre forms a stem around which the spiral fibres are coiled (figs. 1, 3, 19, 20, 25, 38, &c.).

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[*] Philosophical Transactions for 1862, page 894.

[†] See a paper in my 'Archives,' "On the Branching of Nerve-trunks, and of the subdivision of the individual fibres composing them," vol. iv. p. 127.

The nuclei in connexion with the spiral fibres are well represented in figs. 1, 19, 20, 26. Sometimes they are still more numerous, and I have seen as many as twelve nuclei in the lower part of one of these cells. Several are imbedded in the very substance of the matter of which the cell is composed, but they lie for the most part near the surface. They vary a little in size, and divide longitudinally and transversely. By the division of a nucleus, and the subsequent formation of a fibre from each of the two resulting nuclei, a single spiral fibre is continued onwards from a certain point as two fibres; but it is, I think, probable that in some cases the fibre itself may divide into two, quite irrespective of the nuclei in connexion with it. I have seen instances in which the straight fibre passes through a fissure in the spiral fibre. The connexion of the spiral fibres with the surface of the body of the cell is well seen in figs. 1, 3, 19, 22, 26, 31, & 42.

If figs. 3, 20 *a*, *b*, 24, 25, & 31 be compared with figs. 1, 4, 20 *c*, 23, & 25, a great difference in the extent of the spiral portion of the fibre will be noticed; and it has been already shown that at an early period there is no spiral fibre at all, but that the part of the fibre which is to become spiral pursues a course at first opposite, then perhaps at right angles to, and afterwards parallel with the other fibre (figs. 10, 13, 16, 19, 20, & 24). It has also been shown that, as a general rule, those ganglion-cells which have the longest stems, or are separated by the greatest distance from the general mass of the ganglion (fig. 6), are the oldest cells. Now these are the very cells in which the spiral exhibits the greatest number of coils; and from numerous observations I am convinced that the number of coils increases as the fibre advances in age (figs. 2, 3, & 4).

Moreover numerous observations prove that the quantity of matter constituting the body of the cell varies greatly in different cases; and it is almost certain that as these cells advance in age they diminish in bulk, while the so-called nucleus is absolutely, as well as relatively, smaller in the oldest cells. The fundus of the cell, with its large nucleus, exhibits the same characters for a long period; but at the lower, narrow portion of the cell great alterations are observed. In some cells scarcely anything is left except the large nucleus, external to which is a little "formed matter," from which several fibres proceed. Old ganglion-cells are represented in figs. 4 & 25.

Now, as has been shown (figs. 9, 10, 11, 12, 13, 14, & 17), there is only one large nucleus at an early period of development of the cell; while in a fully formed cell there may be from ten to twenty smaller oval nuclei at the lower part of the cell, some of which are connected with the spiral fibres, as well as the large circular one at the fundus of the cell. There is also very frequently a nucleus in the straight fibre near its origin (figs. 1 & 19). It seems pretty certain that all these smaller nuclei are developed from the larger one; but this question will receive further consideration in the sequel.

There are then, in the fully developed ganglion-cell, "nuclei" connected with the *straight fibre* as well as with the *spiral fibre or fibres*; and it must be borne in mind that there are nuclei connected with the dark-bordered fibres, near their origin in some instances, and near their distribution in all tissues, as well as in connexion with the pale fibres distributed with them. And there is also the very large nucleus with nucleoli at the fundus of the cell.

5. *Of the essential nature of the changes occurring during the formation of all nerve-cells, and of the formation of the spiral fibres.*

Although ganglion-cells are formed according to the three different processes just described, it will be found, upon careful consideration, that the changes which occur are, in their essential nature, the same in all. In every case it has been shown that what is commonly called the "nucleus" takes an important part in the process; and in the drawings (figs. 5, 9, 15, & 38) the large size of the "nucleus" in the young cells cannot fail to be noticed.

In considering the actual changes which occur, it will be better to call the matter which is coloured red by an ammoniacal solution of carmine "*germinal matter*," and the colourless matter around it and continuous with it "*formed material.*" In young specimens the germinal matter is sometimes seen to gradually pass into the formed material; but in fully formed cells of all kinds there appears a line of demarcation.

Neither ganglion-cell, nor nerve-fibre, nor indeed any *living* tissue exists without there being *living germinal matter* in connexion with it. On the other hand, masses of germinal matter exist before either ganglion-*cells* or distinct *fibres* are formed. If figs. 5, 9, 14, 15, 16, & 38, which represent young ganglion-cells, be contrasted with figs. 1, 3, 4, 20, & 27, which are taken from fully formed or old cells, the different proportion of "*germinal matter*" to the "*formed material*" at different ages will be observed; and it is to be noticed that the youngest cells (fig. 9) consist almost entirely of germinal matter, while in the fully formed cell there is at least from ten to twenty times as much "*formed material*" as there is of "*germinal matter*" (fig. 1).

In the fully formed cell the germinal matter (nucleus) exhibits a line around it, but there is no cell-wall or other structure between it and the granular material which surrounds it. In fact, by the use of high powers, an actual *continuity of structure* may usually be demonstrated. The smaller centres (nucleoli) also seem to be distinct from the germinal matter in which they lie. Even in these so-called "nucleoli" still smaller spherical bodies, to the number of three or four (nucleoluli), are sometimes to be seen distinctly (figs. 26 & 27). These centres are evidently formed one within the other.

The last or smallest centres are most darkly coloured. In the "nucleoli" the colour is not so intense. The "nuclei" again are still paler, but nevertheless the colour is very decided indeed. The matter more external is very faintly coloured, or it remains perfectly colourless. So that in this, as in many other instances elsewhere, it is to be noticed that the outer part of each cell, or that in *actual contact with the colouring solution*, is not coloured, while the intensity of the colour gradually increases as we pass towards the *innermost part* of the germinal matter, although this may be situated at the greatest distance from the colouring solution. To reach the *nucleus* and *nucleolus*, it is obvious the solution must pass *through a considerable thickness of tissue. The colour is, however, deposited here in greatest quantity.*

Again, it is to be noticed that in the younger cells (figs. 9 & 38), which are more or less darkly *coloured* over their whole extent, the one or two distinct nuclei seen in the fully formed cells are not to be demonstrated, although it often happens that a vast number of very darkly coloured spots are to be discerned, each being imbedded in matter more faintly coloured than itself. In the young cell every part of the germinal matter, or, in ordinary phraseology, the "entire cell," possesses equal power, and, as we have seen, may divide and give rise to the production of several separate cells; but when the formed matter is produced on the surface; the cell, as a whole, no longer possesses this power, which is restricted to the so-called "*nucleus*" or "*nucleolus*," which may divide and give rise to the formation of new cells.

Suppose new cells are to be developed from a fully formed cell: the outer colourless or formed material, the cell-wall, if present, and the surrounding connective tissue *take no part in the process*; but the active changes are effected by the "*living germinal matter*" ("*nucleus*") alone.

How, then, is the formed material of the cell produced? The observations just made seem to me to lead to but one conclusion—that the *formed material* results from changes occurring in the *germinal matter.* I hold that all the formed material was once in the state of germinal matter, and that whenever the ganglion-cell increases in size, or the fibres in connexion with it increase in length, except of course when artificially stretched, a certain amount of *germinal matter undergoes conversion into formed material.* The changes which take place in the formation of nerve-fibres occur in a similar order; but as the relations of structure produced are more simple, the alterations may be studied more readily and described more clearly.

Figs. 29 & 30 represent a portion of a dark-bordered fibre in course of development. It consists of nuclei in connexion with fibres. The fibre is seen to be thinnest about midway between the respective nuclei (*b*). The fibre grows at the points marked *a*, and at these points only. The oldest parts of the fibre are the narrowest portions, marked *b*. These are narrow because at the time of their formation the masses of germinal matter were so much smaller than they are now (fig. 30). The nuclei in connexion with these fibres may divide, and other new fibres may be produced; and a similar process occurs in the nuclei of the ganglion-cell which are connected with the formation of the spiral fibres.

Fig. 29 gives the appearance which this fibre would have presented if examined at an earlier period. Now, although this figure literally represents but my own view of the matter, it is only just that I should state that fibres presenting every degree of change have been actually observed in the same specimen, so that there can be little doubt as to the general truth of the facts brought forward, although differences of opinion may be entertained with reference to some of the explanations I have ventured to offer.

My observations upon various tissues in different stages of development have convinced me that the growth of the cells or elementary parts is a much more simple process than is generally supposed, and consists merely in a certain proportion of germinal matter undergoing conversion into formed material, while at the same time pabulum passes into the germinal matter, and the wonderful properties or powers possessed by this substance are communicated to it. By this formation of new germinal matter the proportion of the latter converted into formed material, and the formed material which is destroyed and removed, may be completely compensated for.

There is a certain relation between the proportion of germinal matter and formed material of the cell, which varies at different ages and under different circumstances, as I have shown. The rate at which pabulum undergoes conversion into germinal matter varies according to the facility with which it comes into contact with the living matter. The formed material offers a greater impediment to its passage in old than in young cells, so that under normal conditions, the process of growth occurs more and more slowly as the cell advances in age. In the young cell, more inanimate pabulum becomes *living* matter, and more *living* matter becomes *formed material* than in the adult, and in the latter, more than in the old cell.

Next, then, for consideration is the question of the mode of formation of the spiral fibre. Now it must, I think, be admitted that there is a great accumulation of evidence in favour of the general conclusion that *all living matter possesses a power of movement*. It seems to me that not one step in growth can be explained unless the particles of living matter move by virtue of some inherent force or power which acts independently of, and is capable of overcoming, the force of gravitation. The movements of living matter have been observed in many of the lower forms of living structures. I have described the phenomenon as it may be seen in the mucous corpuscles and young epithelial cells of the nasal and bronchial mucous membranes; and although I have not seen the movements in the living matter of the tissues generally, there seems to me the strongest evidence that such movements actually occur*. In these peculiar ganglion-cells we have, I think, very convincing evidence that movements have taken place uninterruptedly since the earliest changes occurring in their formation. I have endeavoured to show that the cell, when fully formed, does not occupy the same spot as it did when its development commenced; and upon consideration it will appear that it is not possible that many of the ganglion-cells could have been developed in the position in which they are found in the fully formed state.

The spiral seems to result partly from a sort of splitting and subsequent condensation of the lower portion of the cell itself, and partly by growing from the nuclei connected with the fibres, while at the same time the fundus moves away, and spiral after spiral is left around the central fibre, which is of course gradually increasing in length also. I think it doubtful if the entire cell *rotates*, because the central fibre does not appear to be twisted; but it is obviously possible that the *outer portion* of the cell might rotate slowly round the inner portion without causing any twisting of the fibre, the mass of which the cell is composed being in the natural state very soft and plastic.

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* See a paper in my 'Archives,' vol. iv. p. 150, "New Observations upon the Movements of the *living* or *germinal* matter of the tissues of Man and the higher animals."

It must be borne in mind that at first the two fibres of the ganglion-cell are parallel to each other, and that the cell while altering its position continues to grow. As the cell moves away, its fundus or large extremity preceding, the fibres projecting from it increase in length—are drawn out, as it were.

There is a fact in favour of rotation which I have observed so often that it may be regarded as constant—that, in peripheral parts where a dark-bordered fibre is being developed, a fine fibre passes spirally around it; and this may be accounted for in precisely the same way as I have attempted to explain the production of the spiral fibre of the ganglion-cells. The arrangement described is represented in figs. 29 & 30. It is constant, but can only be demonstrated positively at an early period of development of the dark-bordered fibre. The frequent crossing and twisting of fibres around one another amongst ganglion-cells, and the strange crossing over and under observed in the case of all fibres in the trunks of nerves, must also be due to a corresponding change of position between contiguous fibres after they have been formed, but at an early period of their life's history. The arrangement of the fine fibre, represented in figs. 29 & 30, is very remarkable; and I have seen very many specimens exhibiting the same points. It must also be noticed that the nuclei of the dark-bordered fibre are *much nearer together than the nuclei of the fine fibre.* This fact is also constant in the case of such nerves near their ultimate distribution. I am not yet able to give a satisfactory explanation of the fact, but it would seem to show either that the fine fibre has grown very much faster than the dark-bordered fibre, or that the fine fibre was developed, and perhaps in an active state, at a period anterior to the development of the dark-bordered fibre. These points are of the utmost interest, and well deserve the most searching and minute investigation; for it is certain that the settlement of many of the questions raised, and but very imperfectly considered here, must lead to the establishment of new general principles of wide application.

6. *The sense in which the term "nucleus" is employed in this paper.*

Although I have for convenience made use of the ordinary word "*nucleus*," it must be understood that it is used only in a general sense; for I maintain that the matter around the nucleus differs from that of the nucleus itself only in having reached a further stage of existence. My meaning will be readily understood by the following statement, which is supported by evidence already adduced (pages 548, 549). The whole of the germinal matter, of which the young cell is almost entirely composed, may divide and subdivide, and from it any number of new cells may be produced. Nor is it necessary that a "*nucleus*" should be present in the detached portion. The "nucleus" is often not to be distinguished until *some time afterwards*. This fact may be observed in the germination of pus and mucus. Neither "nucleus" nor "nucleolus," therefore, are bodies possessing *peculiar powers or actions upon matter around them*, nor is the "nucleus" essential to the being or to the *multiplication of a "cell" or elementary part*. Nuclei are but new centres which appear in preexisting germinal matter; and in these again new centres may arise, and so on, centre within centre. In some of the nerve-cells there is but one such centre, in others more than one. In some the "nucleus" is dividing (figs. 11 & 12). The terms *nucleus, nucleolus, nucleolulus* are arbitrary, and indicate germinal centres, which have appeared one after and one within the other. These consist of living matter in different phases of existence.

After a time the germinal matter of which a young "cell" is composed, at its outer part undergoes conversion into formed material. *This formed material cannot produce new formed material*. It may undergo *physical* and *chemical* changes, but it is no longer the seat of *vital changes*. The germinal matter which remains (nucleus) may still, up to a certain period, give rise to the production of *new cells*. The more the formed material around it increases the greater is the impediment to the passage of nutrient matter, and the more slowly it lives; so that, instead of new cells being produced, the germinal matter that remains gradually undergoes conversion into formed material; and it is doubtful if the germinal matter, at its outer part, where this conversion is actually occurring, could under any circumstances give rise to the production of new cells. It has reached a later stage of being, and has lost this power. Such *vital power*, however, undoubtedly still exists in that part of the germinal matter which in these nerve-cells is known as the "nucleolus." In passing from without inwards, in the case of a fully formed cell (fig. 1) we meet with matter in different stages of existence, which exhibits a difference in power. Most externally is the formed matter, which possesses no power of *formation* or *reproduction* whatever; next we come to matter in which the vital powers of reproduction still exist, but to a limited degree. It may increase; but from it no defined complex structure like a ganglion-cell could be produced. It is gradually undergoing conversion into formed matter. Within this, again, is germinal matter, which possesses the power of increase, and of giving rise, under certain conditions, to the production of perfect ganglion-cells. This matter (nucleus, nucleolus) still retains the *power possessed by the entire mass, of which the embryo cell was composed*, before it exhibited the wonderful structure evident in its fully developed state; and from it new cells might be developed.

7. *Of the fibres in the nerve-trunks continuous with the straight and spiral fibres of the ganglion-cells.*

The ganglion-cells I have described are not connected with one peculiar kind of nerve-fibre only. In the frog it is probable that the bundle of very fine fibres correspond to the grey or gelatinous fibres of mammalia. I have shown that bundles of very fine fibres, many of which are destitute of dark-bordered fibres, are connected with ganglion-cells*. Some of these very fine fibres with the ganglion-cell are represented in figs. 22 & 27, magnified 1800 diameters; and these remain as very fine fibres throughout every part of their course.

There is, however, no difficulty in proving that many of the pear-shaped ganglion-cells in the frog are connected with dark-bordered fibres. I have traced a dark-bordered fibre into the ganglion-cell, at the lower part of which it is sometimes convoluted in such a way that its actual connexion with the substance of the cell is not demonstrable. An example of this is represented in fig. 26. It will be asked in this case if the dark-bordered fibre exhibits an axis-cylinder quite up to the ganglion-cell. I have seen a distinct and very fine axis-cylinder, in the case of some old cells in the ganglia on the posterior roots of the nerves in the frog, within three thousandths of an inch of the ganglion-cell; but nearer to the cell the so-called axis-cylinder seems not to be distinct from the material of which the white substance is composed, as I have shown to be the case in all dark-bordered fibres near their distribution. In fact the distinction of a fibre into white substance and axis-cylinder is only to be demonstrated in nerve-fibres which have been developed for a very considerable period of time; and wherever this distinction is observed, we may be sure that the nerve-fibre is of considerable age.

As a general rule the straight fibre is thicker than the spiral fibres, and there is no difficulty in obtaining specimens in which the dark-bordered fibre can be traced on as the straight fibre of the ganglion-cell. The spiral fibres are often very fine, and are connected together here and there. Sometimes several unite to form one very fine fibre, which can be followed for some distance upon, and very close to, the outer part of a wide dark-bordered fibre. But I have many specimens in which both straight and spiral fibres are exceedingly thin and of equal diameter, others in which the straight is much thicker than the spiral fibre, and a few in which the diameter of the spiral fibre or fibres is even greater than that of the straight fibre.

I have traced the *spiral fibres* continuous with *a dark-bordered* fibre in several instances. A very conclusive specimen showing this fact is represented in Plate XXXIII. fig. 1, in which both straight and spiral fibre were continuous with dark-bordered fibres; also in fig. 19, in which the connexion between the spiral fibre and the dark-bordered fibre is represented.

So that, with regard to the fibres connected with these ganglion-cells in the frog, it may be remarked—

1. That in some instances very fine fibres, not more than the $\frac{1}{60,000}$th of an inch in diameter, are alone continuous with both the straight and spiral fibres of the ganglion-cell.

2. That a dark-bordered fibre may be traced to the ganglion-cell as the straight fibre, while the spiral fibres are continued on as very fine fibres.

3. That the spiral fibres may be continued onwards as a dark-bordered fibre, which may be wider, at least for some distance, than the fibre continued from the straight fibre.

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* Archives of Medicine, No. XII.

4. That both straight and spiral fibres may be continuous with dark-bordered fibres*.

There is no reason for assuming that the fine fibres in all cases are but an early stage of development of the dark-bordered fibres; for the two classes of fibres undoubtedly exist at every period of life of the animal, even in old age. I conclude, therefore, that these peculiar ganglion-cells in the frog are connected with both classes of fibres; and in mammalia the apparently spherical ganglion-cells, which answer to these pear-shaped cells of the frog, are connected with both grey and dark-bordered fibres.

It is exceedingly difficult to follow an individual fibre for any distance in the trunk of a nerve. I have seen nerve-fibres dividing in the trunks near to ganglia, and on two or three occasions I have almost convinced myself that one of the fibres resulting from the subdivision passed to a ganglion-cell, while the other passed on in the trunk of the nerve; but I have not succeeded in preserving an undoubted specimen of such an arrangement. The difficulty of following one individual fibre is much increased by the fact that the nerves vary so much in thickness within a very short distance. Dark-bordered fibres may be very distinct, and within a distance of two thousandths of an inch may be so thin as to be scarcely demonstrable, swelling out again a little further on. This renders it necessary to isolate an individual fibre from its neighbours before its course can be traced with positive certainty for any great distance. A good example of the varying diameter of dark-bordered fibres within a short distance is represented in fig. 39.

The observations recorded in page 549 seem to show that a ganglion-cell may be developed upon a nerve-fibre already formed. Hence these ganglion-cells cannot be regarded as centres from which two nerve-fibres proceed direct to their peripheral distribution, but as *centres placed at a part of a circuit which existed as a complete circuit before the ganglion-cell was developed in connexion with it.* It is impossible to discuss this most interesting and important question without entering into the consideration of the connexion of nerve-fibres with other centres, especially with the nerve-cells in the spinal cord and parts above; so I will not pursue it further. But I would state that I have not succeeded in finding ganglia from which fibres proceed in one direction only: and, that I may not be misunderstood upon this point, let me say that I have never seen a ganglion, in connexion with the nervous system of any creature, the fibres of which proceed in but one direction only, as is now believed to be the case by many observers. From every ganglion I have ever seen, fibres proceed to their destinations in at least two different directions; and from the majority of ganglia, even in the case of those very small ones which consist only of three or four cells, fibres often pass away in three or four different directions (fig. 5). And in every case in which I have been able to obtain a separate ganglion-cell well prepared, I have seen at least two fibres; and although these may run parallel to each other for some distance, they have been so often observed to pass in opposite directions when they reach the nerve-trunk, that I believe myself justified in expressing a very positive opinion that such is always the case.

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* I do not feel quite confident that both the fibres proceeding from *one ganglion-cell*, although broad, are true dark-bordered fibres; but it would seem pretty certain that in some cases a dark-bordered fibre is connected with the cell as a spiral, and in others as a straight fibre—a fact which may hereafter be of some importance, as it may afford us a positive index of the direction in which the nerve-current circulates in these elaborate organs.

8. *Of the ganglion-cells of the heart.*

Although the description given in this paper will apply to the ganglion-cells of ganglia in different parts of the frog, including those distributed to the heart and lungs, I feel it necessary to refer particularly to the cardiac ganglia of the frog, because Kölliker has recently made some very confident statements with regard to the structure, arrangement, and action of these ganglia, which my observations fail to confirm. Nor is the difference between us one of interpretation; we are at variance as to actual facts. With reference to the ganglia of the heart this observer says, "It may be particularly mentioned that the origin of nerve-fibres from unipolar cells, and the rarity of the double origin of fibres, may be especially well seen in *one* place, namely, in the septum of the heart of the frog, where also R. Wagner admits the fact. . . . . . *Here also we may most readily convince ourselves that there are many apolar cells without processes, as is most plainly shown in the heart-ganglia*, and in the small ganglia upon the urinary bladder of the *Bombinator*, in which, as also in similar ganglia of the frog, *the matter is as clear as possible*"[*].

But in his Croonian Lecture last year, his opinion as to apolar cells has become curiously modified. Although the existence of such cells was as "clear as possible" in 1860, great doubts are expressed upon the matter only two years afterwards:—"These cells, that is to say, all of them which are connected with nerve-fibres and whose connexions can be clearly made out, are *unipolar*, or send out *but a single fibre*, and that in *a peripheral direction, without having any connexion with the transcurrent fibres of the vagus. Bipolar or multipolar cells are not to be seen: some apparently apolar cells present themselves, but concerning these it may be doubted whether they are unipolar cells whose issuing fibre is in some way hidden from view*"[†].

These positive statements are not illustrated by a single drawing, nor does Professor Kölliker give any reasons for modifying his views as to the presence of apolar cells.

As the result of very numerous observations, I have to state—

1. That there are no apolar cells, either in the ganglia of the heart or in those of the bladder of the frog, although Kölliker asserted that their existence was as "clear as possible."

2. The unipolar cells of Kölliker really have two or more fibres proceeding from them; so that his statement, that "all the cells connected with nerve-fibres send out but a single fibre," is not a fact.

3. Some fibres certainly pass in a central direction; so that Kölliker's assertion, that all the fibres pass in a peripheral direction, is not true.

If these ganglion-cells be examined, it will be found that the fibres proceeding from some pass for a certain distance in a peripheral direction, while others pursue the very opposite (Plates XXXIX., XL. figs. 41 & 42). Very many lying at the side of a nerve-trunk pass transversely towards the central part of the bundle of fibres. The arrangement and structure of the ganglion-cells of the heart differ in no essential particulars from those I have described in other ganglia. I have succeeded in demonstrating in several instances the straight fibre passing in one direction in the trunk of the nerve, and the continuation of the spiral fibre pursuing an opposite course. In some of these ganglion-cells the spiral is reduced to two or three coils (fig. 42), as is observed elsewhere, but I cannot but conclude that every cell has at least two fibres.

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[*] Manual of Microscopic Anatomy, April 1860.

[†] Croonian Lecture, May 1st, 1862.

Nor can I agree with Professor KÖLLIKER in the statement that the ganglion-cells have no connexion with the "transcurrent" fibres of the vagus. Although I have not been able to demonstrate how many fibres of the vagus are connected with the ganglion-cells, nevertheless, looking generally at the course of the fibres, and at the number of the cells, and considering the facts observed in other ganglia, I regard it as very probable that many of them are connected with the cells.

9. *Of the ganglion-cells and nerve-fibres of the arteries.*

In the nerve-trunks running near the branches of the arteries of the palate of the frog are numerous ganglion-cells. These ganglion-cells are often situated at the angle of division of the nerve-trunks. Some of the fibres from the small ganglia lying near arteries may be traced to the coats of the arteries, and some fine nerve-fibres resulting from their subdivision may even be followed amongst the muscular fibre-cells of arteries not more than the $\frac{1}{1000}$th of an inch in diameter.

Ganglia and ganglion-cells are found in considerable number in connexion with the arteries distributed to the different viscera of the abdomen, heart, and lungs, and very many are found close to the small arteries which supply the bladder of the frog. In many cases small ganglia and separate ganglion-cells are imbedded in the external or areolar coat of the artery.

In fig. 46 a small ganglion in course of development upon one of the iliac arteries of the frog is represented; and several fine branches of nerve-fibres can be followed amongst the muscular fibre-cells. I have seen very fine nerve-fibres beneath the circular muscular fibre-cells, and apparently lying just external to their lining membrane, composed of longitudinal fibres with elongated nuclei—an observation which confirms a statement of LUSCHKA'S. I have not succeeded in satisfying myself that nerve-fibres are ever distributed to the lining membrane of an artery, although, from the appearances I have observed, I cannot assert that this is not the case. In the auricle of the heart and at the commencement of the large cavæ very fine nerve-fibres are certainly distributed very near indeed to the internal surface, being separated from the blood only by a very thin layer of transparent tissue (connective tissue).

The distribution of nerve-fibres to the coats of a small artery about the $\frac{1}{800}$th of an inch in diameter is represented in fig. 45. In all cases (and I have examined vessels in almost all the tissues of the frog) not only are nerve-fibres distributed in considerable number upon the external surface of the artery, ramifying in the connective tissue, but I have also followed the fibres amongst the circular fibres of the arterial coat. The nerves can be as readily followed in the external coat as in the fibrous tissues generally; and the appearance of the finest nucleated nerve-fibres, already alluded to, enables one to distinguish them most positively from the fibres of connective tissue in which they ramify.

These nerves invariably form networks with wide meshes. I have demonstrated such an arrangement over and over again. A similar disposition may be seen in the auricle of the frog, in the coats of the venæ cavæ near their origin from the auricle, among the striped muscular fibres of the lymphatic hearts of the posterior extremities of the frog, and in other situations. **Kölliker** confesses that he has not succeeded in observing distinct terminations to the nerves distributed to the vessels of muscles. This observer has made the very positive assertion that some arteries are completely destitute of nerves, and, apparently without having given much attention to the subject, says, "hence it is evident that the walls of the arteries are not in such essential need of nerves as is usually supposed." Professor **Kölliker** seems to conclude, in too many cases, that what he has failed to see does not exist. It is easy to demonstrate nerves in considerable number on the arteries of the frog generally, though these nerves, and more especially those ramifying in the coats of the vessels of mammalia and birds, are still considered by many authorities in Germany to be fibres of connective tissue.

The nerves which supply the small arterial branches in the voluntary muscles of the frog come from the very same fibres which are distributed to the muscles. I have seen a dark-bordered fibre divide into two branches, one of which ramified upon an adjacent vessel, while the other was distributed to the elementary fibres of the muscle.

10. *Of the connexion of the ganglion-cells with each other.*

In figs. 5, 15, & 16, which represent ganglion-cells at an early period of development, several are seen connected together; in fact, the matter of which the several cells are composed is continuous. This must be the case, at least for a certain time, because a number of cells may be formed by the division of one (figs. 5, 11, 12, & 15).

After a time, as the new cells separate further and further from each other, the intervening matter which connects them becomes thinner and thinner, and forms what would be properly termed a fibre, figs. 15 & 16; and as the cells move away from the line where their formation commenced, these connecting fibres become finer and finer, and at last could not be distinguished from fibres of connective tissue. It is probable, in many instances, that all continuity of structure between some of the cells ceases; but it is to be remarked in all cases, that the nerve-fibres in the substance of a ganglion cross each other in various directions, and it is certain that fibres from several different cells run in the same bundle which leaves the ganglion. From what I have observed, I think it almost certain that, in many cases, ganglion-cells of one ganglion are connected by fibres with cells of another ganglion.

11. *Of the "capsule" of the ganglion-cell, and of the connective tissue and its corpuscles.*

From what has already been stated, it will be inferred that there is no actual cell-wall or special capsule at an early period of development of the ganglion-cell. The cell may form protrusions at various parts of its surface, like a young epithelial cell, a mucus- or a pus-corpuscle; and each offset may give rise to the formation of a new cell. After a time the cell is seen to be surrounded by, or imbedded in, a transparent substance, which in some cases exhibits a definite outline and might be termed a cell-wall (fig. 25), while in others it would be more correctly described as a matrix. In old ganglia there is a quantity of this tissue, which accumulates and becomes condensed as age advances; and it exhibits a fibrous character with nuclei imbedded in it. In consequence of the condensation of this tissue, it is often very difficult to demonstrate the anatomy of the cells in animals of mature and advanced age.

In order to understand the formation of this texture, it is necessary to examine ganglia and ganglion-cells at different ages.

Figs. 9, 14, & 15 show that at an early period the ganglion-cell consists simply of one oval mass of germinal matter, surrounded with a little formed material, from each extremity of which a fibre proceeds. The whole is imbedded in a little transparent tissue; and in similar tissue run the very fine nerve-fibres which proceed from the young ganglion-cell (figs. 10, 17, 18, 22, & 23).

Figs. 4, 27, & 28, on the other hand, show fully formed ganglion-cells which are imbedded in a tissue exhibiting striations, and a few fine fibres which resist the action of acetic acid. Around and at a short distance from the ganglion-cells there are several oval nuclei (connective-tissue corpuscles).

Figs. 4, 27, & 28 exhibit the characters of old ganglion-cells. The fibrous appearance of the matrix is more distinct, the fibres which resist the action of acetic acid are more numerous, and there are more nuclei around the ganglion-cell and around the fibres proceeding from it, but these nuclei are not connected with either.

And the important fact that the so-called connective-tissue corpuscles outside the ganglion-cells and outside the nerve-fibres are but faintly coloured with carmine, while those nuclei in connexion with the ganglion-cell and nerve-fibres, although separated from the solution by a greater distance, are more intensely coloured, must not be lost sight of. This was very distinct in the specimen represented in figs. 27 & 28.

Before any attempt is made to explain these facts, it is necessary to consider more particularly the relation of nerve-fibres and nerve-cells to the connective tissue and connective-tissue corpuscles. Although it is undoubtedly true that in preparations mounted in certain fluids it is not possible to distinguish the finest nerve-fibres from connective tissue, this distinction can be most clearly made out in some of my specimens; for example, in figs. 17 & 18 the nerve-fibre can be very readily distinguished as it runs amongst the connective tissue, and the true nature of the fibre is placed beyond question by the presence of the ganglion-cell. All fibres which can be followed for a considerable distance, which refract like true nerve-fibres and exhibit an appearance more or less granular, especially if they can be followed into ganglion-cells, must clearly be pronounced nerves. The finest nerve-fibres may often be followed amongst the connective tissue for a long distance, and their relation to other structures most positively determined. Fig. 43 represents a portion of a very fine nerve-fibre running amongst connective-tissue corpuscles, and crossing one of the processes of a pigment-corpuscle; and it is unquestionably distinct from the last two structures. In the cornea, as I have before stated, the nerves may be followed in their numerous ramifications amongst the corneal corpuscles and their processes, and it can be seen that the latter are not connected with the nerves as KÜHNE supposes. The nerve-fibres and the corneal tissue grow together, but, although closely related, they remain structurally distinct from one another.

But although there is no structural connexion between the nuclei of a true fibrous tissue (pericardium, tendon, cornea, sclerotic, &c.) and the nerve-fibres and nerve-nuclei ramifying in it, there is some difficulty in deciding upon this question in the case of certain forms of indefinite connective tissue immediately surrounding nerve-fibres and ganglia, and it is often not very easy to decide whether a given nucleus really belongs to a nerve-fibre or should be considered as a connective-tissue corpuscle. For example: what is the nature of the nuclei near the nerve-fibre and ganglion-cell in fig. 18? These are undoubtedly, as they now appear, connective-tissue corpuscles; but how were they formed? We know that during the growth of nerves and ganglion-cells new nuclei are formed, and some of these, which lie on the surface of the fibre or cell, produce connective tissue. The nuclei under consideration are, I believe, of this nature; and I consider it probable that they belonged to a nerve-fibre at an earlier period, or at any rate resulted from the division of nuclei which were concerned in the formation of nerve-fibre. So I believe that those close to the ganglion-cell were formed by it. At the lower part of the cell may be seen three small nuclei, which are probably of the same nature. From them, up to a certain period, new nuclei might have been developed and true nerve-fibres might have proceeded; but the nuclei can now only produce a low form of connective tissue, which accumulates around the more important structures. I consider that the ganglion-cell delineated in fig. 17, from the same specimen and not very far from the part represented in fig. 18, represents an earlier stage of development than the last ganglion-cell.

But I have already shown that in many cases delicate fibres from true nerve-fibres, after gradually becoming very fine, are lost in the connective tissue. I have also shown that fibres of connective tissue result from the degeneration of nerve-fibres; and it has been proved conclusively that connective tissue results from the wasting of nerve-fibres in disease. Nor is the nervous the only tissue which by normal wasting or abnormal degeneration leaves what is termed connective tissue. A structure so special as a uriniferous tube, or a portion of the cell-containing network of the liver may waste, and all that represents it will be what is termed connective tissue and connective-tissue corpuscles.

It seems to me that the almost structureless or delicately fibrous matrix in which both nerve-fibres and nerve-cells are imbedded is the result of changes which have taken place before the nerve-fibres and cells there present have made their appearance, and if these very nerve-fibres and cells had been allowed to remain for a longer period in the living animal, they would have become surrounded with more connective tissue. Both fibres and cells might become altered and waste: all the fatty and other constituents having been absorbed, what we term "connective tissue" alone would remain. Connective tissue and connective-tissue corpuscles are produced from the very same masses of germinal matter as those from which nerve-cells and nerve-fibres are developed; and I think it must be admitted that many fibres which resist the action of acetic acid, and which are generally regarded as consisting of yellow elastic tissue, were once nerves.

Nevertheless true nerve-fibres in which the nerve-current passes do not lose themselves in the connective tissue or blend with it, nor are they connected with its corpuscles, but they form networks, as already described. A normal nerve-fibre can always be distinguished from a fibre of connective tissue.

All the structures existing in the adult ganglion were at an early period represented only by masses of germinal matter (nuclei), surrounded or separated from each other by, or imbedded in, a little soft formed material. At a very early period of development the so-called nuclei of the nerve-fibres are very close together. Nerve-centres at an early period of development bear little resemblance to the perfectly developed structure,—a remark which is illustrated in the most striking manner in the case of the particular ganglion-cells which have been described in this memoir. Nor would it be possible to prove the real nature of such a structure as that represented in figs. 5, 11, 12, & 15, if seen amongst the tissues of an embryo, imbedded as it would be in embryonic tissue almost as rich in nuclei as the structure itself. Even when nerve-tissues have reached the period of development when their essential anatomical characters are well marked, and when they perform their characteristic actions, it is often very difficult, and if the ordinary processes of preparation be employed, impossible, to demonstrate positively the arrangement of the nerve-fibres, although we may be quite positive, as for example in the case of voluntary muscles, that nerve-fibres are there. Nuclei can be seen which certainly do not belong to capillaries, and these nuclei lie transversely or obliquely across the muscular fibres, and often several may be seen following each other in lines; but only in very favourable cases can any fibre at all be made out, and with the greatest care and the highest powers a very faint and slightly granular band only can be seen. Nor can the fibres be traced to undoubted nerve-fibres: and it is even difficult to be certain of the nature of what will eventually become the large trunks consisting of dark-bordered fibres; so closely do they resemble vessels, and so numerous are the nuclei. But by using transparent injection the vessels may be made out positively, and by adopting certain precautions in preparation, which it would be tedious to refer to here, many such difficult points have been definitely settled; and I have traced the changes which occur in the minute structure of many tissues from the earliest period at which they could be recognized up to their fully developed state. In the case of ganglia and nerve-fibres, we have at an early period what would be termed nuclei and granular matter around and between them; then we have the fully developed structure (cell-contents, spiral fibre, &c.), still bearing the same relation to the germinal matter which produced it; and lastly, when this structure has wasted, we have its remains represented by connective tissue and masses of germinal or living matter, no longer capable of producing special tissue, but only giving rise to the simple, transparent, more or less fibrous material, or connective tissue. These masses of living matter are usually known as the connective corpuscles; but in indefinite connective tissue, neither fibrous material nor corpuscles are developed as a special tissue destined for a special purpose in the economy: it is merely the remains of higher tissues, which have been in great part removed; or it is formed by germinal matter which is not capable of giving rise to any special structure. On the other hand, the so-called connective tissue of the cornea, of tendon, &c., is developed as a special tissue, and it may be said to fulfil a special purpose.

The structure of the cells in mammalia corresponding to the pear-shaped cells of the frog is a subject worthy of separate consideration; but I may mention that in several instances I have seen a fibre prolonged from the cell, corresponding to the straight fibre of the pear-shaped ganglion-cells of the frog—that the "nucleated fibres" which seem almost to encircle many of the cells correspond to the nucleated spiral fibre or fibres described in this memoir. That these nucleated fibres are true nerve-fibres, and not, as generally supposed in Germany, "nucleated connective tissue," is rendered evident by careful observation of the changes occurring during the development of the ganglia, and, I think, clearly demonstrated by the observations recorded in this paper. I regard it as certain that if these nucleated fibres surrounding the mammalian ganglion-cells are connective tissue, both the fibres I have described in the frog's ganglion-cell are of the same nature.

It is possible that, for many years to come, some observers will persist in terming everything in which they fail to demonstrate distinct structure connective tissue, and all nuclei which are not seen in their specimens to be in connexion with positive vessels, positive nerve-fibres, or other well-defined tissues besides fibrous tissues, connective-tissue corpuscles; but there is little doubt that when the changes occurring during the development of special tissues shall have been patiently worked out by the use of high powers and better means of preparation, opinions on the connective-tissue question will be completely changed. The idea of the necessity for a supporting tissue or framework will be given up, and many structures now included in "connective tissue" will be isolated, just as new chemical substances year after year are being discovered in the indefinite "extractive matters."

It is remarkable how positively many authorities deny the existence of structures which they have failed to demonstrate. Such a course is only justifiable on the presumption that the art of demonstrating structure has arrived at perfection; but we know, on the contrary, that it is but in its infancy. Surely it is premature to maintain that the vessels of the umbilical cord are destitute of nerve-fibres because we may have failed to demonstrate them—that the fibres of voluntary muscle only receive nervous supply at one point, because authorities will not admit that nerve-fibres may exist which are too delicate or too fine to be demonstrated by the means they may have employed—that the spindle-shaped fibres of organic muscle generally are not supplied with nerves, because they cannot find them—that the fibres prolonged from the large cells in the cord and in the brain are not continued into fibres, because they have failed to trace them for any considerable distance.

I think I can convince any one, by positive demonstration, that the three last positions are utterly untenable; while there is every reason to believe that certain elongated nuclei and fibres, to be seen amongst the muscular fibre-cells of the umbilical arteries and on the smaller vessels of the placenta, really belong to the nervous tissue.

Moreover many observers seem to have determined in their own minds what appearance a fibre should present to be entitled to be regarded as a nerve (that it must exhibit the double contour), and then they arbitrarily assert that a fibre which does not present these characters cannot be nervous; and even if it be continuous with an undoubted nerve-fibre, it is put down as connective tissue. The alterations which are produced in undoubted nerve-fibres by stretching, pressure, and the influence of water must not be forgotten.

It is true that during the last few years pale fibres have been admitted to exist in some situations besides the Pacinian corpuscle; but few observers will be prepared to admit the existence of a very extensive distribution of delicate pale nucleated nerve-fibres in every part of the peripheral nervous system, or that the active portion of all nerve-fibres exhibits the same essential anatomical characters, and that in all cases complete circuits exist, while free ends are nowhere to be found: yet these general conclusions are justified by facts which have been demonstrated*.

Conclusions.

The following are some of the most important general conclusions I have arrived at in the course of this inquiry:—

1. That in all cases nerve-cells are connected with nerve-fibres, and that a cell probably influences only the fibres with which it is structurally continuous.

2. That *apolar* and *unipolar* nerve-cells do not exist, but that all nerve-cells have at least two fibres in connexion with them.

3. That in certain ganglia of the frog there are large pear-shaped nerve-cells, from the lower part of which two fibres proceed:—*a*, a *straight fibre* continuous with the

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* I have seen numerous very fine nuclei connected together with exceedingly minute fibres in the tissues of many of the lower animals, especially insects. In this class, I am certain, nerve-fibres exist far too minute to be seen by any power yet made.

I regard very fine fibres and nuclei amongst the contractile tissue of the common Actinia as nervous, and I have seen a texture presenting similar characters ramifying in the muscular tissue of the Starfish and Sea-Urchin.

central part of the body of the cell; and *b*, a *fibre or fibres continuous* with the circumferential part of the cell, which is coiled *spirally* round the straight fibre.

4. These two fibres, after lying very near to, and in some cases, when the spiral is very lax, nearly parallel with each other, at length pass towards the periphery in opposite directions.

5. Ganglion-cells exhibit different characters according to their age. In the youngest cells neither of the fibres exhibits a spiral arrangement; in fully formed cells there is a considerable extent of spiral fibre; but in old cells the number of coils is much greater.

6. These ganglion-cells may be formed in three ways:—

a. From a granular mass like that which forms the early condition of all structures.

b. By the division or splitting up of a mass like a single ganglion-cell, but before the mass has assumed the complete and perfect form.

c. By changes occurring in what appears to be the nucleus of a nerve-fibre.

7. During the development of a ganglion-cell, there is reason to believe that the entire cell moves away from the point where its formation commenced, so that the fibres connected with it will become elongated.

8. There are "nuclei" in the body of the cell; and there are "nuclei" connected with the spiral, and also with the straight fibre. The nuclei in the cell are found upon its surface, and also in its substance.

9. The matter of which the "nucleus" is composed has been termed by me "germinal matter." From it alone growth takes place; and in all cases the matter (formed material) of which the nerve-fibre consists was once in the state of germinal matter.

10. The "nucleolus" consists of germinal matter. It may be regarded as a new centre which originates in a preexisting centre.

11. The ganglion-cells of the frog are connected with dark-bordered fibres, and also with fine fibres.

12. Contrary to the statement of KÖLLIKER, that apolar cells and unipolar cells are to be demonstrated in the cardiac ganglia, all the cells in these ganglia have two or more fibres emanating from them.

13. The muscular coat of all arteries of the frog, and probably of other animals, is supplied with nerve-fibres.

14. Nerve-fibres are not connected with the connective-tissue corpuscles.

15. The so-called nucleated capsule of the ganglion-cells in the ganglia of mammalia usually consists of nerve-fibres, many of which are connected with the cell.

15. As nerve-fibres grow old the soluble matters are absorbed, leaving a fibrous material which is known as connective tissue. A corresponding change is observed in other textures, both in health and disease.

EXPLANATION OF THE PLATES.

The dimensions of each object delineated can be ascertained by reference to the scales at the bottom of each page, magnified by the *same power* as the object itself. These scales, however, have not been measured quite correctly; the upper one being a little too long, while the lower is a little too short.

The drawings, with the exception of figs. 32 to 37, are accurate copies of nature.

PLATE XXXIII.

Contains figures illustrating the structure of the ganglion-cells connected with the sympathetic of the frog.

PLATES XXXIV. & XXXV.

Illustrate the changes occurring during the development of the ganglion-cells in the fully-formed frog, as observed in various ganglia of the sympathetic near the arteries supplying various internal organs of the frog, in the ganglia on the posterior roots of the nerves, the ganglia from which the heart, bladder, palate, and other organs receive their supply of nerve-fibres.

The changes occurring during the development of a granular mass of germinal matter into perfect tissues (nervous, vascular, muscular, fibrous, cartilaginous, osseous, glandular, &c.) can be studied more satisfactorily in the adult frog than in the embryo; for a complete history of the changes may be deduced from careful observations upon a small portion of an organ or structure in the same animal. For example, nerve-cells in every stage of development, from a small mass of germinal matter (nucleus) to the fully developed complex nerve-cell with its *straight and spiral fibre*, can be demonstrated even in one single microscopic ganglion (figs. 5, 6, 7, 8).

PLATE XXXVI.

Shows the relation of ganglion-cells and nerve-fibres to connective tissue and its corpuscles (figs. 17 & 18), and illustrates the connexion between the matter of which the body of the ganglion-cell consists and that which enters into the composition of the nerve-fibre.

PLATE XXXVII.

Exhibits the structure of several different forms of ganglion-cells, all of which possess two or more fibres.

PLATE XXXVIII.

Illustrates the relation of numerous very fine fibres to a single ganglion-cell, the connexion between some of the fibres of which the bundle is formed, and the relation of the compound bundle to the cord of connective tissue in which it is imbedded. In figs. 29 & 30, the manner in which a fine nerve-fibre is coiled spirally round a dark-bordered fibre at an early period of formation of the latter is represented.

PLATE XXXIX.

Contains—

1. Figures of ganglion-cells exhibiting particular characters (figs. 31 & 38).
2. A series of drawings showing the changes which, the author believes, take place in the production of the ganglion-cell with the spiral fibre (figs. 32 to 37).
3. A copy of a bundle of nerve-fibres in which the diameter of each fibre is greatly reduced at the point where the bundle passes through constricted apertures (fig. 39).
4. A drawing of a small compound nerve-trunk with a finer trunk coming off from it at right angles (fig. 40). It will be observed that the fine trunk is composed of fibres which pursue opposite directions in the large trunk, passing as it were towards the centre and towards the periphery.
5. A drawing of one of the pneumogastric nerves of the frog near the auricle of the heart. Numerous ganglion-cells are connected with the trunk of the nerve by very fine fibres, which are soon lost, but some pursue a direction towards the heart, while others pass towards the brain. The trunk of the nerve is at the lower part of the figure. The arrow points towards the heart. The bundle of fibres marked *b* connects the trunk of the nerve with that on the opposite side.

PLATE XL.

Fig. 42 represents two of the ganglion-cells, and the fibres connected with them, from the pneumogastric nerve (fig. 41). The course of some of the fibres can be traced in this drawing.

The relation of a very fine nerve-fibre to the connective-tissue corpuscles, and to a portion of one of the processes of a pigment-cell, is shown in fig. 43, which is magnified nearly three thousand diameters. The distribution of nerve-fibres in the tissue external to capillary vessels is illustrated in figs. 44 & 47. The ramification of fine nerve-fibres upon the muscular coat of a small artery from the bladder of the *Hyla* is seen in fig. 45, and in fig. 46 a portion of a branch of the iliac artery, with some small ganglion-cells and nerve-fibres connected with them.

XXVII. *On the Rigidity of the Earth.* By W. THOMSON, *LL.D., F.R.S., Professor of Natural Philosophy in the University of Glasgow.*

Received April 14,—Read May 15, 1862.

1. THAT the earth cannot, as many geologists suppose, be a liquid mass enclosed in only a thin shell of solidified matter, is demonstrated by the phenomena of precession and nutation. Mr. HOPKINS*, to whom is due the grand idea of thus learning the physical condition of the interior from phenomena of rotatory motion presented by the surface, applied mathematical analysis to investigate the rotation of rigid ellipsoidal shells enclosing liquids, and arrived at the conclusion that the solid crust of the earth must be not less than 800 or 1000 miles thick. Although the mathematical part of the investigation might be objected to, I have not been able to perceive any force in the arguments by which this conclusion has been controverted, and I am happy to find my opinion in this respect confirmed by so eminent an authority as Archdeacon PRATT†.

2. It has always appeared to me, indeed, that Mr. HOPKINS might have pressed his argument further, and have concluded that no continuous liquid vesicle at all approaching to the dimensions of a spheroid 6000 miles in diameter can possibly exist in the earth's interior without rendering the phenomena of precession and nutation very sensibly different from what they are.

3. Considerations regarding the velocities of long waves in deep sea, of tidal waves and of earthquake waves, and the harmonic vibrations of a liquid globe, having recently led me to think of the relative values of gravitation and elasticity in giving rigidity to the earth's figure, I was surprised to find that the former would have a larger share in this effect than the latter, unless the average substance of the earth had a very high degree of rigidity. For instance, I found that a homogeneous incompressible liquid globe of the same density as the mean density of the earth, if changed to a spheroidal form and then left free, influenced only by mutual gravitation of its parts, would perform simple harmonic vibrations in $47^m\ 12^s$ half-period‡. A steel globe of the same dimensions, without mutual gravitation of its parts, could scarcely oscillate so rapidly, since the velocity of plane waves of distortion in steel is only about 10,140 feet per second, at which rate a space equal to the earth's diameter would not be travelled in less than $1^h\ 8^m\ 40^s$.

4. Hence it is obvious that, unless the average substance of the earth is more rigid than steel, its figure must yield to the distorting forces of the moon and sun, not incomparably less than it would if it were fluid. To illustrate this conclusion, I have investi-

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[*] Philosophical Transactions, years 1839, 1840, 1842.

[†] Figure of the Earth, edit. 1860, § 85.

[‡] This will be demonstrated in a mathematical paper which the author hopes soon to communicate to the Royal Society. See §§ 55–58 of "Dynamical Problems, &c." following the present paper in the Transactions.

gated the deformation experienced by a homogeneous elastic spheroid under the influence of any arbitrarily given disturbing forces*. I thus find that if $2h'$ denote the difference between the longest and shortest diameters of the tidal spheroid, calculated on the supposition that the substance is of homogeneous (and therefore incompressible) fluid, and $2h$ the difference between the longest and shortest diameters of the spheroid into which the same mass, if of homogeneous incompressible solid matter, would be deformed from a naturally spherical figure when exposed to the same lunar or solar disturbing influence, we have (see § 34, Appendix to this paper)

$$h=\frac{h'}{1+\frac{19}{2}\frac{n}{gwr}},$$

where w denotes the mass of unit volume, and n the "rigidity" of the substance (see § 71 of the paper following the present in the Transactions); and g denotes the force of gravity on a unit of mass at the surface, and r the radius of the globe.

5. The density of iron or steel (7·8 times that of water) does not differ very much from the mean density of the earth (5·6 times that of water according to CAVENDISH'S experiment, or 6·6 according to the Astronomer Royal's). The rigidity of iron, according to experiments of my brother, Professor JAMES THOMSON†, is 10,800,000 lbs. per square inch. Since the weight of 1 lb. at Glasgow, where the experiment was made, is 32·2 British absolute units of force, we must multiply by 32·2 to reduce to kinetic measure as to force; and we must multiply by 144 to make the unit of area a square foot instead of a square inch. We thus find, in consistent absolute measure,

$$n=501\times10^8,\text{—}$$

the unit of mass being 1 lb., the unit of space 1 foot, and the unit of force that force which, acting on one pound of matter during a mean solar second of time, generates a velocity of 1 foot per second. In terms of the same units we have $r=20{,}887{,}700$; $g=32·14$, being about the average over all the earth; and for iron or steel $w=487$. Hence

$$h=\frac{h'}{1+\frac{19}{2}\cdot\frac{501\times10^8}{3308\times10^8}}=\frac{h'}{2·44}=·41h'.$$

Of glass, the rigidity is, according to WERTHEIM, about one-fifth of the value we have just used as that of iron; and therefore if the earth were homogeneous of its actual mean density, and had throughout the same rigidity as that of glass, the result would be $h=·78h'$.

6. Hence it appears that if the rigidity of the earth, on the whole, were only as much as that of steel or iron, the earth as a whole would yield about two-fifths as much to the tide-producing influences of the sun and moon as it would if it had no rigidity at all; and it would yield by more than three-fourths of the fluid yielding, if its rigidity were no more than that of glass.

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* The solution of this problem will be found in the paper referred to above (see §§ 47, 48).

† Cambridge and Dublin Mathematical Journal, 1848.

7. Such a deformation as this would be quite undiscoverable by any direct geodetical or astronomical observations; but if it existed, it would largely influence the actual phenomena of the tides and of precession and nutation.

§§ 8–20. *Effect of the Earth's Elastic Yielding on the Tides.*

8. To find the effect of the earth's elastic yielding on the tides, let 2H denote the difference between the greatest and least diameters of the spheroidal surface perpendicular to the resultant of the lunar or solar disturbing force*, and terrestrial gravitation supposed perfectly symmetrical about the centre, then $\frac{H}{r}$ will be the ellipticity of that spheroid; and we shall call it the *ellipticity of level produced by the lunar or solar influence on a rigid earth.* It may be remarked that H is the height of high above low water in the "equilibrium tide" of an ocean of infinitely small density covering a rigid earth.

9. Let H′ denote the height of the equilibrium tide for an ocean of density $\frac{1}{N}$ of the earth's mean density, the earth being still supposed *perfectly* rigid and covered by the ocean. Then the terrestrial gravitation level will be disturbed (as is proved in the theory of the attraction of ellipsoids) from the spherical surface to the spheroidal surface of ellipticity $\frac{3}{5}\cdot\frac{1}{N}\cdot\frac{H'}{r}$, by the attraction of the ocean in its altered figure. The ellipticity of level induced by lunar or solar influence must be added to this to give the ellipticity of actual level, which is of course the ellipticity of the free equilibrium surface of the ocean, or according to our notation $\frac{H'}{r}$. Hence

$$\frac{H'}{r}=\frac{H}{r}+\frac{3}{5}\cdot\frac{1}{N}\cdot\frac{H'}{r},$$

by which we find

$$H'=\frac{H}{1-\frac{3}{5}\cdot\frac{1}{N}}.$$

For sea-water the value of N is about $\frac{1}{5\cdot5}$; and therefore

$$H'=\frac{9\cdot2}{8\cdot2}H=1\cdot12H,$$

or only 12 per cent. more than for an ocean of infinitely small density.

10. What we have denoted above by h' is the value of H′ for N=1; and therefore

$$h'=\frac{5}{2}H,$$

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* This "disturbing force" is of course the resultant of the actual attraction of either body on a unit of mass in any position, and a force equal and opposite to its attraction on a unit of mass at the earth's centre.

and

$$h=\frac{5}{2}\cdot\frac{\mathrm{H}}{1+\frac{19}{2}\cdot\frac{n}{gwr}}.$$

11. Now, according to a proposition regarding the attraction of ellipsoids already used, we have $\frac{3}{5}\cdot\frac{h}{r}$ for the ellipticity in the terrestrial gravitation level produced by the ellipticity of deformation $\frac{h}{r}$ experienced in consequence of want of perfect rigidity. Hence the ellipticity of the terrestrial gravitation level, as disturbed by lunar or solar influence, is $\frac{3}{5}\cdot\frac{h}{r}+\frac{\mathrm{H}}{r}$. This will be the absolute tidal equilibrium ellipticity of an ocean of infinitely small density covering the elastic globe; but since there is a tidal ellipticity $\frac{h}{r}$ induced in the solid itself, the height from low tide to high tide of fluid relatively to solid (that is to say, the difference of depth between high water and low water) will be

$$\left(\frac{3}{5}h+\mathrm{H}\right)-h,$$

or

$$\mathrm{H}-\frac{2}{5}h;$$

or, according to the value of h just found (§ 10),

$$\frac{\frac{19}{2}\cdot\frac{n}{gwr}\mathrm{H}}{1+\frac{19}{2}\cdot\frac{n}{gwr}}.$$

12. This result expresses strictly the height of the equilibrium tide of a liquid of infinitely small density covering an elastic solid globe. It may be regarded as a better expression of the true tidal tendency on the actual ocean than the slightly different result calculated with allowance for the effect of the attraction of the altered watery figure constituting the equilibrium spheroid, and its influence on the figure of the elastic solid; since the impediments of land and the influence of the sea-bottom render the actual ocean surface altogether different from that of the equilibrium spheroid.

13. Hence the actual tidal tendency, which would be H if the earth were perfectly rigid, is in reality

$$\frac{\frac{19}{2}\cdot\frac{n}{gwr}\cdot\mathrm{H}}{1+\frac{19}{2}\cdot\frac{n}{gwr}},$$

where n denotes what we may call the *earth's tidal effective rigidity*, being the "rigidity" of a homogeneous incompressible solid globe of equal mass which, with an ocean equal and similar to the earth's, would exhibit the same tides.

14. If, for example, we give n the value for iron or steel above indicated, the formula becomes $\cdot59\times\mathrm{H}$. The comparison between theory and observation, owing to the

extreme complexity of the circumstances, has been hitherto so imperfect that we cannot say it disproves this result; and therefore, from tidal phenomena hitherto observed, we cannot infer that the earth is more effectively rigid than steel.

15. The value of n for glass, according to WERTHEIM, is $2160000 \times 144 \times 32{\cdot}2$, in British absolute units; and it reduces the formula to $\frac{1}{4\frac{1}{2}}$H. Now, imperfect as the comparison between theory and observation as to the absolute height of the tides has been hitherto, it is scarcely possible to believe that the height is in reality only two-ninths of what it would be if, as has hitherto been universally assumed in tidal investigations, the earth were perfectly rigid. It seems therefore nearly certain, with no other evidence than is afforded by the tides, that the tidal effective rigidity of the earth must be greater than that of glass.

16. Any approach to a close testing of the absolute amount of the tidal influence can scarcely be expected of either of the two great Kinetic[*] theories—the Oceanic theory of LAPLACE, or the Channel theory of AIRY,—as applied to diurnal or semidiurnal tides; but notwithstanding the strong contempt which has been expressed by the last-mentioned naturalist[†] (no doubt justly as regards false applications of it) for the Equilibrium theory[‡], we may look to it confidently for good information when it is applied to test the difference between mean fortnightly variations of level at two well-chosen stations, one in a low latitude, and the other in a high latitude. (See Note at the end of this paper.)

17. The fortnightly tide[**] at each pole gives high water when the moon's declination (Δ), whether north or south, is greatest, and low water when she crosses the equator; and the whole difference in level produced by it would be

$$H' \sin^2 \Delta$$

if the earth were all covered with water. The mean daily level at the equator, on the same supposition, would vary by half that amount, being low water when the moon is furthest from the equator, and high when she crosses the equator. But, owing to the actual distribution of land and water, either of those variations may be diminished by an amount which it is impossible to estimate theoretically; but then the other must be increased by nearly the same amount. And if a denote the mean height of the sea-level above a fixed mark at the earth's north pole, about times when the moon's declination is greatest, b the corresponding mean of observations about times when she is crossing the equator, a' and b' corresponding means derived from observation at an equatorial station, and ${}_{\prime}\mathrm{H}$ something intermediate between H and H′, we must have

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[*] Dynamics meaning properly the science of force, and there being precedents of the very highest kind, for instance, in DELAUNAY's 'Mécanique Rationnelle' of 1861, and ROBISON's 'Mechanical Philosophy' of 1804, in favour of using the term according to its proper meaning—and the modern corrupt usage, which has confined it to the branch of dynamical science in which relative motion is considered, being excessively inconvenient and vexatious,—it has been proposed to introduce the term "kinetics" to express this branch; so that dynamics may be defined simply as the "science of force," and divided into the two branches, Statics and Kinetics. The introduction of this new term, derived from κίνησις, *motion*, or act of moving, does not interfere with AMPÈRE's term, now universally accepted, "kinematics" (from κίνημα), the *science of movements*.

[†] "Naturalist. A person well versed in Natural Philosophy."—JOHNSON's Dictionary. Armed with this authority, chemists, electricians, astronomers, and mathematicians may surely claim to be admitted along with merely descriptive investigators of nature to the honourable and convenient title of Naturalist, and refuse to accept so un-English, unpleasing, and meaningless a variation from old usage as "physicist."

[‡] Encyclopædia Metropolitana, "Tides and Waves," §§ 64, 539, &c.

[**] AIRY's 'Tides and Waves,' § 45.

$$a-b+b'-a'=\tfrac{3}{2}\,{}_{\prime}\mathrm{H}\sin^2\Delta,$$

whatever be the distribution of land and water over the earth, only provided the fortnightly tide follows sensibly the equilibrium law, which, for moderately well-chosen stations, we may suppose it must do.

18. If, instead of being at a pole and at the equator, the stations are in latitudes respectively l and l', we should have

$$a-b+b'-a'=\tfrac{3}{2}\,{}_{\prime}\mathrm{H}\sin^2\Delta(\sin^2 l-\sin^2 l').$$

Now if we suppose the moon's mass to be $\frac{1}{75}$ of the earth's, we have $\mathrm{H}=1{\cdot}92$ foot. As $\mathrm{H}'=1{\cdot}12\,\mathrm{H}$, and as there is more area of water than of land over the earth, we cannot be far wrong in taking ${}_{\prime}\mathrm{H}=1{\cdot}08\,\mathrm{H}=2{\cdot}04$ feet.

The greatest value of Δ is 28° 37′; and hence, in the most favourable lunations,

$$a-b+b'-a'={\cdot}713\text{ foot}\times(\sin^2 l-\sin^2 l').$$

19. Iceland and Teneriffe, in nearly the same longitude, and in latitudes 63° 20′ and 28° 30′, would probably be very favourable stations. For them $\sin^2 l-\sin^2 l'={\cdot}571$; and therefore

$$a-b+b'-a'=0{\cdot}407\text{ foot},$$

or about 4·9 inches.

It is probable that carefully made and reduced observations, with proper allowance for barometric disturbances, at two such stations, would not only detect this tide, but would give a tolerably accurate determination of its amount.

20. It would be, for Iceland and Teneriffe, as found above, 4·9 inches if the earth were perfectly rigid; or 3 inches if the tidal effective rigidity is only that of steel; or about an inch if the tidal effective rigidity is only that of glass.

There seems no more hopeful way to ascertain how rigid the earth really is, than to make careful observations with a view to determining the fortnightly tide with all possible accuracy. It is possible also that very accurate observations on the semi-diurnal tides in a deep inland lake of great extent, or at distant points of the Mediterranean sea-board with only deep water intervening, might help to solve this question.

§§ 21–32. *Effects of Elastic Yielding on Precession and Nutation.*

21. If we suppose the sun, the moon, and the earth's centre to be reduced to rest at any moment, the two former to be held fast in their places, and their attractions on the latter to be balanced by an infinitely great mass held at the proper infinitely great distance in the proper direction; the tide-generating distribution of force, and the couple

tending to turn the earth round an equatorial diameter, by which precession and nutation are produced, would be left precisely as they are.

22. If the sun and moon be carried round the earth at rest, according to their true relative orbital motions, and if the infinitely distant mass be shifted continually so as always to balance their attractions on the earth's centre of gravity, all the phenomena of tides, of nutation, and of precession, will take place precisely as they do in reality.

23. If, now, merely as an artifice to avoid mathematical calculations, we suppose the earth's rotation to be stopped, and instead a repulsive force, from an infinite fixed line coinciding with the earth's axis in the first place, to be introduced—the force to vary directly as the distance from this fixed line, and to amount to $\frac{1}{289}$ of gravity at the equator,—the figure of the earth will remain unchanged.

24. Let us first suppose the earth to be either perfectly fluid, or to be a perfectly elastic homogeneous and isotropic solid, of spherical figure when undisturbed, and the sun and moon to be held fast in any positions. It will clearly remain in perfect equilibrium in the circumstances defined in §§ 21, 23, whether the sun and moon are both in the plane of its equator or not; because if it is not in equilibrium it must commence rotating with a continually accelerated motion about some axis; the fixed repelling line not being supposed to impede its motion, but merely to continue repelling according to the stated law.

25. But either sun or moon, if not in the plane of the equator, and the corresponding part of the infinitely distant balancing attractor, will exercise a couple upon the earth, by attracting the near protuberant equatorial parts more, and the remote less, than the centre. How then is this couple balanced? Clearly it is by the repulsion of the fixed repelling line on the tidal deformation produced by the sun or moon, as the case may be. Considering for simplicity only one, the sun for instance, we perceive that the equilibrium tide will produce a small elliptic deviation, superimposed on the great polar and equatorial ellipticity, the longer axis of this smaller superimposed ellipticity being in the line through the sun. Now without this the spheroid of revolution would experience no resultant action from the repelling line; but with it the actual resultant spheroid will experience, from the repelling line, a couple tending to carry away from this line the longer axis of the superimposed ellipticity. This is exactly opposite to the couple produced by the attraction of the sun; and as there can be no resultant action, we see that the equilibrium is maintained by the balancing of these two couples.

26. Hence in reality we conclude that the couple due to a disturbing body in any position attracting a rotating fluid spheroid, or elastic isotropic body naturally spherical and rendered oblate by rotation, is balanced by the couple of centrifugal force on the crowns of the tidal elongation produced by the disturbing body, provided the rotation is not so fast as to render the tidal deformation sensibly different from what it would be if the disturbing body rotated with the same angular velocity as the spheroid. This condition will, it is intended, be shown, in a subsequent communication to the Royal Society, to be essentially fulfilled when the rotation is slow enough to allow the first approximation to be used as in the ordinary investigations regarding the figure of the earth.

27. Let us now suppose the earth to be an elastic spheroid of nearly on the whole the

same figure in all its surfaces of equal density as a rotating fluid, and therefore on the whole nearly free from distorting stress in its interior. If under the lunar and solar influences it were to yield and experience nearly as much tidal elongation as it would if quite fluid, the couple to which precession and nutation are due would be very nearly balanced by the centrifugal force on the crowns of the tidal elongation, and consequently precession and nutation would be very much less than if the earth were perfectly rigid. For instance, if the earth were a homogeneous incompressible elastic solid of the same rigidity as glass, it would, as stated above, experience seven-ninths of the tidal deformation of a fluid globe of the same density. Hence seven-ninths of the couple would be balanced, and precession and nutation would be reduced to two-ninths by the elastic yielding. Even if the rigidity were as much as that of steel, the precession and nutation would not be more than three-fifths of their full amount for a perfectly rigid spheroid.

28. The close agreement between the observed amounts of precession and nutation, and the results of theory on the hypothesis of perfect rigidity, renders it impossible to believe that there is enough of elastic yielding to influence the phenomena to any considerable extent. It is worthy of remark, however, that in general the theoretical estimates of the amount of precession have been somewhat above the true amount demonstrated by observation. It seems not altogether improbable that this discrepance is genuine, and is to be explained by some small amount of deformation experienced by the solid parts of the earth, under lunar and solar influence.

29. But the only possible ground on which it could be maintained that the earth as a whole is less rigid than a solid steel globe of the same dimensions, is to assume that there is an enormous liquid vesicle, or a solid nucleus separated by a fluid layer from the outer crust, in the interior, and that the loss of precessional effective moment of inertia, owing to this portion not being carried round in the precessional movement, is almost exactly compensated by a diminution of the generating couple in very nearly the same proportion by elastic yielding. Although, considering HALLEY'S theory of the secular variation of terrestrial magnetism, and the general accordance of its results with the actual phenomena as demonstrated by the best observations made up to the present time*, it would be most rash to say that it is very improbable there is a solid iron nucleus sunk to the centre of a hollow central vesicle of fluid in the earth, yet it seems to me excessively improbable that the defect of moment of inertia due to fluidity in the earth's interior bears approximately the same ratio to the whole moment of inertia, as the actual elastic yielding bears to the yielding which would take place if the earth were perfectly fluid. Avoiding conjectural assumptions, however, I conclude that either this proportion is approximately fulfilled, or both the following propositions are true:—

I. The defect of moment of inertia, owing to fluidity in the interior, is small in comparison with the whole moment of inertia of the earth.

II. The deformation experienced by the earth, owing to lunar and solar influence, is small in comparison with what it would experience if it were perfectly fluid.

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* General SABINE, Proceedings of the Royal Society, 1862.

30. It is easily seen that the first of these propositions is not opposed to HALLEY's theory. For instance, if there were a spheroidal iron core* 2000 miles diameter, cool and magnetic to within 100 miles of its surface, sunk to the centre of a spheroidal space of lighter fluid 3000 miles diameter, enclosed within a solid crust 2500 miles thick, the moment of inertia would be only about one-half per cent. less than it would be if the whole were rigid. Far less magnetism than such a nucleus could retain would be sufficient in amount to account for either the whole of "terrestrial magnetism" as manifested at the surface, or for the secular variations of it which have been observed. Whatever may be thought of the probability of this hypothesis, the barest possibility that it may be true renders it an interesting problem for mathematicians to find the precessional movement of a rigid spheroid sunk in a lighter liquid enclosed within a rigid spheroidal shell. A solution, founded on the supposition that all the bounding surfaces are truly elliptic and of small ellipticity, and that the fluid portion is of uniform density, is to be obtained with ease in an extremely simple form, and might be useful as a guide for speculation.

31. That the "tidal effective rigidity" (§ 13), and what we may similarly call the "precessional effective rigidity" of the earth, may be both several times as much as that of iron (which would make the phenomena both of tides and of precession and nutation sensibly the same as if the earth were perfectly rigid), it is enough that the actual rigidity should be several times as great as the actual rigidity of iron, throughout 2000 or more miles' thickness of crust.

32. At the surface, and for many miles below the surface, the rigidity is certainly very much less than that of iron (how much less might be estimated if we had trustworthy data as to the velocity of natural or artificial earthquake waves through short distances); and therefore at great depths the rigidity must be enormously greater than at the surface. That both the rigidity and the resistance to compression should be much greater several hundred miles down than at the surface, seems a natural consequence of the enormous pressure experienced at those great depths by the matter of the earth.

Note on the Fortnightly Tide.

33. In water 10,000 feet deep (which is considerably less than the general depth of the Atlantic, as demonstrated by the many soundings taken within the last few years, especially those along the whole line of the Atlantic telegraph cable, from Valencia to Newfoundland) the velocity of long free waves is 567 feet per second†. At this rate the time of advancing through 57° (or a distance equal to the earth's radius) would be only ten hours. Hence it may be presumed that, at least at all islands of the Atlantic, the fortnightly tide should follow sensibly the equilibrium law.

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* An ancient cold iron meteorite which may have entered a nebula of smaller bodies and formed the nucleus of our present earth, which under such circumstances could not but be built up and heated by attracting them to itself.

† AIRY, § 170.

"In the Philosophical Transactions, 1839, p. 157, Mr. Whewell shows that the "observations of high and low water at Plymouth give a mean height of water increasing "as the moon's declination increases, and amounting to three inches when the moon's "declination is 25°. This is the same direction as that corresponding in the expression "above to a high latitude. The effect of the sun's declination is not investigated from "the observations. In the Philosophical Transactions, 1840, p. 163, Mr. Whewell has "given the observations of some most extraordinary tides at Petropaulofsk in Kamschatka, and at Novo-Arkhangelsk in the island of Sitkhi on the west coast of North "America. From the curves in the Philosophical Transactions, as well as from the "remaining curves relating to the same places (which, by Mr. Whewell's kindness, "we have inspected), there appears to be no doubt that the mean level of the water at "Petropaulofsk and Novo-Arkhangelsk rises as the moon's declination increases. We "have no further information on this point."—Airy's 'Tides and Waves,' § 533.

Appendix, added January 2, 1864.

34. Let the difference of longest and shortest radii, which would be produced by lunar and solar influence in the two cases—of the earth supposed a homogeneous incompressible fluid tending to the spherical shape by gravitation alone, and supposed a homogeneous incompressible elastic solid without mutual gravitation but tending in virtue of its elasticity to the spherical figure—be denoted by h' and h'' respectively; and let h be the difference of greatest and least radii when both gravity and elasticity act jointly to maintain the spherical figure. We shall have obviously

$$\frac{1}{h}=\frac{1}{h'}+\frac{1}{h''}.$$

For the distorting force, being balanced by elasticity and by gravity jointly, may be divided into two parts, one $\frac{h}{h''}$ of the whole, balanced by elasticity alone, and the other $\frac{h}{h'}$, balanced by gravity alone; and therefore $\frac{h}{h''}+\frac{h}{h'}=1$.

But, by § 53 of the following mathematical investigation regarding elastic spheroids, we have $h''=\frac{3}{2}\frac{\mathrm{m}}{c^3}\frac{5w}{19n}r^3$, where m denotes the mass of the disturbing body, and c its distance from the earth's centre. With the same notation we have, by the aid of § 51 of the same paper, $\mathrm{H}=\frac{3}{2}\frac{\mathrm{m}}{c^3}\frac{r^2}{g}$, where H has the meaning defined above in § 8 of the present paper; and therefore, § 10, $h'=\frac{5}{2}\cdot\frac{3}{2}\frac{\mathrm{m}}{c^3}\frac{r^2}{g}$. From this and the value above for h'', we have $\frac{h'}{h''}=\frac{19n}{2gwr}$, and, as we have just seen that $h=\frac{h'}{1+\frac{h'}{h''}}$, we have the result used in § 4.

XXVIII. *Dynamical Problems regarding Elastic Spheroidal Shells and Spheroids of Incompressible Liquid.* *By Professor* W. Thomson, *LL.D., F.R.S.*

Received August 22,—Read November 27, 1862.

1. The theory of elastic solids in equilibrium presents the following general problem:—

A solid of any shape being given, and displacements being arbitrarily produced or forces arbitrarily applied over its whole bounding surface, it is required to find the displacement of every point of its substance. The chief object of the present communication is to show the solution of this problem for the case of a shell consisting of isotropic elastic material, and bounded by two concentric spherical surfaces, with the natural restriction that the whole alteration of figure is very small.

2. Let the centre of the spherical surfaces be taken as origin, and let x, y, z be the rectangular coordinates of any particle of the solid, in its undisturbed position, and $x+\alpha$, $y+\beta$, $z+\gamma$ the coordinates of the same particle when the whole is in equilibrium under the given superficial disturbing action. Then, by the known equations of equilibrium of elastic solids, we have

$$\left.\begin{aligned} n\left(\frac{d^2\alpha}{dx^2}+\frac{d^2\alpha}{dy^2}+\frac{d^2\alpha}{dz^2}\right)+m\frac{d}{dx}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)=0,\\ n\left(\frac{d^2\beta}{dx^2}+\frac{d^2\beta}{dy^2}+\frac{d^2\beta}{dz^2}\right)+m\frac{d}{dy}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)=0,\\ n\left(\frac{d^2\gamma}{dx^2}+\frac{d^2\gamma}{dy^2}+\frac{d^2\gamma}{dz^2}\right)+m\frac{d}{dz}\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)=0, \end{aligned}\right\} \quad \ldots \ldots \quad (1)$$

$m-\frac{1}{3}n$ and n denoting the two coefficients of elasticity, which may be called respectively the *elasticity of volume*, and the *rigidity*. A demonstration of these equations, with definitions of the coefficients, will be found in § 71 of an Appendix to the present communication.

3. For brevity let

$$\delta=\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}, \quad \ldots \ldots \quad (2)$$

so that δ shall denote the cubic dilatation at the point (x, y, z) of the solid. Also, for brevity, let the operation $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$ be denoted by ∇^2. Then the preceding equations become

$$\left.\begin{aligned} n\nabla^2\alpha+m\frac{d\delta}{dx}=0,\\ n\nabla^2\beta+m\frac{d\delta}{dy}=0,\\ n\nabla^2\gamma+m\frac{d\delta}{dz}=0. \end{aligned}\right\} \quad \ldots \ldots \quad (3)$$

4. In certain cases, especially the ideal one of an incompressible elastic solid, the following notation is more convenient:—

p the mean normal pressure per unit of area on all sides of any small portion of the solid, round the point x, y, z. Then (below, § 21)

$$p=-(m-\tfrac{1}{3}n)\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right); \quad . \; . \; . \; . \; . \; . \quad (4)$$

and the equations of equilibrium become

$$\left.\begin{aligned} n\nabla^2\alpha-\frac{m}{m-\frac{1}{3}n}\frac{dp}{dx}&=0,\\ n\nabla^2\beta-\frac{m}{m-\frac{1}{3}n}\frac{dp}{dy}&=0,\\ n\nabla^2\gamma-\frac{m}{m-\frac{1}{3}n}\frac{dp}{dz}&=0.\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

5. If the solid were incompressible, we should have $m=\infty$ and

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0,$$

which must be taken instead of (4), and, along with (5), would constitute the four differential equations required for the four unknown functions α, β, γ, p[*].

6. To solve the general equations (3) or (5), take $\frac{d}{dx}$ of the first, $\frac{d}{dy}$ of the second, and $\frac{d}{dz}$ of the third; and add. We have thus

$$(n+m)\nabla^2\delta=0, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

or, which is in general sufficient,

$$\nabla^2\delta=0. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

If, now, an appropriate solution of this equation for δ is found, the three equations (3) may be solved by known methods, the first of them for α, the second for β, and the third for γ,—the arbitrary part of the solution in each case being merely a solution of the equation $\nabla^2u=0$. These arbitrary parts must be determined so as to fulfil equation (2) and the prescribed surface conditions.

The complete particular determination of δ cannot, however, in most cases be effected without regard to α, β, γ; and the order of procedure which has been indicated is only convenient for determining the proper forms for general solutions of the equations.

7. First, then, to solve the equation in δ generally, we may use a theorem belonging to the foundation of LAPLACE's remarkable analysis of the attraction of spheroids, which may be enunciated as follows.

If the equation $\nabla^2\delta=0$ is satisfied for every point between two concentric spheres of radii a (greater) and a' (less), the value of δ for any point of this space, at distance r from the centre, may be expressed by the double series

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[*] See Professor STOKES's paper "On the Friction of Fluids in Motion, and the Equilibrium and Motion of Elastic Solids," Cambridge Philosophical Society's Transactions, April, 1845.

$$\begin{array}{l} V_0 \;\; + V_1 \;\; + V_2 \;\; + \&c. \\ + V'_0 r^{-1} + V'_1 r^{-3} + V'_2 r^{-5} + \&c., \end{array}$$

of which the first part converges at least as rapidly as the geometrical progression

$$\frac{r}{a}, \left(\frac{r}{a}\right)^2, \left(\frac{r}{a}\right)^3, \ldots$$

and the second at least as rapidly as

$$\frac{a'}{r}, \left(\frac{a'}{r}\right)^2, \left(\frac{a'}{r}\right)^3, \ldots,$$

—if V_i, V'_i denote homogeneous functions of x, y, z of the order i, each satisfying, continuously, for all values of x, y, z, the equation

$$\nabla^2 V = 0.$$

A proof of this proposition is given in THOMSON and TAIT'S 'Natural Philosophy,' chap. i. Appendix B. It is also there shown, what I believe has been hitherto overlooked, that V_i, V'_i, as above defined, cannot but be rational and integral, if i is any positive integer.

8. To avoid circumlocution, we shall call any homogeneous function of (x, y, z) which satisfies the equation

$$\nabla^2 V = 0$$

a "spherical harmonic function," or, more shortly, a "spherical harmonic." Thus V_i and V'_i, as defined in § 7, are spherical harmonics of degree or order i; and $V'_i r^{-2i-1}$, being also a solution of $\nabla^2 V = 0$, is a spherical harmonic of degree $-(i+1)$. We shall sometimes call the latter a spherical harmonic of inverse order i. Thus u_i being any spherical harmonic of integral degree i, and therefore necessarily a rational integral function of this degree, $u_i r^{-2i-1}$ is a spherical harmonic of degree $-(i+1)$, or of inverse order i.

If we put $-(i+1) = j$, and denote this last function by φ_j, then we have

$$\varphi_j r^{-2j-1} = u_i;$$

and thus it appears that the relation between a spherical harmonic of positive degree i and of negative degree j is reciprocal. The general (well known) proposition on which this depends is that if V_i is any homogeneous function of (x, y, z) of degree i, positive or negative, integral or fractional, $V_i r^{-2i-1}$ is also a solution of the equation $\nabla^2 V = 0$ (see THOMSON and TAIT'S 'Natural Philosophy,' chap. i. Appendix B.).

A spherical harmonic of integral whether positive or negative degree, satisfying the differential equation continuously for all values of the variables, will be called an "entire spherical harmonic," because such functions are suited for the solution of acoustical and other physical problems regarding entire spheres or entire spherical shells.

A spherical harmonic function of (x, y, z) will be called a "spherical surface-harmonic" when the point (x, y, z) lies anywhere on a spherical surface having its centre at the origin of coordinates. A *spherical surface-harmonic* is therefore a function of two variables, angular coordinates of a point on a spherical surface. If Y_i denote such a function of order i, positive and integral, then $Y_i r^i$ and $Y_i r^{-i-1}$ are what we now call simply *spherical harmonics*; but sometimes we shall call them, by way of distinction, "spherical solid harmonics." Functions Y_i, or spherical surface-harmonics of integral orders, have been generally called "LAPLACE'S coefficients" by English writers.

9. From the theorem enunciated in § 7, we see that the general solution of our problem, so far as δ is concerned, is this:—

$$\delta=\Sigma_{i=0}^{i=\infty}(V_i+V'_i r^{-2i-1}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

10. Now because the equation $\nabla^2 u=0$ is linear, it follows that differential coefficients of any solution, with reference to x, y, z, or linear functions of such differential coefficients, are also solutions. Hence the terms V_i and $V'_i r^{-2i-1}$, of δ, give harmonics of the degrees $i-1$ and $-(i+2)$, in $\frac{d\delta}{dx}$, $\frac{d\delta}{dy}$, $\frac{d\delta}{dz}$. To solve equations (3) we have therefore only to solve

$$\nabla^2 u=\phi_n,$$

where ϕ_n denotes an entire spherical harmonic of any positive or negative degree, n. Trying

$$u=Ar^2\phi_n,$$

which is obviously the right form, we have

$$\nabla^2 u=A\left\{r^2\nabla^2\phi_n+4\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)\phi_n+\phi_n\nabla^2(r^2)\right\}.$$

But, because ϕ_n is a homogeneous function of x, y, z of degree n,

$$\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)\phi_n=n\phi_n;$$

and because it is a spherical harmonic,

$$\nabla^2\phi_n=0.$$

We have also

$$\nabla^2(r^2)=6,$$

by differentiation. Hence

$$\nabla^2 u=A.2(2n+3)\phi_n,$$

and therefore the complete solution of the equation

$$\nabla^2 u=\phi_n$$

is

$$u=V+\frac{r^2}{2(2n+3)}\phi_n,$$

where V denotes any solution of the equation

$$\nabla^2 V=0.$$

11. Hence, by taking for ϕ_n the terms of $\frac{d\delta}{dx}$, $\frac{d\delta}{dy}$, $\frac{d\delta}{dz}$ above referred to (§ 10), and giving n its proper value, $i-1$, or $-(i+2)$, for each term as the case may be, we find, for the complete solution of (3), the following:—

$$\left.\begin{aligned}\alpha&=\Sigma\left\{u_i+u'_i r^{-2i-1}-\frac{mr^2}{n\,.\,2(2i+1)}\frac{d}{dx}(\mathrm{V}_i-\mathrm{V}'_i r^{-2i-1})\right\},\\ \beta&=\Sigma\left\{v_i+v'_i r^{-2i-1}-\frac{mr^2}{n\,.\,2(2i+1)}\frac{d}{dy}(\mathrm{V}_i-\mathrm{V}'_i r^{-2i-1})\right\},\\ \gamma&=\Sigma\left\{w_i+w'_i r^{-2i-1}-\frac{mr^2}{n\,.\,2(2i+1)}\frac{d}{dz}(\mathrm{V}_i-\mathrm{V}'_i r^{-2i-1})\right\},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where u_i, u'_i, v_i, v'_i, w_i, w'_i denote six harmonics, each of degree i.

12. But in order that these formulæ may express the solution of the original equations (1), the functions u, v, &c. must be related to the functions V so as to satisfy equations (2) and (3). Now, taking account of the following formula,

$$\frac{d}{dx}\left(r^2\frac{d\phi_i}{dx}\right)+\frac{d}{dy}\left(r^2\frac{d\phi_i}{dy}\right)+\frac{d}{dz}\left(r^2\frac{d\phi_i}{dz}\right)=2\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)\phi_i+r^2\nabla^2\phi_i\,,$$

which becomes simply

$$2i\phi_i\,,$$

if ϕ_i is a spherical harmonic of any degree i (whether positive or negative, integral or fractional), we derive from (9) by differentiation, and selection of terms of order i, and of order inverse i (or degree $-i-1$),

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=\Sigma\left\{\psi_i+\psi'_i r^{-2i-1}-\frac{m}{n(2i+1)}\left[i\mathrm{V}_i+(i+1)\mathrm{V}'_i r^{-2i-1}\right]\right\},$$

where, for brevity, we put

$$\left.\begin{aligned}&\psi_i=\frac{du_{i+1}}{dx}+\frac{dv_{i+1}}{dy}+\frac{dw_{i+1}}{dz}\,,\\ \text{and}\qquad&\psi'_i r^{-2i-1}=\frac{d(u'_{i-1}r^{-2i+1})}{dx}+\frac{d(v'_{i-1}r^{-2i+1})}{dy}+\frac{d(w'_{i-1}r^{-2i+1})}{dz}\,.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Hence, to satisfy (2) and (8),

$$\mathrm{V}_i=\psi_i-\frac{mi}{n(2i+1)}\mathrm{V}_i$$

and

$$\mathrm{V}'_i=\psi'_i-\frac{m(i+1)}{n(2i+1)}\mathrm{V}'_i\,,$$

from which we find

$$\left.\begin{aligned}\mathrm{V}_i&=\frac{n(2i+1)}{(2n+m)i+n}\psi_i\,,\\ \mathrm{V}'_i&=\frac{n(2i+1)}{(2n+m)i+n+m}\psi'_i\,.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

13. Using these in (9), we conclude

$$\left.\begin{aligned}\alpha&=\Sigma_{i=0}^{i=\infty}{}^{*}\left\{u_{i+1}+u'_{i-1}r^{-2i+1}-\frac{mr^2}{2}\frac{d}{dx}\left[\frac{\psi_i}{(2n+m)i+n}-\frac{\psi'_i r^{-2i-1}}{(2n+m)i+n+m}\right]\right\},\\ \beta&=\Sigma_{i=0}^{i=\infty}\left\{v_{i+1}+v'_{i-1}r^{-2i+1}-\frac{mr^2}{2}\frac{d}{dy}\left[\frac{\psi_i}{(2n+m)i+n}-\frac{\psi'_i r^{-2i-1}}{(2n+m)i+n+m}\right]\right\},\\ \gamma&=\Sigma_{i=0}^{i=\infty}\left\{w_{i+1}+w'_{i-1}r^{-2i+1}-\frac{mr^2}{2}\frac{d}{dz}\left[\frac{\psi_i}{(2n+m)i+n}-\frac{\psi'_i r^{-2i-1}}{(2n+m)i+n+m}\right]\right\}\end{aligned}\right\} \quad . \quad (12)$$

for a complete solution of the general equations (1), the equations of equilibrium of an isotropic elastic solid. The circumstances for which this solution is appropriate will be understood when the general proposition of § 7 is duly considered.

14. It remains to show how the harmonics u_i, v_i, w_i, u'_i, v'_i, w'_i are to be determined so as to satisfy the superficial conditions. Let us first suppose these to be that the displacement of every point of the bounding surface is given arbitrarily. Let ΣA_i, ΣB_i, ΣC_i be the harmonic series[†], expressing the three components of the displacement at any point of the outer surface of the shell, and $\Sigma A'_i$, $\Sigma B'_i$, $\Sigma C'_i$ the corresponding expressions for the given condition of the inner surface. Thus the surface-equations of condition to be fulfilled are

$$\left.\begin{aligned} r=a&\begin{cases}\alpha=\Sigma A_i,\\ \beta=\Sigma B_i,\\ \gamma=\Sigma C_i,\end{cases}\\ r=a'&\begin{cases}\alpha=\Sigma A'_i,\\ \beta=\Sigma B'_i,\\ \gamma=\Sigma C'_i,\end{cases}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where a and a' denote the radii of the outer and inner surfaces respectively, and A_i, B_i, C_i, A'_i, B'_i, C'_i spherical surface-harmonics of the order i.

15. Now collecting from the series (12) of § 13, which constitute the general expressions for α, β, γ, those terms which, being either solid spherical harmonics of degrees i and $-i-1$, or such functions multiplied by r^2, give, at the boundary, surface-harmonics of the order i, and equating the terms of this order on the two sides of equations (13), we have

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[*] For the case $i=0$, the terms u'_{i-1}, v'_{i-1}, w'_{i-1} may be omitted; but their full interpretation would be to express a displacement without deformation. Thus u'_{-1}, being of degree -1, cannot but be $\frac{A}{r}$, where A is a constant; and therefore $u'_{i-1}r^{-2i+1}$ becomes A when $i=0$.

[†] That is, series of terms each of which is a spherical surface-harmonic of integral order i. That any function, arbitrarily given over an entire spherical surface, may be so expressed, is a well-known theorem. A demonstration of it is given in THOMSON and TAIT's 'Natural Philosophy,' chap. i. Appendix B, § s.

$$\left.\begin{aligned}
&u_i+u_i'r^{-2i-1}-\frac{mr^2}{2}\frac{d}{dx}\left[\frac{\psi_{i+1}}{(2n+m)i+3n+m}-\frac{\psi_{i-1}'r^{-2i+1}}{(2n+m)i-n}\right]\begin{cases}=\mathrm{A}_i \text{ when } r=a,\\ =\mathrm{A}_i' \text{ when } r=a',\end{cases}\\
&v_i+v_i'r^{-2i-1}-\frac{mr^2}{2}\frac{d}{dy}\left[\frac{\psi_{i+1}}{(2n+m)i+3n+m}-\frac{\psi_{i-1}'r^{-2i+1}}{(2n+m)i-n}\right]\begin{cases}=\mathrm{B}_i \text{ when } r=a,\\ =\mathrm{B}_i' \text{ when } r=a',\end{cases}\\
&w_i+w_i'r^{-2i-1}-\frac{mr^2}{2}\frac{d}{dz}\left[\frac{\psi_{i+1}}{(2n+m)i+3n+m}-\frac{\psi_{i-1}'r^{-2i+1}}{(2n+m)i-n}\right]\begin{cases}=\mathrm{C}_i \text{ when } r=a,\\ =\mathrm{C}_i' \text{ when } r=a'.\end{cases}
\end{aligned}\right\}\quad . \quad . \quad (15)$$

16. These six equations would suffice to determine the six harmonics u_i, v_i, w_i, u_i', v_i', w_i', if ψ_{i+1} and ψ_{i-1}' were known. For, since each of those six functions is a homogeneous function of x, y, z of order i, each of them divided by r^i is a function of angular coordinates relative to the centre, and independent of r; and therefore if, for instance, we denote u_i by $r^i\varpi$ and u_i' by $r^i\varpi'$, we have two unknown quantities ϖ and ϖ' to be determined by the two equations of condition relative to α for the outer and the inner surface. These equations may be written as follows, if we further denote $\frac{d\psi_{i+1}}{dx}$ by $r^i\vartheta$, and $\frac{d(\psi_{i-1}'r^{-2i+1})}{dx}$ by $r^{-i-1}\vartheta'$, because these are homogeneous functions of the orders i and $-i-1$ respectively:

$$\varpi a^{2i+1}+\varpi'=\mathrm{A}_i a^{i+1}+\frac{ma^{2i+3}}{2[(2n+m)i+3n+m]}\vartheta-\frac{ma^2}{2[(2n+m)i-n]}\vartheta',$$

$$\varpi a'^{2i+1}+\varpi'=\mathrm{A}_i' a'^{i+1}+\frac{ma'^{2i+3}}{2[(2n+m)i+3n+m]}\vartheta-\frac{ma'^2}{2[(2n+m)i-n]}\vartheta'.$$

Resolving these equations for ϖ and ϖ', and returning to the original notation instead of ϖ, ϖ', ϑ, ϑ',

$$u_i=\frac{(a^{i+1}\mathrm{A}_i-a'^{i+1}\mathrm{A}_i')r^i+(a^{2i+3}-a'^{2i+3})\mathrm{M}_{i+2}\frac{d\psi_{i+1}}{dx}-(a^2-a'^2)\mathrm{M}_{i-2}'\frac{d(\psi_{i-1}'r^{-2i+1})}{dx}r^{2i+1}}{a^{2i+1}-a'^{2i+1}},$$

$$u_i'=\frac{(aa')^{i+1}(a^i\mathrm{A}_i'-a'^i\mathrm{A}_i)r^i-(aa')^{2i+1}(a^2-a'^2)\mathrm{M}_{i+2}\frac{d\psi_{i+1}}{dx}-(aa')^2(a^{2i-1}-a'^{2i-1})\mathrm{M}_{i-2}'\frac{d(\psi_{i-1}'r^{-2i+1})}{dx}r^{2i+1}}{a^{2i+1}-a'^{2i+1}},$$

where, for brevity,

$$\left.\begin{aligned}
\mathrm{M}_{i+2}&=\frac{m}{2[(2n+m)i+3n+m]},\\
\mathrm{M}_{i-2}'&=\frac{m}{2[(2n+m)i-n]}.
\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Introducing, also for brevity, the following notation,

$$\left.\begin{aligned}
\mathfrak{A}_i&=\frac{a^{i+1}\mathrm{A}_i-a'^{i+1}\mathrm{A}_i'}{a^{2i+1}-a'^{2i+1}},\\
\mathfrak{A}_i'&=\frac{(aa')^{i+1}(a^i\mathrm{A}_i'-a'^i\mathrm{A}_i)}{a^{2i+1}-a'^{2i+1}},
\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\left.\begin{aligned}
&\mathfrak{M}_{i+2}=\frac{a^{2i+3}-a'^{2i+3}}{a^{2i+1}-a'^{2i+1}}\mathrm{M}_{i+2}, &&\mathfrak{M}'_{i-2}=\frac{a^2-a'^2}{a^{2i+1}-a'^{2i+1}}\mathrm{M}'_{i-2},\\
&\mathfrak{N}_{i+2}=\frac{(aa')^{2i+1}(a^2-a'^2)}{a^{2i+1}-a'^{2i+1}}\mathrm{M}_{i+2}, &&\mathfrak{N}_{i-2}'=\frac{(aa')^2(a^{2i-1}-a'^{2i-1})}{a^{2i+1}-a'^{2i+1}}\mathrm{M}_{i-2}',
\end{aligned}\right\}\quad . \quad . \quad . \quad (18)$$

we have the expressions for u_i and u'_i given below. Dealing with the equations of condition relative to β and γ, and introducing an abbreviated notation B_i, B'_i, C_i, C'_i corresponding to (17), we find similar expressions for v_i, v'_i, w_i, w'_i, as follows:—

$$\left.\begin{aligned} u_i &= \mathfrak{A}_i r^i + \mathfrak{M}_{i+2}\frac{d\psi_{i+1}}{dx} - \mathfrak{M}'_{i-2}\frac{d(\psi'_{i-1}r^{-2i+1})}{dx}r^{2i+1}, \\ v_i &= \mathfrak{B}_i r^i + \mathfrak{M}_{i+2}\frac{d\psi_{i+1}}{dy} - \mathfrak{M}'_{i-2}\frac{d(\psi'_{i-1}r^{-2i+1})}{dy}r^{2i+1}, \\ w_i &= \mathfrak{C}_i r^i + \mathfrak{M}_{i+2}\frac{d\psi_{i+1}}{dz} - \mathfrak{M}'_{i-2}\frac{d(\psi'_{i-1}r^{-2i+1})}{dz}r^{2i+1}, \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (19)$$

$$\left.\begin{aligned} u'_i &= \mathfrak{A}'_i r^i - \mathfrak{N}_{i+2}\frac{d\psi_{i+1}}{dx} - \mathfrak{N}'_{i-2}\frac{d(\psi'_{i-1}r^{-2i+1})}{dx}r^{2i+1}, \\ v'_i &= \mathfrak{B}'_i r^i - \mathfrak{N}_{i+2}\frac{d\psi_{i+1}}{dy} - \mathfrak{N}'_{i-2}\frac{d(\psi'_{i-1}r^{-2i+1})}{dy}r^{2i+1}, \\ w'_i &= \mathfrak{C}'_i r^i - \mathfrak{N}_{i+2}\frac{d\psi_{i+1}}{dz} - \mathfrak{N}'_{i-2}\frac{d(\psi'_{i-1}r^{-2i+1})}{dz}r^{2i+1}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (20)$$

17. It only remains to determine the functions ψ and ψ', which we can do by combining these last equations with (10) of § 12. Thus, changing i into $i+1$ in (17) and into $i-1$ in (18), applying equations (10) of § 12, and taking advantage of the following properties,

$$\nabla^2\psi_{i+2}=0,\ \nabla^2(\psi'_i r^{-2i-1})=0, \text{ \&c.},$$

$$x\frac{d(\psi'_i r^{-2i-1})}{dx}+y\frac{d(\psi'_i r^{-2i-1})}{dy}+z\frac{d(\psi'_i r^{-2i-1})}{dz}=-(i+1)\psi'_i,$$

and

$$x\frac{d\psi_i}{dx}+y\frac{d\psi_i}{dy}+z\frac{d\psi_i}{dz}=i\psi_i,$$

we find

$$\left.\begin{aligned} \psi_i &= \frac{d(\mathfrak{A}_{i+1}r^{i+1})}{dx}+\frac{d(\mathfrak{B}_{i+1}r^{i+1})}{dy}+\frac{d(\mathfrak{C}_{i+1}r^{i+1})}{dz}+(2i+3)(i+1)\mathfrak{M}'_{i-1}\psi'_i, \\ \psi'_i &= \left\{\frac{d(\mathfrak{A}'_{i-1}r^{-i})}{dx}+\frac{d(\mathfrak{B}'_{i-1}r^{-i})}{dy}+\frac{d(\mathfrak{C}'_{i-1}r^{-i})}{dz}\right\}r^{2i+1}+(2i-1)i\mathfrak{N}_{i+1}\psi_i. \end{aligned}\right\} \quad . \; . \; . \quad (21)$$

These equations, used to determine the two unknown functions ψ_i and ψ'_i, give

$$\left.\begin{aligned} \psi_i &= \frac{\Theta_i+(2i+3)(i+1)\mathfrak{M}'_{i+1}\Theta'_i}{1-(2i+3)(2i-1)(i+1)i\mathfrak{M}_{i+1}\mathfrak{N}_{i+1}}, \\ \psi'_i &= \frac{(2i-1)i\mathfrak{N}_{i+1}\Theta_i+\Theta'_i}{1-(2i+3)(2i-1)(i+1)i\mathfrak{M}'_{i+1}\mathfrak{N}_{i+1}}, \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

where, for brevity,

$$\left.\begin{aligned} \Theta_i &= \frac{d(\mathfrak{A}_{i+1}r^{i+1})}{dx}+\frac{d(\mathfrak{B}_{i+1}r^{i+1})}{dy}+\frac{d(\mathfrak{C}_{i+1}r^{i+1})}{dz}, \\ \Theta'_i &= \left\{\frac{d(\mathfrak{A}'_{i-1}r^{-i})}{dx}+\frac{d(\mathfrak{B}'_{i-1}r^{-i})}{dy}+\frac{d(\mathfrak{C}'_{i-1}r^{-i})}{dz}\right\}r^{2i+1}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (23)$$

18. The functions ψ and ψ' being expressed in terms of the data of the problem by equations (22), (23), (17), (18), (16), we have only to use (19) and (20) in (12) to find the following expression of the complete solution:—

$$\left.\begin{aligned}\alpha&=\Sigma\left\{\mathfrak{A}_i r^i+\mathfrak{A}'_i r^{-i-1}+(\mathfrak{M}_i-\mathfrak{N}_i r^{-2i+3}-\mathrm{M}_i r^2)\frac{d\psi_{i-1}}{dx}-(\mathfrak{M}'_i r^{2i+5}+\mathfrak{N}'_i-\mathrm{M}'_i r^2)\frac{d(\psi'_{i+1}r^{-2i-3})}{dx}\right\},\\ \beta&=\Sigma\left\{\mathfrak{B}_i r^i+\mathfrak{B}'_i r^{-i-1}+(\mathfrak{M}_i-\mathfrak{N}_i r^{-2i+3}-\mathrm{M}_i r^2)\frac{d\psi_{i-1}}{dy}-(\mathfrak{M}'_i r^{2i+5}+\mathfrak{N}'_i-\mathrm{M}'_i r^2)\frac{d(\psi'_{i+1}r^{-2i-3})}{dy}\right\},\\ \gamma&=\Sigma\left\{\mathfrak{C}_i r^i+\mathfrak{C}'_i r^{-i-1}+(\mathfrak{M}_i-\mathfrak{N}_i r^{-2i+3}-\mathrm{M}_i r^2)\frac{d\psi_{i-1}}{dz}-(\mathfrak{M}'_i r^{2i+5}+\mathfrak{N}'_i-\mathrm{M}'_i r^2)\frac{d(\psi'_{i+1}r^{-2i-3})}{dz}\right\},\end{aligned}\right\} \quad (24)$$

19. This solution leads immediately, through an extreme case of its application, to the solution of the general problem for a plate of elastic substance between two infinite parallel planes:—Given the displacement of every point of its surface, required the displacement of any interior point. For if we give infinite values to a and a', and keep $a-a'$ finite, the spherical shell becomes an infinite plane plate.

20. It is, however, less easy to deduce the result in this way from the solution for the spherical shell, than to apply directly the general method of § 6 to the case of the infinite plane plate. We shall return to this subject (§ 31, below), when the details of the investigation will be sufficiently indicated.

21. A very important part of the general problem proposed in § 1 remains to be considered,—that in which not the displacement, but the arbitrarily applied force, is given all over the surface. To express the surface-equations of condition for such data, we must use the formulæ expressing the stress (or force of elasticity) in any part of an elastic solid in terms of the strain (or deformation) of the substance. These are

$$\left.\begin{aligned}&\mathrm{P}=(m+n)\frac{d\alpha}{dx}+(m-n)\left(\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right);\\ &\mathrm{Q}=(m+n)\frac{d\beta}{dy}+(m-n)\left(\frac{d\gamma}{dz}+\frac{d\alpha}{dx}\right);\\ &\mathrm{R}=(m+n)\frac{d\gamma}{dz}+(m-n)\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}\right);\\ &\mathrm{S}=n\left(\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right);\ \mathrm{T}=n\left(\frac{d\gamma}{dx}+\frac{d\alpha}{dz}\right);\ \mathrm{U}=n\left(\frac{d\alpha}{dy}+\frac{d\beta}{dx}\right);\end{aligned}\right\} \quad \ldots\ldots \quad (25)$$

where P, Q, R are the normal tractions (which when negative are pressures) on the faces of a unit cube respectively perpendicular to the lines of reference OX, OY, OZ, and S, T, U the tangential forces along the faces respectively parallel, and in the directions in these planes respectively perpendicular, to OX, OY, OZ (see Appendix, § 70).

22. In terms of these we have the following expressions for the components F, G, H of the force on a unit area perpendicular to any line whose direction cosines are f, g, h:—

$$\left.\begin{aligned}\mathrm{F}&=\mathrm{P}f+\mathrm{U}g+\mathrm{T}h,\\ \mathrm{G}&=\mathrm{U}f+\mathrm{Q}g+\mathrm{S}h,\\ \mathrm{H}&=\mathrm{T}f+\mathrm{S}g+\mathrm{R}h\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (26)$$

(see "Elements of a Mathematical Theory of Elasticity," Philosophical Transactions for 1856, p. 481).

23. Using the expressions (25) in (26), we find

$$F=(m-n)\left(\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right)f+n\left(\frac{d\alpha}{dx}f+\frac{d\alpha}{dy}g+\frac{d\alpha}{dz}h\right)+n\left(\frac{d\alpha}{dx}f+\frac{d\beta}{dx}g+\frac{d\gamma}{dx}h\right), \quad . \quad (27)$$

and symmetrical expressions for G and H.

24. If now we suppose f, g, h to denote the direction-cosines of the normal at any point x, y, z of the surface of an elastic solid, the surface condition, when force, not displacement, is given, will be expressed by equating F, G, H respectively to three functions of the coordinates of a point in the surface, quite arbitrary except in so far as they must balance one another in order that equilibrium in the body may be possible; and therefore they must fulfil the following integral equations:—

$$\iint F d\Omega=0,\quad \iint G d\Omega=0,\quad \iint H d\Omega=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$\iint (Hy-Gz)d\Omega=0,\quad \iint (Fz-Hx)d\Omega=0,\quad \iint (Gx-Fy)=0, \quad . \quad . \quad . \quad (29)$$

where $d\Omega$ denotes an element of the surface at the point (x, y, z), and the double integrals include the whole surface of application of the forces F, G, H.

25. For our case of the spherical shell, with origin of coordinates at its centre, we have

$$f=\frac{x}{r},\quad g=\frac{y}{r},\quad h=\frac{z}{r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and the last triple term in the expression (27) for F may be conveniently written thus:—

$$\frac{n}{r}\frac{d(\alpha x+\beta y+\gamma z)}{dx}-\frac{n\alpha}{r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Then, for brevity, putting

$$\alpha x+\beta y+\gamma z=\zeta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

and

$$x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}=r\frac{\mathrm{d}}{dr}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where $\frac{\mathrm{d}}{dr}$ prefixed to any function of x, y, z will denote its rate of variation per unit of length in the radial direction; and using (2) of § 3, we have, by (30) and the symmetrical equations for G and H,

$$\left.\begin{aligned} Fr&=(m-n)\delta.x+n\left\{\left(r\frac{\mathrm{d}}{dr}-1\right)\alpha+\frac{d\zeta}{dx}\right\},\\ Gr&=(m-n)\delta.y+n\left\{\left(r\frac{\mathrm{d}}{dr}-1\right)\beta+\frac{d\zeta}{dy}\right\},\\ Hr&=(m-n)\delta.z+n\left\{\left(r\frac{\mathrm{d}}{dr}-1\right)\gamma+\frac{d\zeta}{dz}\right\}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

26. It is to be remarked that these equations express such functions of (x, y, z), the coordinates of any point P of the solid, that $F.\varpi$, $G.\varpi$, $H.\varpi$ are the three components of

the force transmitted across an infinitely small area ω perpendicular to OP, while, for any point of either the outer or the inner bounding spherical surface, $F\omega$, $G\omega$, $H\omega$ are the three components of the force applied to an infinitely small element ω of this surface.

27. To reduce the surface-equations of condition derived from these expressions to harmonic equations, let us consider homogeneous terms of degree i of the complete solution, which we shall denote by α_i, β_i, γ_i, and let δ_{i-1}[*], ζ_{i+1} denote the corresponding terms of the other functions. Thus we have

$$\left.\begin{aligned} Fr&=\Sigma\left\{(m-n)\delta_{i-1}x+n(i-1)\alpha_i+n\frac{d\zeta_{i+1}}{dx}\right\},\\ Gr&=\Sigma\left\{(m-n)\delta_{i-1}y+n(i-1)\beta_i+n\frac{d\zeta_{i+1}}{dy}\right\},\\ Hr&=\Sigma\left\{(m-n)\delta_{i-1}z+n(i-1)\gamma_i+n\frac{d\zeta_{i+1}}{dz}\right\}.\end{aligned}\right\} \tag{35}$$

28. The second of the three terms of order i in these equations, when the general solution of § 13 is used, become at the boundary each explicitly the sum of two surface harmonics of orders i and $i-2$ respectively. To bring the other parts of the expressions to similar forms, it is convenient that we should first express ζ_{i+1} in terms of the general solution (12) of § 13, by selecting the terms of algebraic degree i. Thus we have

$$\alpha_i=u_i-\frac{mr^2}{2[(2n+m)i-n-m]}\frac{d\psi_{i-1}}{dx}, \tag{36}$$

and symmetrical expressions for β_i and γ_i, from which we find

$$\alpha_i x+\beta_i y+\gamma_i z=\zeta_{i+1}=u_i x+v_i y+w_i z-\frac{(i-1)mr^2\psi_{i-1}}{2[(2n+m)i-n-m]}.$$

Hence, by the proper formulæ [see (42) below] for reduction to harmonics,

$$\zeta_{i+1}=-\frac{1}{2i+1}\left\{\frac{(2i-1)[m(i-1)-2n]}{2[(2n+m)i-n-m]}r^2\psi_{i-1}+\varphi_{i+1}\right\}, \tag{37}$$

where

$$\varphi_{i+1}=r^{2i+3}\left\{\frac{d(u_i r^{-2i-1})}{dx}+\frac{d(v_i r^{-2i-1})}{dy}+\frac{d(w_i r^{-2i-1})}{dz}\right\}, \tag{38}$$

and (as before assumed in § 12)

$$\psi_{i-1}=\frac{du_i}{dx}+\frac{dv_i}{dy}+\frac{dw_i}{dz}. \tag{39}$$

Also, by (11) of § 12, or directly from (36) by differentiation, we have

$$\delta_{i-1}=\frac{n(2i-1)}{(2n+m)i-n-m}.\psi_{i-1}. \tag{40}$$

Substituting these expressions for δ_{i-1}, α_i, and ζ_{i+1} in (35), we find

---

[*] When $i-1$ is positive, δ_{i-1} will express the same function as V_{i-1} of § 9 above. The suffixes now introduced have reference solely to the algebraic degree, positive or negative, of the functions, whether harmonic or not, to the symbols for which they are applied.

$$\mathrm{F}r=\Sigma\left\{n(i-1)u_i+\frac{n(2i-1)[(m-2n)i+2m+n]}{(2i+1)[(m+2n)i-m-n]}x\psi_{i-1}-\frac{n[2i(i-1)m-(2i-1)n]}{(2i+1)[(m+2n)i-m-n]}r^2\frac{d\psi_{i-1}}{dx}-\frac{n}{2i+1}\,\frac{d\phi_{i+1}}{dx}\right\}. \quad (41)$$

This is reduced to the required harmonic form by the obviously proper formula

$$x\psi_{i-1}=\frac{1}{2i-1}\left\{r^2\frac{d\psi_{i-1}}{dx}-r^{2i+1}\frac{d(\psi_{i-1}r^{-2i+1})}{dx}\right\} \quad . \ . \ . \ . \ . \ . \ . \quad (42)$$

Thus, and dealing similarly with the expressions for Gr and Hr, we have, finally,

$$\left.\begin{aligned}\mathrm{F}r&=n\Sigma\left\{(i-1)u_i-2(i-2)\mathrm{M}_i r^2\frac{d\psi_{i-1}}{dx}-\mathrm{E}_i r^{2i+1}\frac{d(\psi_{i-1}r^{-2i+1})}{dx}-\frac{1}{2i+1}\,\frac{d\phi_{i+1}}{dx}\right\},\\ \mathrm{G}r&=n\Sigma\left\{(i-1)v_i-2(i-2)\mathrm{M}_i r^2\frac{d\psi_{i-1}}{dy}-\mathrm{E}_i r^{2i+1}\frac{d(\psi_{i-1}r^{-2i+1})}{dx}-\frac{1}{2i+1}\,\frac{d\phi_{i+1}}{dy}\right\},\\ \mathrm{H}r&=n\Sigma\left\{(i-1)w_i-2(i-2)\mathrm{M}_i r^2\frac{d\psi_{i-1}}{dz}-\mathrm{E}_i r^{2i+1}\frac{d(\psi_{i-1}r^{-2i+1})}{dx}-\frac{1}{2i+1}\,\frac{d\phi_{i+1}}{dz}\right\},\end{aligned}\right\} \quad . \ . \quad (43)$$

where [as above, (16) of § 16.]

and now further

$$\left.\begin{aligned}\mathrm{M}_i&=\frac{1}{2}\,\frac{m}{(m+2n)i-m-n},\\ \mathrm{E}_i&=\frac{(m-2n)i+2m+n}{(2i+1)[(m+2n)i-m-n]}.\end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (44)$$

29. To express the surface conditions by harmonic equations, let us suppose the superficial values of F, G, H to be given as follows:

and

$$\left.\begin{aligned}&\left.\begin{aligned}\mathrm{F}&=\Sigma\mathrm{A}_i,\\ \mathrm{G}&=\Sigma\mathrm{B}_i,\\ \mathrm{H}&=\Sigma\mathrm{C}_i,\end{aligned}\right\}\text{when } r=a,\\ &\left.\begin{aligned}\mathrm{F}&=\Sigma\mathrm{A}'_i,\\ \mathrm{G}&=\Sigma\mathrm{B}'_i,\\ \mathrm{H}&=\Sigma\mathrm{C}'_i,\end{aligned}\right\}\text{when } r=a',\end{aligned}\right\}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (45)$$

where A_i, B_i, C_i, A'_i, B'_i, C'_i denote surface harmonics of order i. Now the terms of algebraic degree i, exhibited in the preceding expressions (43) for Fr, Gr, Hr, become, at either of the concentric spherical surfaces, sums of surface harmonics of orders i and $i-2$, when i is positive, and of orders $-i-1$ and $-i-3$ when i is negative. Hence, selecting all the terms which lead to surface harmonics of order i, and equating to the proper terms of the data (45), we have

$$\frac{n}{r}\left\{\begin{aligned}&(i-1)u_i-(i+2)u_{-i-1}-2i\mathrm{M}_{i+2}r^2\frac{d\psi_{i+1}}{dx}+2(i+1)\mathrm{M}_{-i+1}r^2\frac{d\psi_{-i}}{dx}\\ &\quad-\mathrm{E}_i r^{2i+1}\frac{d(\psi_{i-1}r^{-2i+1})}{dx}-\mathrm{E}_{-i-1}r^{-2i-1}\frac{d(\psi_{-i-2}r^{2i+3})}{dx}-\frac{1}{2i+1}\left(\frac{d\phi_{i+1}}{dx}-\frac{d\phi_{-i}}{dx}\right)\end{aligned}\right\}=\left\{\begin{aligned}&\mathrm{A}_i \text{ when } r=a,\\ &\mathrm{A}'_i \text{ when } r=a',\end{aligned}\right\} \quad (46)$$

and symmetrical equations relative to y and z.

30 These equations might be dealt with exactly as formerly with the equations (15)

of § 15. But the following order of proceeding is more convenient. Commencing with the first of the surface equations (46), multiplying it by $\left(\frac{r}{a}\right)^i$, attending to the degree of each term, and taking advantage of the principle that, if ψ be any homogeneous function of x, y, z, of degree t, the function of angular coordinates, or of the ratios $x:y:z$, which it becomes at the spherical surface $r=a$, is the same as $\left(\frac{a}{r}\right)^t\psi$ for any value of r, we have

$$\frac{n}{a}\left\{\begin{array}{l}(i-1)u_i-(i+2)\left(\frac{r}{a}\right)^{2i+1}u_{-i-1}-2i\mathrm{M}_{i+2}a^2\frac{d\psi_{i+1}}{dx}+2(i+1)\mathrm{M}_{-i+1}a^2\left(\frac{r}{a}\right)^{2i+1}\frac{d\psi_{-i}}{dx}\\ \quad-\mathrm{E}_ir^{2i+1}\frac{d(\psi_{i-1}r^{-2i+1})}{dx}-\mathrm{E}_{-i-1}a^{-2i-1}\frac{d(\psi_{-i-2}r^{2i+3})}{dx}-\frac{1}{2i+1}\left[\frac{d\phi_{i+1}}{dx}-\left(\frac{r}{a}\right)^{2i+1}\frac{d\phi_{-i}}{dx}\right]\end{array}\right\}=\mathrm{A}_i\left(\frac{r}{a}\right)^i, \quad (47)$$

where the second member, and each term of the first member, is now a homogeneous function of degree i, of x, y, z (being in fact a solid spherical harmonic of degree and order i). Taking $\frac{d}{dx}$ of this, and $\frac{d}{dy}$ and $\frac{d}{dz}$ of the two symmetrical equations, adding, taking into account equations (38) and (39), and taking advantage of the equation $\nabla^2\mathrm{V}=0$ for the solid harmonic functions concerned, we have

$$\left.\begin{array}{l}\frac{n}{a}\left\{[i-1+(2i+1)i\mathrm{E}_i]\psi_{i-1}-2(i+1)a^{-2i-1}r^{2i-1}\phi_{-i}-2(i+1)(2i+1)i\mathrm{M}_{-i+1}\left(\frac{r}{a}\right)^{2i+1}\psi_{-i}\right\}\\ \qquad=\frac{1}{a^i}\left\{\frac{d(\mathrm{A}_ir^i)}{dx}+\frac{d(\mathrm{B}_ir^i)}{dy}+\frac{d(\mathrm{C}_ir^i)}{dz}\right\}.\end{array}\right\} \quad (48)$$

Again, multiplying (47) by $a^{-2}r^{-2i-1}$, and taking $r^{2i+3}\frac{d}{dx}$ of the result, dealing similarly with the two symmetrical equations, and adding, we have

$$\left.\begin{array}{l}\frac{n}{a}\left\{2ia^{-2}\phi_{i+1}-[i+2-(2i+1)(i+1)\mathrm{E}_{-i-1}]\left(\frac{r}{a}\right)^{2i+3}\psi_{-i-2}+2i(i+1)(2i+1)\mathrm{M}_{i+2}\psi_{i+1}\right\}\\ \qquad=\frac{r^{2i+3}}{a^{i+2}}\left\{\frac{d(\mathrm{A}_ir^{-i-1})}{dx}+\frac{d(\mathrm{B}_ir^{-i-1})}{dy}+\frac{d(\mathrm{C}_ir^{-i-1})}{dz}\right\}.\end{array}\right\} \quad (49)$$

Changing i into $i-2$ in this equation, we have

$$\left.\begin{array}{l}\frac{n}{a}\left\{2(i-2)a^{-2}\phi_{i-1}-[i-(2i-3)(i-1)\mathrm{E}_{-i+1}]\left(\frac{r}{a}\right)^{2i-1}\psi_{-i}+2(i-2)(i-1)(2i-3)\mathrm{M}_i\psi_{i-1}\right\}\\ \qquad=\frac{r^{2i-1}}{a^i}\left\{\frac{d(\mathrm{A}_{i-2}r^{-i+1})}{dx}+\frac{d(\mathrm{B}_{i-2}r^{-i+1})}{dy}+\frac{d(\mathrm{C}_{i-2}r^{-i+1})}{dz}\right\}.\end{array}\right\} \quad (50)$$

Precisely similar equations, derived from the inner surface condition of the shell, are obtained by changing a, A, B, C into a', A′, B′, C′. We thus have (48), (50), and the two corresponding equations for the inner surface, in all four equations, to determine the four unknown functions ψ_{i-1}, ψ_{-i}, ϕ_{i-1}, ϕ_{-i}, in terms of the data which appear in the second members. The equations being simple algebraic equations, we may regard these four functions as explicitly determined. In other words, we may suppose ϕ_i and ψ_i known for every positive or negative integral value of i. Then equation (47), the two equations symmetrical with it, and the others got by changing A, a, &c. into A$'$, a', &c., give u_i, v_i, w_i explicitly in terms of known functions, and the expressions (36) for α_i, β_i, γ_i complete the solution of the problem.

31. The solution for the infinite plane plate is of course included in the general solution for the spherical shell, as remarked above for the case in which surface displacements, not surface forces, were given; but, as in that case, it will be simpler and practically easier to work out the problem *ab initio*, taking advantage of the appropriate Fourier forms. The relative ease of the independent investigation is indeed still greater in the case in which the surface forces are given than in the other case, since the general expressions for the surface forces assume simple forms when the surface is plane, and require no such transformation as that which we have found necessary, and which has constituted the special difficulty of the problem, when the surface was spherical. The problem of the plane plate presents many questions of remarkable interest and practical importance; and although the object and limits of the present paper preclude any detailed investigation of special cases, we may make a short digression to work out the general solution.

32. Let the origin of coordinates be taken in one side of the plate and the axis OX perpendicular to it. Then, according to the general expressions (25) of § 21, the three components of the force per unit of area, in or parallel to either side of the plate, are respectively

$$\left.\begin{aligned}&\text{parallel to OX, } \mathrm{P}=(m+n)\frac{d\alpha}{dx}+(m-n)\left(\frac{d\beta}{dy}+\frac{d\gamma}{dz}\right),\\&\text{parallel to OY, } \mathrm{U}=n\left(\frac{d\alpha}{dy}+\frac{d\beta}{dx}\right),\\&\text{parallel to OZ, } \mathrm{T}=n\left(\frac{d\alpha}{dz}+\frac{d\gamma}{dx}\right).\end{aligned}\right\}\quad . \; . \; . \; . \; . \quad (51)$$

The surface condition to be fulfilled is that each of these functions shall have an arbitrarily given value at every point of each infinite plane side of the plate.

33. From the indications of § 6 above, it is easily seen that the following assumptions are correct for a general solution of the equations of internal equilibrium, and convenient for the application at present proposed,

$$\alpha=u+x\frac{d\phi}{dx},$$

$$\beta=v\;+x\frac{d\phi}{dy},$$

$$\gamma=w+x\frac{d\phi}{dz},$$

where u, v, w, and ϕ denote functions of (x, y, z) which each fulfil the equation $\nabla^2\mathrm{V}=0$. From these, by differentiation, and by taking $\nabla^2\phi=0$ into account, we have

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}+\frac{d\phi}{dx},$$

or

if

$$\left.\begin{aligned} \delta&=\psi+\frac{d\phi}{dx} \\ \psi&=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

and δ be used with the same signification as above (§ 2). Also, by differentiation and application of the equations $\nabla^2 u=0$, $\nabla^2\frac{d\phi}{dx}=0$, we find

$$\nabla^2\alpha=2\frac{d^2\phi}{dx^2},\quad \nabla^2\beta=2\frac{d^2\phi}{dxdy},\quad \nabla^2\gamma=2\frac{d^2\phi}{dxdz}.$$

Hence, to satisfy the general equations of internal equilibrium (3) of § 3, we must have

$$\frac{d\phi}{dx}=-\frac{m}{m+2n}\psi.$$

Hence the general solution becomes

$$\left.\begin{aligned} \alpha&=u-\frac{mx}{m+2n}\psi, \\ \beta&=v-\frac{mx}{m+2n}\frac{d\int\psi dx}{dy}, \\ \gamma&=w-\frac{mx}{m+2n}\frac{d\int\psi dx}{dz}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where u, v, w are any functions whatever which satisfy the general equation $\nabla^2 V=0$, and ψ is given by (52); and where, further, it must be understood that $\int\psi dx$ must be so assigned as to satisfy the equation $\nabla^2 V=0$, which ψ itself satisfies by virtue of (52).

34. The general form of the solution of $\nabla^2 V=0$, convenient for the present application, is clearly

$$\varepsilon^{\pm px}\genfrac{}{}{0pt}{}{\sin}{\cos}(sy)\genfrac{}{}{0pt}{}{\sin}{\cos}(tz),$$

where p, s, t are three constants subject to the equation

$$p^2=s^2+t^2.$$

If now we suppose, as a particular case, the surface condition to be that

$$\left.\begin{aligned} &\left.\begin{aligned} P&=A\sin(sy)\sin(tz), \\ U&=B\cos(sy)\sin(tz), \\ T&=C\sin(sy)\cos(tz), \end{aligned}\right\}\text{when } x=0, \\ \text{and}\quad &\left.\begin{aligned} P&=A'\sin(sy)\sin(tz), \\ U&=B'\cos(sy)\sin(tz), \\ T&=C'\sin(sy)\cos(tz), \end{aligned}\right\}\text{when } x=a, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

where A, B, C, A′, B′, C′ are six given constants, we must clearly have

$$\left.\begin{aligned} u&=(f\varepsilon^{-px}+f'\varepsilon^{px})\sin(sy)\sin(tz),\\ v&=(g\varepsilon^{-px}+g'\varepsilon^{px})\cos(sy)\sin(tz),\\ w&=(h\varepsilon^{-px}+h'\varepsilon^{px})\sin(sy)\cos(tz),\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(55)$$

where f, g, h, f', g', h' are six constants to be determined by six linear equations obtained directly from (54), (51), (53), (52), (55). But, by proper interchanges of sines and cosines, we have in (54) a representation of the general terms of the series or of the definite integrals, representing, according to FOURIER's principles, the six arbitrary functions, whether periodic or non-periodic, by which P, U, T are given over each of the two infinite plane sides. Hence the solution thus indicated is complete.

35. To complete the theory of the equilibrium of an elastic spheroidal shell, we must now suppose every point of the solid substance to be urged by a given force. The problem thus presented will be reduced to that already solved, by the following simple investigation.

36. Let X, Y, Z be the components of the force per unit of volume on the substance at any point x, y, z. (That is to say, let qX, qY, qZ be the three components of the actual force on a volume q, infinitely small in all its dimensions, enclosing the point (x, y, z). Not to unnecessarily limit the problem, we must suppose X, Y, Z to be each an absolutely arbitrary function of x, y, z.

37. When we remember that x, y, z are the coordinates of the undisturbed position of any point of the substance, and differ by the infinitely small quantities α, β, γ from the actual coordinates of the same point of the substance in the body disturbed by the applied forces, we perceive that $\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz$ need not be the differential of a function of three independent variables. It actually will not be a complete differential if the case be that of the interior kinetic equilibrium of a rigid body starting from rest under the influence of given constant forces applied to its surface, and having for their resultant a couple in a plane perpendicular to a principal axis. Nor will $\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz$ be a complete differential in the interior of a steel bar-magnet held at rest under the influence of an electric current directed through one half of its length, as we perceive when we consider FARADAY's beautiful experiment showing rotation to supervene in this case when the magnet is freed from all mechanical constraint.

38. The equations of elastic equilibrium are of course now

$$\left.\begin{aligned} n\nabla^2\alpha+m\frac{d\delta}{dx}&=-\mathrm{X},\\ n\nabla^2\beta+m\frac{d\delta}{dy}&=-\mathrm{Y},\\ n\nabla^2\gamma+m\frac{d\delta}{dz}&=-\mathrm{Z}.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(56)$$

Let ϖ, ϱ, σ denote some three particular solutions of the equations

$$\left.\begin{aligned}\nabla^2\varpi&=-\mathrm{X},\\ \nabla^2\varrho&=-\mathrm{Y},\\ \nabla^2\sigma&=-\mathrm{Z}.\end{aligned}\right\} \quad \dots\dots\dots\dots \quad (57)$$

These, ϖ, ϱ, σ, we may regard as known functions, being derivable from X, Y, Z by known methods (THOMSON and TAIT'S 'Natural Philosophy,' chap. vi.). Then, if we assume

$$\left.\begin{aligned}\alpha-\frac{\varpi}{n}&=\alpha_{,},\\ \beta-\frac{\varrho}{n}&=\beta_{,},\\ \gamma-\frac{\sigma}{n}&=\gamma_{,},\end{aligned}\right\} \quad \dots\dots\dots\dots \quad (58)$$

and

$$\frac{d\alpha_{,}}{dx}+\frac{d\beta_{,}}{dy}+\frac{d\gamma_{,}}{dz}=\delta_{,}, \quad \dots\dots\dots\dots \quad (59)$$

the equations (56) of interior equilibrium become

$$\left.\begin{aligned}n\nabla^2\alpha_{,}+m\frac{d\delta_{,}}{dx}&=-\frac{m}{n}\frac{d\xi}{dx},\\ n\nabla^2\beta_{,}+m\frac{d\delta_{,}}{dy}&=-\frac{m}{n}\frac{d\xi}{dy},\\ n\nabla^2\gamma_{,}+m\frac{d\delta_{,}}{dz}&=-\frac{m}{n}\frac{d\xi}{dz},\end{aligned}\right\} \quad \dots\dots\dots\dots \quad (60)$$

where ξ is a known function given by the equation

$$\xi=\frac{d\varpi}{dx}+\frac{d\varrho}{dy}+\frac{d\sigma}{dz}. \quad \dots\dots\dots\dots \quad (61)$$

Now, as we verify in a moment by differentiation, equations (59) and (60) are satisfied by

$$\left.\begin{aligned}\alpha_{,}&=\frac{-m}{n(m+n)}\frac{d\vartheta}{dx},\\ \beta_{,}&=\frac{-m}{n(m+n)}\frac{d\vartheta}{dy},\\ \gamma_{,}&=\frac{-m}{n(m+n)}\frac{d\vartheta}{dz},\end{aligned}\right\} \quad \dots\dots\dots\dots \quad (62)$$

if ϑ is some particular solution of

$$\nabla^2\vartheta=\xi. \quad \dots\dots\dots\dots \quad (63)$$

Hence (58), (57), (62), (63), (61) express a particular solution of (56).

39. We conclude that the general solution of (56) may be expressed thus:—

$$\left.\begin{aligned}\alpha&=\frac{1}{n}\left(\varpi-\frac{m}{m+n}\frac{d\vartheta}{dx}\right)+{}'\alpha,\\ \beta&=\frac{1}{n}\left(\varrho-\frac{m}{m+n}\frac{d\vartheta}{dy}\right)+{}'\beta,\\ \gamma&=\frac{1}{n}\left(\sigma-\frac{m}{m+n}\frac{d\vartheta}{dz}\right)+{}'\gamma,\end{aligned}\right\} \quad \dots\dots\dots\dots \quad (64)$$

where

$$\left.\begin{aligned}\varpi&=\nabla^{-2}\mathrm{X},\\ \varrho&=\nabla^{-2}\mathrm{Y},\\ \sigma&=\nabla^{-2}\mathrm{Z},\\ \vartheta&=\nabla^{-2}\left(\frac{d\varpi}{dx}+\frac{d\varrho}{dy}+\frac{d\sigma}{dz}\right),\end{aligned}\right\}\quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (65)$$

according to an abbreviated notation, which explains itself sufficiently; and $'\alpha$, $'\beta$, $'\gamma$ denote a general solution of the equations

$$\left.\begin{aligned}n\nabla^{2}{}'\alpha+m\frac{d}{dx}\left(\frac{d\,'\alpha}{dx}+\frac{d\,'\beta}{dy}+\frac{d\,'\gamma}{dz}\right)&=0,\\ n\nabla^{2}{}'\beta+m\frac{d}{dy}\left(\frac{d\,'\alpha}{dx}+\frac{d\,'\beta}{dy}+\frac{d\,'\gamma}{dz}\right)&=0,\\ n\nabla^{2}{}'\gamma+m\frac{d}{dz}\left(\frac{d\,'\alpha}{dx}+\frac{d\,'\beta}{dy}+\frac{d\,'\gamma}{dz}\right)&=0.\end{aligned}\right\}\quad . \; . \; . \; . \; . \; . \; . \quad (66)$$

40. This solution is applicable of course to an elastic body of any shape. It enables us to determine the displacement of every point of it when any given force is applied to every point of its interior, and either displacements or forces are given over the whole surface, if we can solve the general problem for the same shape of body with arbitrary superficial data, but no force on the interior parts. For $'\alpha$, $'\beta$, $'\gamma$ are determined by the solution of this problem, to be worked out with the given arbitrary superficial functions modified by the subtraction from them of terms due to the parts of α, β, γ which are explicitly shown in terms of data by equations (64) and (65).

41. Hence the problem of § 35 is completely solved,—whether we have *displacements* given over each of the two concentric spherical bounding surfaces, when the solution of §§ 14–18 determines $'\alpha$, $'\beta$, $'\gamma$; or *forces* given over the boundary, when the solution of §§ 26–30 is available. In the former case the superficial values of the functions

$$\frac{1}{n}\left(\varpi-\frac{m}{m+n}\frac{d\vartheta}{dx}\right),$$

$$\frac{1}{n}\left(\varrho-\frac{m}{m+n}\frac{d\vartheta}{dy}\right),$$

$$\frac{1}{n}\left(\sigma-\frac{m}{m+n}\frac{d\vartheta}{dz}\right),$$

known from equations (65), must be subtracted from the arbitrary functions given as the superficial values of α, β, γ, and the residues, expressed in surface-harmonic series by the known method, will be the harmonic expressions for the superficial values of $'\alpha$, $'\beta$, $'\gamma$. In the latter case, we must first substitute those known functions $\frac{1}{n}\left(\varpi-\frac{m}{m+n}\frac{d\vartheta}{dx}\right)$, &c., instead of α, β, γ respectively in (34), and the values of $\mathrm{F}r$, $\mathrm{G}r$, $\mathrm{H}r$ thus found must be subtracted from the given arbitrary functions representing the true superficial values of Fr, Gr, Hr. The remainders, which we may denote by $'Fr$, $'Gr$, $'Hr$, must then be reduced to harmonic series, as in (45), and used according to the investigation of § 30, to determine $'\alpha$, $'\beta$, $'\gamma$.

42. The general solution (64) and the expression just indicated for the terms to be subtracted from the data so as to find $'Fr$, $'Gr$, $'Hr$, becomes much simplified when, as in some of the most important practical applications, $Xdx+Ydy+Zdz$ is a complete differential. Thus let

$$-X=\frac{dW}{dx}, \quad -Y=\frac{dW}{dy}, \quad -Z=\frac{dW}{dz}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{67}$$

W denoting any function of x, y, z. Then, assuming, as we may do according to (65),

$$\varpi=\frac{d}{dx}\nabla^{-2}W,$$

$$\varrho=\frac{d}{dy}\nabla^{-2}W,$$

$$\sigma=\frac{d}{dz}\nabla^{-2}W,$$

we have by differentiating, &c.,

$$\frac{d\varpi}{dx}+\frac{d\varrho}{dy}+\frac{d\sigma}{dz}=W,$$

and therefore

$$\vartheta=\nabla^{-2}W. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{68}$$

Hence the solution (64) becomes

$$\left.\begin{aligned}\alpha&=\frac{1}{m+n}\frac{d\vartheta}{dx}+'\alpha,\\ \beta&=\frac{1}{m+n}\frac{d\vartheta}{dy}+'\beta,\\ \gamma&=\frac{1}{m+n}\frac{d\vartheta}{dz}+'\gamma.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{69}$$

From this we find

$$\delta=\frac{1}{m+n}W+'\delta,$$

and (§ 25)

$$\zeta=\frac{1}{m+n}r\frac{\mathrm{d}\vartheta}{\mathrm{d}r}+'\zeta,$$

if

$$'\delta=\frac{d'\alpha}{dx}+\frac{d'\beta}{dy}+\frac{d'\gamma}{dz},$$

and

$$'\zeta='\alpha x+'\beta y+'\gamma z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{70}$$

Hence, by (34),

$$Fr=\frac{1}{m+n}\left\{(m-n)Wx+n\left[\left(r\frac{\mathrm{d}}{\mathrm{d}r}-1\right)\frac{d}{dx}+\frac{d}{dx}r\frac{\mathrm{d}}{\mathrm{d}r}\right]\vartheta\right\}+'Fr.$$

But

$$\frac{d}{dx}r\frac{\mathrm{d}}{\mathrm{d}r}=\left(r\frac{\mathrm{d}}{\mathrm{d}r}+1\right)\frac{d}{dx}.$$

Thus for Fr, and the symmetrical expressions, we have

$$\left.\begin{aligned}\mathrm{F}r &= \frac{1}{m+n}\left\{(m-n)\mathrm{W}x+2nr\frac{\mathrm{d}}{dr}\frac{d\vartheta}{dx}\right\}+{}'\mathrm{F}r,\\ \mathrm{G}r &= \frac{1}{m+n}\left\{(m-n)\mathrm{W}y+2nr\frac{\mathrm{d}}{dr}\frac{d\vartheta}{dy}\right\}+{}'\mathrm{G}r,\\ \mathrm{H}r &= \frac{1}{m+n}\left\{(m-n)\mathrm{W}z+2nr\frac{\mathrm{d}}{dr}\frac{d\vartheta}{dz}\right\}+{}'\mathrm{H}r.\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (71)$$

43. These expressions become further simplified if W is a homogeneous function of any positive or negative integral or fractional order $i+1$, in which case we shall denote it by W_{i+1}. For ϑ will be a homogeneous function of order $i+3$, and $\frac{d\vartheta}{dx}$ of order $i+2$. Hence

$$r\frac{\mathrm{d}}{dr}\frac{d}{dx}\vartheta=(i+2)\frac{d\vartheta}{dx}.$$

Hence the preceding become

$$\left.\begin{aligned}\mathrm{F}r &= \frac{1}{m+n}\left\{(m-n)\mathrm{W}_{i+1}\;x+2n(i+2)\frac{d\vartheta}{dx}\right\}+{}'\mathrm{F}r,\\ \mathrm{G}r &= \frac{1}{m+n}\left\{(m-n)\mathrm{W}_{i+1}\;y+2n(i+2)\frac{d\vartheta}{dy}\right\}+{}'\mathrm{G}r,\\ \mathrm{H}r &= \frac{1}{m+n}\left\{(m-n)\mathrm{W}_{i+1}\;z+2n(i+2)\frac{d\vartheta}{dz}\right\}+{}'\mathrm{H}r.\end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (72)$$

44. These expressions are the more readily reduced to the harmonic forms proper for working out the solution, if the interior force potential, W_{i+1}, is itself a harmonic function. We then have (§ 10)

$$\left.\begin{aligned}&\vartheta=\frac{1}{2(2i+5)}r^2\mathrm{W}_{i+1}, \quad \frac{d\vartheta}{dx}=\frac{1}{2i+5}\left(x\mathrm{W}_{i+1}+\frac{1}{2}r^2\frac{d\mathrm{W}_{i+1}}{dx}\right)\\ \text{and}\quad &\mathrm{W}_{i+1}x=\frac{1}{2i+3}\left\{r^2\frac{d\mathrm{W}_{i+1}}{dx}-r^{2i+5}\frac{d(\mathrm{W}_{i+1}r^{-2i-3})}{dx}\right\},\end{aligned}\right\} \quad . \; . \; . \; . \quad (73)$$

which give

$$\mathrm{F}r=\frac{1}{m+n}\left\{\frac{m+(i+1)n}{2i+3}r^2\frac{d\mathrm{W}_{i+1}}{dx}-\frac{m(2i+5)-n}{(2i+3)(2i+5)}r^{2i+5}\frac{d(\mathrm{W}_{i+1}r^{-2i-3})}{dx}\right\}+{}'\mathrm{F}r, \quad . \; . \; . \quad (74)$$

and symmetrical expressions for Gr and Hr. Here the terms to be subtracted from the arbitrary functions given to represent the superficial values of Fr, Gr, and Hr are each explicitly expressed in sums of two surface harmonics of orders i or $-i-1$, and $i+2$ or $-i-3$ respectively, viz., in each case, that one of the two numbers which is not negative.

45. When the shell is in equilibrium under the influence of the forces acting on it through its interior, without any application of force to its surface, we must have

$$\left.\begin{aligned}\mathrm{F}r&=0,\\ \mathrm{G}r&=0,\\ \mathrm{H}r&=0,\end{aligned}\right\}\text{when } r=a \text{ and when } r=a'. \quad . \; . \; . \; . \; . \; . \; . \quad (75)$$

Hence, for the case in which W is a spherical harmonic, the preceding equations give the proper harmonic expressions for $'Fr$, $'Gr$, $'Hr$ at the outer and inner bounding surfaces, for determining $'\alpha$, $'\beta$, $'\gamma$ by the method of §§ 28–30. Thus, using all the same notations, with the exception of $'\alpha$, $'\beta$, $'\gamma$, $'F$, $'G$, $'H$, instead of α, β, γ, F, G, H, and, for the present, supposing $i+1$ to be positive*, we have the complete harmonic expressions of $'F$, $'G$, $'H$, each in two terms, of orders i and $i+2$ respectively. Hence the A, A′, &c. of (45) are given by the following equations:—

$$\left.\begin{aligned}
\frac{A_i}{a^{i+1}}=\frac{A'_i}{a'^{i+1}}&=-\frac{m+(i+1)n}{(2i+3)(m+n)}r^{-i}\frac{dW_{i+1}}{dx},\\
\frac{B_i}{a^{i+1}}=\frac{B'_i}{a'^{i+1}}&=-\frac{m+(i+1)n}{(2i+3)(m+n)}r^{-i}\frac{dW_{i+1}}{dy},\\
\frac{C_i}{a^{i+1}}=\frac{C'_i}{a'^{i+1}}&=-\frac{m+(i+1)n}{(2i+3)(m+n)}r^{-i}\frac{dW_{i+1}}{dz},\\
\frac{A_{i+2}}{a^{i+1}}=\frac{A'_{i+2}}{a'^{i+1}}&=\frac{(2i+5)m-n}{(2i+3)(2i+5)(m+n)}r^{i+3}\frac{d(W_{i+1}r^{-2i-3})}{dx},\\
\frac{B_{i+2}}{a^{i+1}}=\frac{B'_{i+2}}{a'^{i+1}}&=\frac{(2i+5)m-n}{(2i+3)(2i+5)(m+n)}r^{i+3}\frac{d(W_{i+1}r^{-2i-3})}{dy},\\
\frac{C_{i+2}}{a^{i+1}}=\frac{C'_{i+2}}{a'^{i+1}}&=\frac{(2i+5)m-n}{(2i+3)(2i+5)(m+n)}r^{i+3}\frac{d(W_{i+1}r^{-2i-3})}{dz}.
\end{aligned}\right\}\quad . \; . \; . \quad (76)$$

46. The functions derived from A_i, B_i, C_i, &c., which are required for formulæ (48) and (49), are therefore as follows:—

$$\left.\begin{aligned}
&\frac{d(A_ir^i)}{dx}+\frac{d(B_ir^i)}{dy}+\frac{d(C_ir^i)}{dz}=0,\\
&\frac{d(A_ir^{-i-1})}{dx}+\frac{d(B_ir^{-i-1})}{dy}+\frac{d(C_ir^{-i-1})}{dz}=\frac{(i+1)(2i+1)[m+(i+1)n]}{(2i+3)(m+n)}\frac{a^{i+1}}{r^{2i+3}}W_{i+1},\\
&\frac{d(A_{i+2}r^{i+2})}{dx}+\frac{d(B_{i+2}r^{i+2})}{dy}+\frac{d(C_{i+2}r^{i+2})}{dy}=-\frac{(i+2)[(2i+5)m-n]}{(2i+3)(m+n)}a^{i+1}W_{i+1},\\
&\frac{d(A_{i+2}r^{-i-3})}{dx}+\frac{d(B_{i+2}r^{-i-3})}{dy}+\frac{d(C_{i+2}r^{-i-3})}{dz}=0,
\end{aligned}\right\}.\quad (77)$$

with the corresponding expressions relative to A'_i, B'_i, C'_i, &c., obtained simply by changing a into a'.

Hence by (48) and (50), and the two corresponding equations for the inner surface, we infer that each of the four functions ψ_{i-1}, ψ_{-i}, ϕ_{i-1}, ϕ_{-i} vanishes. By the same equations, with i changed into $i+2$, we obtain expressions, all of one harmonic form, direct or reciprocal, as follows, for the four functions of order $i+1$:—

---

* As we shall not in the present paper consider particularly any case of a shell influenced by centres of force in the hollow space within it, which alone could give a potential W_{i+1} of negative degree, we need not write any of the expressions in forms convenient for making $i+1$ negative.

$$\left.\begin{aligned}\psi_{i+1}&=K_{i+1}W_{i+1},\\ \psi_{-i-2}&=K'_{i+1}r^{-2i-3}W_{i+1},\\ \phi_{i+1}&=L_{i+1}W_{i+1},\\ \phi_{-i-2}&=L'_{i+1}r^{-2i-3}W_{i+1},\end{aligned}\right\}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ . \quad (78)$$

K_{i+1}, K'_{i+1}, L_{i+1}, L'_{i+1}, which need not be here explicitly expressed, being four constants obtained from the solution of four simple algebraic equations. Lastly, by the four equations with $(i+4)$ instead of i, we find that ψ_{i+3}, ψ_{-i-4}, ϕ_{i+3}, ϕ_{-i-4} all vanish. Using these results for ψ and ϕ in (47), we see that each of the functions u must be a harmonic congruent with either $\frac{dW_{i+1}}{dx}$ or $\frac{d(W_{i+1}r^{-2i-3})}{dx}$. Hence, by using (78) in (38) and (39) we find

$$\left.\begin{aligned}u_i\ &=\frac{-L_{i+1}}{(2i+1)(i+1)}\frac{dW_{i+1}}{dx}, & u_{-i-1}&=\frac{-K'_{i+1}}{(2i+1)(i+1)}r^{-2i-1}\frac{dW_{i+1}}{dx},\\ u_{i+2}&=\frac{-K_{i+1}}{(2i+5)(i+2)}r^{2i+5}\frac{d(W_{i+1}r^{-2i-3})}{dx}, & u_{-i-3}&=\frac{-L'_{i+1}}{(2i+5)(i+2)}\frac{d(W_{i+1}r^{-2i-3})}{dx};\end{aligned}\right\}\quad (79)$$

and symmetrical expressions for v and w. Finally, using these expressions, (79) and (78), in (36), and the result in (69) with (73), we arrive at an explicit solution of the problem in the following remarkably simple form:—

$$\left.\begin{aligned}\alpha&=\mathfrak{E}_{i+1}\frac{dW_{i+1}}{dx}+\mathfrak{E}'_{i+1}\frac{d(W_{i+1}r^{-2i-3})}{dx},\\ \beta&=\mathfrak{E}_{i+1}\frac{dW_{i+1}}{dy}+\mathfrak{E}'_{i+1}\frac{d(W_{i+1}r^{-2i-3})}{dy},\\ \gamma&=\mathfrak{E}_{i+1}\frac{dW_{i+1}}{dz}+\mathfrak{E}'_{i+1}\frac{d(W_{i+1}r^{-2i-3})}{dz},\end{aligned}\right\}\quad .\ .\ .\ .\ .\ .\ . \quad (80)$$

where

$$\left.\begin{aligned}\mathfrak{E}_{i+1}&=-\frac{L_{i+1}+K'_{i+1}r^{-2i-1}}{(i+1)(2i+1)}+\left\{\frac{1}{2(2i+3)(m+n)}-\tfrac{1}{2}m\frac{K_{i+1}}{(m+2n)i+m+3n}\right\}r^2,\\ \mathfrak{E}'_{i+1}&=-\frac{K_{i+1}r^{2i+5}+L'_{i+1}}{(i+2)(2i+5)}+\left\{\frac{-r^{2i+5}}{(2i+3)(2i+5)(m+n)}+\frac{mr^2}{2}\frac{K'_{i+1}}{(m+2n)i+2m+3n}\right\}.\end{aligned}\right\}\quad . \quad (81)$$

47. In conclusion, let us consider the case of a solid sphere. For this we have

$$\psi_{-i-2}=0,\ \text{and}\ \phi_{-i-2}=0,$$

as we see at once from the character of the problem, or as we find by putting $a'=0$ in the four equations by which in § 46 we have seen that K_{i+1}, K'_{i+1}, L_{i+1}, L'_{i+1} are to be determined. Then, by (48), with i changed into $i+2$, and by (49), we find

$$\left.\begin{aligned}\psi_{i+1}&=-\frac{(i+2)[(m+2n)i+m+3n][m(2i+5)-n]}{(2i+3)n\{m[2(i+2)^2+1]-n(2i+3)\}(m+n)}W_{i+1},\\ \phi_{i+1}&=a^2\frac{(i+1)^2(2i+1)[m(i+3)-n]}{2n\{m[2(i+2)^2+1]-n(2i+3)\}}W_{i+1}.\end{aligned}\right\}\quad .\ .\ .\ . \quad (82)$$

The coefficients of W_{i+1} in these expressions are the values which we must take for K_{i+1} and L_{i+1} respectively in (81); and therefore, after reductions which show $(m+n)$ as a factor of the numerator of each fraction in which it appears at first as a factor of the denominator, we have

$$\left.\begin{aligned} \mathfrak{E}_{i+1} &= -\frac{(i+1)[m(i+3)-n]a^2}{2n\{m[2(i+2)^2+1]-n(2i+3)\}}+\frac{[(i+2)(2i+5)m-(2i+3)n]r^2}{2n(2i+3)\{m[2(i+2)^2+1]-n(2i+3)\}}, \\ \mathfrak{E}'_{i+1} &= \frac{(i+1)mr^{2i+5}}{n(2i+3)\{m[2(i+2)^2+1]-n(2i+3)\}}. \end{aligned}\right\} \quad (83)$$

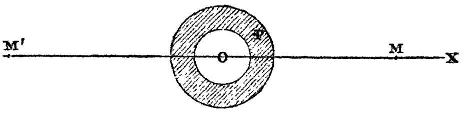
These, substituted in (80), give expressions for α, β, γ which constitute a complete and explicit solution of the problem.

It is easy to verify this result, by testing that (56) (with $-X=\frac{dW_{i+1}}{dx}$, &c.) is satisfied for every point of the solid, and that equations (34) give $F=0$, $G=0$, $H=0$ at the bounding surface, $r=a$.

48. The case of $i=1$ is, as we shall immediately see, of high importance. For it the preceding expressions, (83) and (80), become

$$\left.\begin{aligned} \mathfrak{E}_2 &= \frac{-10(4m-n)a^2+(21m-5n)r^2}{10n(19m-5n)}, \\ \mathfrak{E}'_2 &= \frac{4mr^7}{10n(19m-5n)}, \\ \alpha &= \mathfrak{E}_2\frac{dW_2}{dx}+\mathfrak{E}'_2\frac{d(W_2\,r^{-5})}{dx}, \\ \beta &= \mathfrak{E}_2\frac{dW_2}{dy}+\mathfrak{E}'_2\frac{d(W_2\,r^{-5})}{dy}, \\ \gamma &= \mathfrak{E}_2\frac{dW_2}{dz}+\mathfrak{E}'_2\frac{(dW_2\,r^{-5})}{dz}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (84)$$

49. As an example of the application of §§ 45–48, let us suppose a spherical shell or solid sphere to be equilibrated under the influence of masses collected in two fixed external points[*], and each attracting according to the inverse square of its distance. Let the two masses M, M′ be in the axis OX; and, P being the point whose coordinates are x, y, z, let $PM=D$, $PM'=D'$. Let also $OM=c$, $OM'=c'$.



Then, if m, m′ denote the two masses, for equilibrium we must have

$$\frac{\mathrm{m}}{c^2}=\frac{\mathrm{m}'}{c'^2}.$$

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[*] If our limits permitted, a highly interesting example might be made of the case of a shell under the influence of a single attracting point in the hollow space within it. The effect will clearly be to keep the whole shell sensibly in equilibrium even if the attracting point is excentric; and under stress even if the attracting point is in the centre.

The potential at P, due to the two masses, will be $\frac{\mathrm{m}}{\mathrm{D}}+\frac{\mathrm{m}'}{\mathrm{D}'}$, or, according to the notation of § 42, with, besides, w taken to denote the mass of unit volume of the elastic solid,

$$-\mathrm{W}=w\left(\frac{\mathrm{m}}{\mathrm{D}}+\frac{\mathrm{m}'}{\mathrm{D}'}\right).$$

The known forms in the elementary theory of spherical harmonics give immediately the development of this in a converging infinite series of solid harmonic terms. We have only then to apply the solution of §§ 45, 46 to each term, to obtain a series expressing the required solution.

50. We may work out this result explicitly for the case in which both masses are very distant; and for simplicity we shall suppose one of them infinitely more distant than the other; that is to say, we shall suppose it to exercise merely a constant balancing force on the substance of the shell. We shall then have precisely the same bodily disturbing force as that which the earth experiences from the moon alone, or from the sun alone.

51. Referring to the diagram and notation of § 49, we have

$$\frac{1}{\mathrm{D}}=\frac{1}{c}\left\{1+\frac{x}{c}+\frac{x^2-\frac{1}{2}(y^2+z^2)}{c^2}\right\}$$

if we neglect higher powers of $\frac{x}{c}$, $\frac{y}{c}$, $\frac{z}{c}$ than the square; and

$$\frac{1}{\mathrm{D}'}=\frac{1}{c'}\left(1-\frac{x}{c'}\right)$$

neglecting all higher powers of $\frac{x}{c}$, $\frac{y}{c}$, $\frac{z}{c}$. Hence, taking account of the relation $\frac{\mathrm{m}}{c^2}=\frac{\mathrm{m}'}{c'^2}$ required for equilibrium, we have, for the disturbance potential,

$$-\mathrm{W}=\frac{\mathrm{m}}{c^3}(x^2-\tfrac{1}{2}y^2-\tfrac{1}{2}z^2)w,$$

an irrelevant constant being omitted from the expression which § 49 would give. This being a harmonic of the second degree, we may use it for W_{i+1}, putting $i=1$ in the formulæ of § 47, and thus solve the problem of finding the deformation of a homogeneous spherical shell under the influence of a distant attracting mass and a uniform balancing force. I hope, in a future communication to the Royal Society, to show the application of this result to the case of the lunar and solar influence on a body such as the earth is assumed to be by many geologists—that is to say, a solid crust, constituting a spheroidal shell, of some thickness less than 100 miles, with its interior filled with liquid. The untenability of this hypothesis is, however, sufficiently demonstrated by the considerations adduced in a previous communication ("On the Rigidity of the Earth," read May 8, 1862), in which the following explicit solution of the problem for a homogeneous solid sphere only is used.

52. Using the expression of § 51 for W_2, we have

$$\left.\begin{aligned}
&\frac{d\mathrm{W}_2}{dx}=-2\frac{\mathrm{m}}{c^3}xw,\\
&\frac{d\mathrm{W}_2}{dy}=+\frac{\mathrm{m}}{c^3}yw,\quad \frac{d\mathrm{W}_2}{dz}=+\frac{\mathrm{m}}{c^3}zw,\\
&\frac{d(\mathrm{W}_2r^{-5})}{dx}=+3\frac{\mathrm{m}}{c^3}\frac{(x^2-\frac{3}{2}y^2-\frac{3}{2}z^2)x}{r^{-7}}w,\\
&\frac{d(\mathrm{W}_2r^{-5})}{dy}=+3\frac{\mathrm{m}}{c^3}\frac{(2x^2-\frac{1}{2}y^2-\frac{1}{2}z^2)y}{r^{-7}}w,\quad \frac{d(\mathrm{W}_2r^{-5})}{dz}=+3\frac{\mathrm{m}}{c^3}\frac{(2x^2-\frac{1}{2}y^2-\frac{1}{2}z^2)z}{r^{7}}w.
\end{aligned}\right\}\quad . \qquad (85)$$

These formulæ being substituted for the differential coefficients which appear in (84), we have algebraic expressions for the displacement of any point of the solid.

The condition of the body being symmetrical about the axis of x, we may conveniently assume

$$y=\mathrm{y}\cos\phi,\quad z=\mathrm{y}\sin\phi,$$

$$\beta^2+\gamma^2=\mu^2;$$

so that we shall have (as we see by the preceding expressions)

$$\beta=\mu\cos\phi,$$

$$\gamma=\mu\sin\phi,$$

and μ will denote the component displacement perpendicular to OX. If, further, we assume

$$x=r\cos\theta,$$

$$\mathrm{y}=r\sin\theta,$$

the expressions (84) for the component displacements, with (85) used in them, give

$$\left.\begin{aligned}
\alpha&=w\frac{\mathrm{m}}{c^3}\left\{-2r\mathfrak{E}_2+\frac{3}{2}\frac{\mathfrak{E}_2'}{r^4}(5\cos^2\theta-3)\right\}\cos\theta,\\
\mu&=w\frac{\mathrm{m}}{c^3}\left\{\qquad r\mathfrak{E}_2+\frac{3}{2}\frac{\mathfrak{E}_2'}{r^4}(5\cos^2\theta-1)\right\}\sin\theta.
\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \qquad (86)$$

The values given in (84) for $\mathfrak{E}_2$ and $\mathfrak{E}_2'$ are to be used for any internal point, at a distance r from the centre, in these equations (86), and thus we have the simplest possible expression for the required displacement of any point of the solid.

53. If we resolve the displacement along and perpendicular to the radius, and consider only the radial component, we see that the series of concentric spherical surfaces of the undisturbed globe become spheroids of revolution in the distorted body. The elongation of the axial radius, obtained by putting $\theta=0$ and taking the value of α, is double the shortening of the equatorial radius, obtained by putting $\theta=\frac{1}{2}\pi$ and taking the value of μ; which we might have inferred from the fact, shown by the general equations (80) above, that there can be no alteration of volume on the whole within any one of these surfaces. The expression for the excess of the axial above the equatorial radius is

$$-r\mathfrak{E}_2+\frac{9}{2}\frac{\mathfrak{E}_2'}{r^4},$$

which, if we substitute for $\mathfrak{E}_2$ and $\mathfrak{E}'_2$ their values by (84), becomes

$$3w\frac{\mathrm{m}}{c^3}\,\frac{2(4m-n)a^2-(3m-n)r^2}{10n(19m-5n)}\,r.$$

If in this we take $r=a$, and $m=\infty$, it becomes $\frac{2}{3}\,\frac{\mathrm{m}}{c^3}\,\frac{5}{19n}r^3$, which is the result used in § 34 of the paper on the Rigidity of the Earth, preceding the present in the Transactions.

54. In the case of $a'=0$, the result of § 18 takes the extremely simple form

$$\left.\begin{aligned}
\alpha&=\Sigma\left\{\mathrm{A}_i\left(\frac{r}{a}\right)^i+\frac{m(a^2-r^2)}{2a^i[n(2i-1)+m(i-1)]}\,\frac{d\Theta_{i-1}}{dx}\right\},\\
\beta&=\Sigma\left\{\mathrm{B}_i\left(\frac{r}{a}\right)^i+\frac{m(a^2-r^2)}{2a^i[n(2i-1)+m(i-1)]}\,\frac{d\Theta_{i-1}}{dy}\right\},\\
\gamma&=\Sigma\left\{\mathrm{C}_i\left(\frac{r}{a}\right)^i+\frac{m(a^2-r^2)}{2a^i[n(2i-1)+m(i-1)]}\,\frac{d\Theta_{i-1}}{dz}\right\},
\end{aligned}\right.$$

where

$$\Theta_{i-1}=\frac{d(\mathrm{A}_ir^i)}{dx}+\frac{d(\mathrm{B}_ir)}{dy}+\frac{d(\mathrm{C}_ir^i)}{dz}\,. \qquad \ldots\ldots\ldots \quad (87)$$

This expresses the displacement at any point within a solid sphere of radius a, when its surface is displaced in a given manner ($\Sigma\mathrm{A}_i$, $\Sigma\mathrm{B}_i$, $\Sigma\mathrm{C}_i$). And merely by making i negative we have, in the same formula, the solution of the same problem for an infinite solid with a hollow spherical space every point of the surface of which is displaced to a given distance in a given direction. These solutions are obtained directly, with great ease, by the method of §§ 6–15, or are easily proved by direct verification, without any of the intricacy of analysis inevitable when, as in the general investigations with which we commenced, a shell bounded by two concentric spherical surfaces is the subject.

[Added since the reading of the Paper.]

§§ 55 to 58. *Oscillations of a Liquid Sphere.*

55. Let V be the gravitation potential at any point P(x, y, z), and h the height of the surface (or radial component of its displacement) from the mean spherical surface at a point E in the radius through P. Then, if

$$h=\mathrm{S}_1+\mathrm{S}_2+\cdots \qquad \ldots\ldots\ldots\ldots \quad (88)$$

be the expression for h in terms of spherical surface harmonic functions of the position of E, and if μ be the attraction on the unit of mass exercised by a particle equal in mass to the unit bulk of the liquid, we have, by the known methods for finding the attractions of bodies infinitely nearly spherical (THOMSON and TAIT's 'Natural Philosophy,' chap. vi.),

$$\left.\begin{aligned}
\mathrm{V}&=4\pi a\mu\left\{\tfrac{1}{2}a-\tfrac{1}{6}\frac{r^2}{a}+\Sigma\left(\frac{r}{a}\right)^i\frac{\mathrm{S}_i}{2i+1}\right\} &&\text{when } r<a,\\
\mathrm{V}&=4\pi a\mu\left\{\tfrac{1}{3}\frac{a^2}{r}+\frac{a}{r}\Sigma\left(\frac{a}{r}\right)^i\frac{\mathrm{S}_i}{2i+1}\right\} &&\quad\text{,,}\quad r>a,
\end{aligned}\right.$$

and

$$\mathrm{V}=4\pi a\mu\left\{\tfrac{1}{3}a+\Sigma\frac{\mathrm{S}_i}{2i+1}\right\} \qquad \text{,,}\quad r=a. \qquad \ldots\ldots \quad (89)$$

In these

$$4\pi\mu a=3g, \qquad (90)$$

if g denote the force of gravity at the surface, due to the mean sphere, of radius a.

56. Now for infinitely small motions the ordinary kinetic equations give

$$-\frac{dp}{dx}=\varrho\left(\frac{du}{dt}-\frac{dV}{dx}\right);\quad -\frac{dp}{dy}=\varrho\left(\frac{dv}{dt}-\frac{dV}{dy}\right);\quad -\frac{dp}{dz}=\varrho\left(\frac{dw}{dt}-\frac{dV}{dz}\right); \qquad (91)$$

where ϱ is the mass per unit of volume; u, v, w the component velocities through the fixed point P at time t; and p the fluid pressure. Hence, possible non-periodic motions being omitted, $udx+vdy+wdz$ is a complete differential; and, denoting it by φ, d we have

$$C-p=\varrho\left(\frac{d\varphi}{dt}-V\right). \qquad (92)$$

57. To find the surface conditions,—first, since the pressure has a constant value, Π, at the free surface,

$$p=g\varrho h+\Pi \text{ when } r=a, \qquad (93)$$

the variations of gravity depending on the variations of figure being of course neglected in the infinitely small term $g\varrho h$. And, since $\frac{dh}{dt}$ is the radial component of the velocity at E, we have, when $r=a$,

$$\frac{x}{r}\frac{d\varphi}{dx}+\frac{y}{r}\frac{d\varphi}{dy}+\frac{z}{r}\frac{d\varphi}{dz}=\frac{dh}{dt}. \qquad (94)$$

Now since, the fluid being incompressible, $\nabla^2\varphi=0$, φ may be expanded in a series of solid harmonic functions; let

$$\varphi=\Sigma\Phi_i\left(\frac{r}{a}\right)^i, \qquad (95)$$

where $\Phi_1, \Phi_2, \ldots$ are surface harmonics. Hence, as the successive terms are homogeneous functions of the coordinates (x, y, z), of degrees 1, 2, &c.,

$$\frac{x}{r}\frac{d\varphi}{dx}+\frac{y}{r}\frac{d\varphi}{dy}+\frac{z}{r}\frac{d\varphi}{dz}=\frac{1}{r}\Sigma i\Phi_i\left(\frac{r}{a}\right)^i, \qquad (96)$$

and therefore, by (88) and (94),

$$\frac{dS_i}{dt}=\frac{i}{a}\Phi_i. \qquad (97)$$

58. Eliminating p between (92) with $r=a$ and (93), substituting for V by (89) and (90), differentiating, substituting for $\frac{dS_i}{dt}$ by (97), and comparing harmonic terms of order i, we have

$$\frac{d^2\Phi_i}{dt^2}+\frac{g}{a}i\left(1-\frac{3}{2i+1}\right)\Phi_i; \qquad (98)$$

of which the integral is

$$\Phi_i=A\cos\left\{t\sqrt{\frac{g}{a}i\left(1-\frac{3}{2i+1}\right)}-E\right\}. \qquad (99)$$

Here A is a surface spherical harmonic function of the coordinates of E expressing the maximum value of Φ_i, and E is the epoch (THOMSON and TAIT, § 53) of the simple harmonic function of the time which we find to represent Φ_i. Using this solution in (97) and (88), we see that if the surface be normally displaced according to a spherical harmonic of order i, and left to itself, the resulting motion gives rise to a simple harmonic variation of the normal displacement, having for period

$$2\pi\sqrt{\frac{a}{g}\frac{2i+1}{2i(i-1)}},$$

that is, the period of a common pendulum of length $\frac{(2i+1)a}{2i(i-1)}$. It is worthy of remark that the period of vibration thus calculated is the same for the same density of liquid, whatever be the dimensions of the globe.

For the case of $i=2$, or an ellipsoidal deformation, the length of the isochronous pendulum becomes $\frac{5}{4}a$, or one and a quarter times the earth's radius, for a homogeneous liquid globe of the same mass and diameter as the earth; and therefore for this case, or for any homogeneous liquid globe of about $5\frac{1}{2}$ times the density of water, the half-period is $47^{\mathrm{m}}\,12^{\mathrm{s}}$, which is the result stated in the paper "On the Rigidity of the Earth" (§ 3), preceding the present in the Transactions.

APPENDIX, §§ 59-71.—*General Theory of the Equilibrium of an Elastic Solid.*

59. Let a solid composed of matter fulfilling no condition of isotropy in any part, and not homogeneous from part to part, be given of any shape, unstrained, and let every point of its surface be altered in position to a given distance in a given direction. It is required to find the displacement of every point of its substance in equilibrium. Let x, y, z be the coordinates of any particle, P, of the substance in its undisturbed position, and $x+\alpha, y+\beta, z+\gamma$ its coordinates when displaced in the manner specified; that is to say, let α, β, γ be the components of the required displacement. Then, if for brevity we put

$$\left.\begin{aligned}
A&=\left(\frac{d\alpha}{dx}+1\right)^2+\left(\frac{d\beta}{dx}\right)^2+\left(\frac{d\gamma}{dx}\right)^2,\\
B&=\left(\frac{d\alpha}{dy}\right)^2+\left(\frac{d\beta}{dy}+1\right)^2+\left(\frac{d\gamma}{dy}\right)^2,\\
C&=\left(\frac{d\alpha}{dz}\right)^2+\left(\frac{d\beta}{dz}\right)^2+\left(\frac{d\gamma}{dz}+1\right)^2,\\
a&=\frac{d\alpha}{dy}\frac{d\alpha}{dz}+\left(\frac{d\beta}{dy}+1\right)\frac{d\beta}{dz}+\frac{d\gamma}{dy}\left(\frac{d\gamma}{dz}+1\right),\\
b&=\frac{d\alpha}{d}\left(\frac{d\alpha}{dx}+1\right)+\frac{d\beta}{dz}\frac{d\beta}{dx}+\left(\frac{d\gamma}{dz}+1\right)\frac{d\gamma}{dx},\\
c&=\left(\frac{d\alpha}{dx}+1\right)\frac{d\alpha}{dy}+\frac{d\beta}{dx}\left(\frac{d\beta}{dy}+1\right)+\frac{d\gamma}{dx}\frac{d\gamma}{dy};
\end{aligned}\right\}\quad\dots\dots\quad(100)$$

these six quantities A, B, C, a, b, c, as is known*, thoroughly determine the strain experienced by the substance infinitely near the particle P (irrespectively of any rotation it may experience), in the following manner:—

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* THOMSON and TAIT's 'Natural Philosophy,' § 190 (e) and § 181 (5).

60. Let ξ, η, ζ be the undisturbed coordinates of a particle infinitely near P, relatively to axes through P parallel to those of x, y, z respectively; and let $\xi_{\prime}$, $\eta_{\prime}$, $\zeta_{\prime}$ be the coordinates, relative still to axes through P, when the solid is in its strained condition. Then

$$\xi_{\prime}^2+\eta_{\prime}^2+\zeta_{\prime}^2=\mathrm{A}\xi^2+\mathrm{B}\eta^2+\mathrm{C}\zeta^2+2a\eta\zeta+2b\zeta\xi+2c\xi\eta\,; \quad . \quad . \quad . \quad . \quad (101)$$

and therefore all particles which in the strained state lie on a spherical surface

$$\xi_{\prime}^2+\eta_{\prime}^2+\zeta_{\prime}^2=r_{\prime}^2,$$

are, in the unstrained state, on the ellipsoidal surface,

$$\mathrm{A}\xi^2+\mathrm{B}\eta^2+\mathrm{C}\zeta^2+2a\eta\zeta+2b\zeta\xi+2c\xi\eta=r_{\prime}^2.$$

This, as is well known*, completely defines the homogeneous strain of the matter in the neighbourhood of P.

61. Hence the thermo-dynamic principles by which, in a paper on the Thermoelastic Properties of Matter in the first Number of the 'Quarterly Mathematical Journal' (April 1855), GREEN's dynamical theory of elastic solids was demonstrated as part of the modern dynamical theory of heat, show that if $w.dxdydz$ denote the work required to alter an infinitely small undisturbed volume, $dxdydz$, of the solid, into its disturbed condition, when its temperature is kept constant, we must have

$$w=f(\mathrm{A}, \mathrm{B}, \mathrm{C}, a, b, c), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

where f denotes a positive function of the six elements, which vanishes when $\mathrm{A}-1$, $\mathrm{B}-1$, $\mathrm{C}-1$, a, b, c each vanish. And if W denote the whole work required to produce the change actually experienced by the whole solid, we have

$$\mathrm{W}=\iiint w\,dxdydz, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

where the triple integral is extended through the space occupied by the undisturbed solid.

62. The position assumed by every particle in the interior of the solid will be such as to make this a minimum, subject to the condition that every particle of the surface takes the position given to it, this being the elementary condition of stable equilibrium. Hence, by the method of variation,

$$\delta\mathrm{W}=\iiint\delta w\,dxdydz=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

But, exhibiting only terms depending on $\delta\alpha$, we have

$$\begin{aligned}\delta w=&\left\{2\frac{dw}{d\mathrm{A}}\left(\frac{d\alpha}{dx}+1\right)+\frac{dw}{db}\frac{d\alpha}{dz}+\frac{dw}{dc}\frac{d\alpha}{dy}\right\}\frac{d\delta\alpha}{dx}\\ &+\left\{2\frac{dw}{d\mathrm{B}}\frac{d\alpha}{dy}+\frac{dw}{da}\frac{d\alpha}{dz}+\frac{dw}{dc}\left(\frac{d\alpha}{dx}+1\right)\right\}\frac{d\delta\alpha}{dy}\\ &+\left\{2\frac{dw}{d\mathrm{C}}\frac{d\alpha}{dz}+\frac{dw}{da}\frac{d\alpha}{dy}+\frac{dw}{db}\left(\frac{d\alpha}{dx}+1\right)\right\}\frac{d\delta\alpha}{dz}\\ &+\&\mathrm{c}.\end{aligned}$$

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* THOMSON and TAIT's 'Natural Philosophy,' §§ 155–165.

Hence, integrating by parts, and observing that $\delta\alpha$, $\delta\beta$, $\delta\gamma$ vanish at the limiting surface, we have

$$\delta\mathrm{W}=-\iiint dxdydz\left\{\left(\frac{d\mathrm{P}}{dx}+\frac{d\mathrm{Q}}{dy}+\frac{d\mathrm{R}}{dz}\right)\delta\alpha+\&c.\right\}, \quad . \; . \; . \; . \quad (105)$$

where for brevity P, Q, R denote the factors of $\frac{d\delta\alpha}{dx}$, $\frac{d\delta\alpha}{dy}$, $\frac{d\delta\alpha}{dz}$ respectively, in the preceding expression. In order that $\delta\mathrm{W}$ may vanish, the factors of $\delta\alpha$, $\delta\beta$, $\delta\gamma$ in the expression now found for it must each vanish; and hence we have, as the equations of equilibrium,

$$\left.\begin{array}{l}\frac{d}{dx}\left\{2\frac{dw}{d\mathrm{A}}\left(\frac{d\alpha}{dx}+1\right)+\frac{dw}{db}\frac{d\alpha}{dz}+\frac{dw}{dc}\frac{d\alpha}{dy}\right\}\\ \quad+\frac{d}{dy}\left\{2\frac{dw}{d\mathrm{B}}\frac{d\alpha}{dy}+\frac{dw}{da}\frac{d\alpha}{dz}+\frac{dw}{dc}\left(\frac{d\alpha}{dx}+1\right)\right\}\\ \quad+\frac{d}{dz}\left\{2\frac{dw}{d\mathrm{C}}\frac{d\alpha}{dz}+\frac{dw}{da}\frac{d\alpha}{dy}+\frac{dw}{db}\left(\frac{d\alpha}{dx}+1\right)\right\}=0,\\ \qquad\qquad\&c.\ \&c.,\end{array}\right\} \quad . \; . \; . \; . \; . \; . \quad (106)$$

of which the second and third, not exhibited, may be written down merely by attending to the symmetry.

63. From the property of w that it is necessarily positive when there is any strain, it follows that there must be some distribution of strain through the interior which shall make $\iiint wdxdydz$ *the least possible*, subject to the prescribed surface condition, and therefore that the solution of equations (106), subject to this condition, is possible. If, whatever be the nature of the solid as to difference of elasticity in different directions, in any part, and as to heterogeneousness from part to part, and whatever be the extent of the *change of form and dimensions to which it is subjected, there cannot be any internal configuration of unstable equilibrium, or consequently any but one of stable equilibrium,* with the prescribed surface displacement and no disturbing force on the interior, then, besides being always positive, w must be such a function of A, B, &c. that there can be only one solution of the equations. This is obviously the case when the unstrained solid is homogeneous.

64. It is easy to include, in a general investigation similar to the preceding, the effects of any force on the interior substance, such as we have considered particularly for a spherical shell, of homogeneous isotropic matter, in §§ 35–46 above. It is also easy to adapt the general investigation to superficial data of *force*, instead of displacement.

65. Whatever be the general form of the function f for any part of the substance, since it is always positive it cannot change in sign when A−1, B−1, C−1, a, b, c have their signs changed; and therefore for infinitely small values of these quantities it must be a homogeneous quadratic function of them with constant coefficients. (And it may be useful to observe that for all values of the variables A, B, &c., it must therefore be expressible in the same form, with varying coefficients, each of which is always finite, for

all values of the variables.) Thus, for infinitely small strains, we have GREEN's theory of elastic solids, founded on a homogeneous quadratic function of the components of strain, expressing the work required to produce it. Putting

$$A-1=2e,\ B-1=2f,\ C-1=2g, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (107)$$

and denoting by $\frac{1}{2}(e, e)$, $\frac{1}{2}(f, f)$, ... (e, f), ... (e, a), ... the coefficients, we have

$$\left.\begin{aligned} w=\tfrac{1}{2}\{(e, e)e^2+(f, f)f^2+(g, g)g^2+(a, a)a^2+(b, b)b^2+(c, c)c^2\}& \\ +(e, f)ef+(e, g)eg+(e, a)ea+(e, b)eb+(e, c)ec& \\ +(f, g)fg+(f, a)fa+(f, b)fb+(f, c)fc& \\ +(g, a)ga+(g, b)gb+(g, c)gc& \\ +(a, b)ab+(a, c)ac& \\ +(b, c)bc& \end{aligned}\right\} \quad . \ . \quad (108)$$

The twenty-one coefficients in this expression constitute the twenty-one coefficients of elasticity, which GREEN first showed to be proper and essential for a complete theory of the dynamics of an elastic solid subjected to infinitely small strains.

66. When the strains are infinitely small, the products $\frac{dw}{dA}\frac{d\alpha}{dx}$, $\frac{dw}{db}\frac{d\alpha}{dz}$, &c. are each infinitely small, of the second order. We therefore omit them; and then, attending to (107), we reduce (106) to

$$\left.\begin{aligned} \frac{d}{dx}\frac{dw}{de}+\frac{d}{dy}\frac{dw}{dc}+\frac{d}{dz}\frac{dw}{db}&=0, \\ \frac{d}{dx}\frac{dw}{dc}+\frac{d}{dy}\frac{dw}{df}+\frac{d}{dz}\frac{dw}{da}&=0, \\ \frac{d}{dx}\frac{dw}{db}+\frac{d}{dy}\frac{dw}{da}+\frac{d}{dz}\frac{dw}{dg}&=0, \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (109)$$

which are the equations of interior equilibrium. Attending to (108) we see that $\frac{dw}{de}$ ... $\frac{dw}{da}$ ... are linear functions of e, f, g, a, b, c the components of strain. Writing out one of them as an example, we have

$$\frac{dw}{de}=(e, e)e+(e, f)f+(e, g)g+(e, a)a+(e, b)b+(e, c)c. \quad . \ . \ . \quad (110)$$

And α, β, γ denoting, as before, the component displacements of any interior particle, P, from its undisturbed position $(x\ y, z)$, we have, by (107) and (100),

$$\left.\begin{aligned} &e=\frac{d\alpha}{dx}, \quad f=\frac{d\beta}{dy}, \quad g=\frac{d\gamma}{dz}, \\ &a=\frac{d\beta}{dz}+\frac{d\gamma}{dy}, \quad b=\frac{d\gamma}{dx}+\frac{d\alpha}{dz}, \quad c=\frac{d\alpha}{dy}+\frac{d\beta}{dx}. \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \quad (111)$$

It is to be observed that the coefficients (e, e) (e, f), &c. will be in general functions of (x, y, z), but will be each constants when the unstrained solid is homogeneous.

67. It is now easy to prove directly, for the case of infinitely small strains, that the solution of the equations of interior equilibrium, whether for a heterogeneous or a homogeneous solid, subject to the prescribed surface condition, is unique. For let α, β, γ be components of displacement fulfilling the equations, and let α', β', γ' denote any other functions of (x, y, z) having the same surface values as α, β, γ, and let $e', f', \ldots, w'$ denote functions depending on them in the same way as $e, f, \ldots, w$ depend on α, β, γ. Thus, by TAYLOR's theorem,

$$w'-w=\frac{dw}{de}(e'-e)+\frac{dw}{df}(f'-f)+\frac{dw}{dg}(g'-g)+\frac{dw}{da}(a'-a)+\frac{dw}{db}(b'-b)+\frac{dw}{dc}(c'-c)+\mathrm{H},$$

where H denotes the same homogeneous quadratic function of $e'-e$, &c. that w is of e, &c. If for $e'-e$, &c. we substitute their values by (111), this becomes

$$w'-w=\frac{dw}{de}\frac{d(\alpha'-\alpha)}{dx}+\frac{dw}{db}\frac{d(\alpha'-\alpha)}{dz}+\frac{dw}{dc}\frac{d(\alpha'-\alpha)}{dy}+\&\text{c.}+\mathrm{H}.$$

Multiplying by $dxdydz$, integrating by parts, observing that $\alpha'-\alpha$, $\beta'-\beta$, $\gamma'-\gamma$ vanish at the bounding surface, and taking account of (109), we find simply

$$\iiint(w'-w)dxdydz=\iiint \mathrm{H}dxdydz. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

But H is essentially positive. Therefore every other interior condition than that specified by (α, β, γ), provided only it has the same bounding surface, requires a greater amount of work than w to produce it: and the excess is equal to the work that would be required to produce, from a state of no displacement, such a displacement as superimposed on (α, β, γ) would produce the other. And inasmuch as (α, β, γ) fulfil only the conditions of satisfying (110) and having the given surface values, it follows that no other than one solution can fulfil these conditions.

68. But (as has been remarked by Professor STOKES to the author) when the surface data are of force, not of displacement, or when force acts from without, on the interior substance of the body, the solution is not in general unique, and there may be configurations of unstable equilibrium, even with infinitely small displacement. For instance, let part of the body be composed of a steel bar magnet; and let a magnet be held outside in the same line, and with a pole of the same name in its end nearest to one end of the inner magnet. The equilibrium will be unstable, and there will be positions of stable equilibrium with the inner bar slightly inclined to the line of the outer bar, unless the rigidity of the rest of the body exceed a certain limit.

69. Recurring to the general problem, in which the strains are not supposed infinitely small, we see that, if the solid is isotropic in every part, the function of A, B, C, a, b, c which expresses w must be merely a function of the roots of the equation*

$$(\mathrm{A}-\zeta^2)(\mathrm{B}-\zeta^2)(\mathrm{C}-\zeta^2)-a^2(\mathrm{A}-\zeta^2)-b^2(\mathrm{B}-\zeta^2)-c^2(\mathrm{C}-\zeta^2)+2abc=0, \quad . \quad (113)$$

which (that is the positive values of ζ) are the ratios of elongation along the principal axes of the strain-ellipsoid. It is unnecessary here to enter on the analytical expression of this condition. For the case of $A-1$, $B-1$, $C-1$, a, b, c, each infinitely small, it obviously requires that

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* THOMSON and TAIT's 'Natural Philosophy,' § 181 (11).

and
$$\left.\begin{array}{l}(e,e)=(f,f)=(g,g);\ (f,g)=(g,e)=(e,f);\ (a,a)=(b,b)=(c,c);\\ (e,a)=(f,b)=(g,c)=0;\ (b,c)=(c,a)=(a,b)=0;\\ (e,b)=(e,c)=(f,c)=(f,a)=(g,a)=(g,b)=0.\end{array}\right\} \quad . \quad (114)$$

Thus the twenty-one coefficients are reduced to three—

(e, e), which we may denote by the single letter $\mathfrak{A}$,
(f, g), „ „ „ „ $\mathfrak{B}$,
(a, a), „ „ „ „ n.

It is clear that this is necessary and sufficient for ensuring *cubic isotropy*—that is to say, perfect equality of elastic properties with reference to the three rectangular directions OX, OY, OZ. But for *spherical isotropy*, or complete isotropy with reference to all directions through the substance, it is further necessary that

$$\mathfrak{A}-\mathfrak{B}=2n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

as is easily proved analytically by turning two of the axes of coordinates in their own plane through 45°; or geometrically by examining the nature of the strain represented by any one of the elements a, b, c (a "simple shear") and comparing it with the resultant of c, and $f=-e$ (which is also a simple shear). It is convenient now to put

$$\mathfrak{A}+\mathfrak{B}=2m;\ \text{so that}\ \mathfrak{A}=m+n,\ \mathfrak{B}=m-n; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

and thus the expression for the potential energy per unit of volume becomes

$$2w=m(e+f+g)^2+n(e^2+f^2+g^2-2fg-2ge-2ef+a^2+b^2+c^2). \quad . \quad . \quad . \quad (117)$$

Using this in (108), and substituting for e, f, g, a, b, c their values by (111), we find immediately, for the equations of internal equilibrium, equations the same as (1) of § 2.

70. To find the mutual force exerted across any surface within the solid, as expressed by (26) of § 22, we have clearly, by considering the works done respectively by P, Q, R, S, T, U (§ 21) on any infinitely small change of figure or dimensions in the solid,

$$P=\frac{dw}{de},\ Q=\frac{dw}{df},\ R=\frac{dw}{dg},\ S=\frac{dw}{da},\ T=\frac{dw}{db},\ U=\frac{dw}{dc}. \quad . \quad . \quad . \quad (118)$$

Hence, for an isotropic solid, (117) gives the expression (25) of § 21, which we have used above.

71. To interpret the coefficients m and n in connexion with elementary ideas as to the elasticity of the solid; first let $a=b=c=0$, and $e=f=g=\frac{1}{3}\delta$; in other words, let the substance experience a uniform dilatation, in all directions, producing an expansion of volume from 1 to $1+\delta$. In this case (117) becomes

$$w=\tfrac{1}{2}(m-\tfrac{1}{3}n)\delta^2;$$

and we have

$$\frac{dw}{d\delta}=(m-\tfrac{1}{3}n)\delta.$$

Hence $(m-\frac{1}{3}n)\delta$ is the normal force per unit area of its surface required to keep any portion of the solid expanded to the amount specified by δ. Thus $m-\frac{1}{3}n$ measures the elastic force called out by, or the elastic resistance against, change of volume: and viewed as a *coefficient of elasticity*, it may be called the *elasticity of volume*. What is commonly called the "compressibility" is measured by $\frac{1}{m-\frac{1}{3}n}$.

And let next $e=f=g=b=c=0$; which gives

$$w=\tfrac{1}{2}na^2;\text{ and, by (118), } S=na.$$

This shows that the tangential force per unit area required to produce an infinitely small shear*, amounting to a, is na. Hence n measures the innate power with which the body resists change of shape, and returns to its original shape when force has been applied to change it; that is to say, it measures *the rigidity* of the substance.

[NOTE added, December 1863].

Since this paper was communicated to the Royal Society, the author has found that the solution of the most difficult of the problems dealt with in it, which is the determination of the effect produced on a spherical shell by a prescribed application of force to its outer and inner surfaces, had previously been given by LAMÉ in a paper published in LIOUVILLE's Journal for 1854, under the title "Mémoire sur l'Équilibre l'élasticité des enveloppes sphériques." In the same paper LAMÉ shows how to take into account the effect of internal force, but does not solve the problem thus presented except for the simple cases of uniform gravity and of centrifugal force. The form in which the analysis has been applied in the present paper is very different from that chosen by LAMÉ (who uses throughout polar coordinates); but the principles are essentially the same, being merely those of spherical harmonic analysis, applied to problems presenting peculiar and novel difficulties.

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* THOMSON and TAIT's 'Natural Philosophy,' § 171.

XXIX. *First Analysis of One Hundred and Seventy-seven Magnetic Storms, registered by the Magnetic Instruments in the Royal Observatory, Greenwich, from* 1841 *to* 1857. *By* GEORGE BIDDELL AIRY, *Astronomer Royal.*

Received November 28,—Read December 17, 1863.

1. IN a paper which the Royal Society have printed in their Philosophical Transactions for 1862, I gave a series of curves exhibiting to the eye the diurnal inequalities of Terrestrial Magnetism in the three directions of Westerly Force, Northerly Force, and Nadir Force, as inferred from eye-observations and photographic registers at the Royal Observatory from 1841 to 1857. The paper, or the works to which it refers, exhibits also the secular change and the annual inequality through that period, and the lunar inequalities as inferred from the period 1848 to 1857. These results were obtained by excluding the observations of certain days (of whch a list was given) on which the motions of the magnetometers were so violent that it was difficult to draw a mean curve through the magnetic curve of the day. In the present paper I propose to give the principal results deducible from the days omitted in the former paper. But before entering into the details of the numerical investigations, I think it desirable to explain the principles upon which both parts of the investigations have been conducted.

2. The methods commonly employed in late years for measuring and classifying the effects of magnetic disturbance have been, in my judgment, very valuable to the science, especially in its earlier stages. But familiarity through many past years with magnetic photograms has strongly impressed me with the feeling that a different method ought now to be employed, taking account of relations of disturbances which perhaps could not be known at the introduction of the ancient method. I may thus describe the general ideas which have guided me:—First, that there is no such thing as a day really free from disturbance, and no reason in the nature of things for separating one or more days from the general series. There is abundant reason for such separation on the ground of convenience of reduction; but when the reduction has been effected by suitable process, the results of the separated days ought to be combined with those of the unseparated days in the formation of general means (the numerical necessity for which I propose to consider in the close of this paper),—the reduction of the separated days serving also to throw great light upon the nature of the acting forces on those days, which forces in all probability are acting, though in different degrees, on other days. Second, that, with our present knowledge of the character of magnetic disturbances, I cannot think myself justified in separating any single magnetic indication, or any series of indications defined only by their magnitude; nor do I entertain the belief that any

special value could attach to the results which I might derive from observations from which such indications have been removed. The study of the photograms shows clearly that the successive indications at successive moments of the same day are a connected series; there is no such thing as a sudden display of force in any element; the sharpest salience which is exhibited on a generally smooth curve occupies at least an hour in its development (I believe, never less, although the individual saliences in a continued storm are of shorter duration), and during this time the force has been gradually increasing and gradually diminishing. Under these circumstances, I cannot think it right that I should cut off a part of that salience, with the belief of obtaining results, that can possess any philosophical value, from the part which is left. And I come to the conclusion that each disturbed day must be considered in its entirety, and that our attention ought to be given in the first instance to the devising of methods by which the complicated registers of each of those days, separately considered, can be rendered manageable, and in the next place to the discussion of the laws of disturbance which they may aid to reveal to us, and to the ascertaining of their effects on the general means in which they ought to be included.

3. The discrimination of the classes of days which (on the one hand) are treated by the general process in the "Results of Magnetical Observations, 1859," and of those which (on the other hand) are to be treated by the methods of this Memoir, has been effected entirely by the judgment of the Superintendent of Computations as to the certainty and accuracy with which he could draw a mean line through the disturbed curves. I do however entirely recognize the propriety of defining the "disturbed days" by some numerical limit, when it can be conveniently done: but, the day being defined, I then think that the entire disturbed day or storm ought to be treated as a coherent whole; and that the laws of disturbance and the amalgamation with general means ought to be deduced from it, as already mentioned, without reference to any numerical limit.

4. The records of disturbances from 1848 to 1857 are taken from the photograms; and the value of these, I believe, is unimpeachable. The instruments appear to have been in the highest state of efficiency; I do not think that there is the least doubt on the indications of any disturbed day. And (as the effect of adjustments made expressly for that purpose) the traces of the most violent motions are in general perfectly preserved—an advantage which is possessed, I believe in a peculiar degree, by the photograms of the Royal Observatory. Some sheets may be lost from defects in the paper, defects in the chemical process, &c.; but none, I believe, from rapidity and violence of motion of the magnets. The indications for every salient point of the curves have been translated into numbers which are printed in the "Results of Magnetical Observations" for each year; and those numbers are used as the basis of the following calculations. For the years 1841–1847, in which observations were made by eye, it will be seen in the printed Observations that no opportunity was lost, on the slightest appearance of disturbance, of following most carefully the indications of all the magnetometers: and in fact, as regards both the number of days of such observations and the number of

observations on each day, the observations taken are far more numerous than was necessary. The judgment of the Superintendent has been exercised in making such a selection of days and such a limitation of records for each day as should make the adopted register for the period 1841–1847 harmonize well with that for the period 1848–1857.

In the following investigations, whenever one instrument has exhibited such signs of disturbance that its indications were thought unfit for treatment in the former Reductions and are therefore included in this Analysis, the indications of the two other instruments are also included in this Analysis.

5. In deciding on the method of making the disturbed curves more manageable, the following was my train of ideas. As the photographic curve usually consists of a series of lines (very little curved) highly inclined to the time-abscissa and leading alternately upwards and downwards, if each of these lines be bisected and the bisecting points be joined, the joining lines will form a polygon of much less violent character than the original. If these joining lines be bisected and the bisecting points joined, we shall have a polygon of still smoother character, with angles sensibly corresponding to the original times, excepting only the first and the last. If the double process be repeated, the polygon will be still smoother, but wanting points corresponding to the two first and two last observations. And thus we shall have a mean curve containing all the long waves of the original curve, and freed from the irregularities of short period, whose values, however, can be measured. Numerically, each step of the process is represented by taking, for the numerical value of a new ordinate, the arithmetical mean of the numerical values of adjacent ordinates, or, still more easily, by adding the adjacent ordinates, adding the adjacent sums thus formed, and dividing by 4, and repeating this operation. An instance will make this process clear.

Readings for Northerly Force (corrected for temperature) in the Magnetic Storm of 1854, March 6.

Göttingen Time	Reading	1st Sum	2nd Sum.	¼th.	3rd Sum.	4th Sum.	⅛th or Adopted
h m							
0 0	·1153	·2306					
1 8	1153	2322	·4628	·1157	·2314		
1 32	1169	2308	4630	1157	2308	·4622	·1155
1 50	1139	2295	4603	1151	2301	4609	1152
2 7	1156	2306	4601	1150	2304	4605	1151
2 30	1150	2309	4615	1154	2309	4613	1153
2 44	1159	2312	4621	1155	2310	4619	1155
2 58	1153	2310	4622	1155	2311	4621	1155
3 30	1157	2314	4624	1156	2315	4626	1157
4 5	1157	2320	4634	1159	2320	4635	1159
4 12	1163	2323	4643	1161	2323	4643	1161
4 45	1160	2325	4648	1162	2323	4646	1161
5 23	1165	2320	4645	1161	2313	4636	1159
6 15	1155	2286	4606	1152	2298	4611	1153
6 39	1131	2299	4585	1146	2303	4601	1150
7 6	1168	2329	4628	1157	2320	4623	1156
7 15	1161	2324	4653	1163	2321	4641	1160
7 24	1163	2309	4633	1158	2310	4631	1158
7 32	1146	2299	4608	1152	2298	4608	1152
7 45	1153	2284	4583	1146	2289	4587	1147
8 25	1131	2287	4571	1143	2292	4581	1145
9 17	1156	2308	4595	1149	2305	4597	1149
9 45	1152	2316	4624	1156	2315	4620	1155
10 40	1164	2318	4634	1159	2324	4639	1160
11 23	1154	2341	4659	1165	2340	4664	1166
11 50	1187	2358	4699	1175	2350	4690	1172
12 8	1171	2343	4701	1175	2344	4694	1174
12 20	1172	2331	4674	1169	2333	4677	1169
12 39	1159	2325	4656	1164	2327	4660	1165
13 8	1166	2328	4653	1163	2325	4652	1163
13 17	1162	2320	4648	1162	2326	4651	1163
13 45	1158	2335	4655	1164	2334	4660	1165
20 0	1177	2345	4680	1170	2340	4674	1169
21 0	1168	2335	4680	1170	2336	4676	1169
22 3	1167	2328	4663	1166	2328	4664	1166
22 25	1161	2321	4649	1162	2319	4647	1162
22 46	1160	2308	4629	1157	2308	4627	1157
22 55	1148	2296	4604	1151	2291	4599	1150
23 4	1148	2265	4561	1140	2272	4563	1141
23 30	1117	2261	4526	1132			
23 59	1144						

The Adopted Numbers are those to be compared with the Original Reading, in order to ascertain what portion of the Original Reading is to be ascribed to Irregularities: and the Adopted Numbers are also to be compared with the Monthly Means deduced from the days of easy reduction, in order to ascertain what portion is to be considered as Wave-Disturbance. Thus we finally obtain the following separation of numbers, whose aggregate represents the Original Reading:—

Component parts of Northerly Force in the Magnetic Storm of 1854, March 6.

Göttingen Time.	Monthly Mean.	Wave-Disturbance.		Irregularities.	
h m					
1 32	·1158		—·0003	+·0014	
1 50	1158		— 6		—·0013
2 7	1158		— 7	+ 5	
2 30	1160		— 7		— 3
2 44	1160		— 5	+ 4	
2 58	1161		— 6		— 2
3 30	1162		— 5	0	
4 5	1162		— 3		— 2
4 12	1162		— 1	+ 2	
4 45	1162		— 1		— 1
5 23	1162		— 3	+ 6	
6 15	1162		— 9	+ 2	
6 39	1163		— 13		— 19
7 6	1163		— 7	+ 12	
7 15	1163		— 3	+ 1	
7 24	1163		— 5	+ 5	
7 32	1163		— 11		— 6
7 45	1163		— 16	+ 6	
8 25	1163		— 18		— 14
9 17	1163		— 14	+ 7	
9 45	1164		— 9		— 3
10 40	1164		— 4	+ 4	
11 23	1164	+·0002			— 12
11 50	1165	+ 7		+ 15	
12 8	1165	+ 9			— 3
12 20	1165	+ 4		+ 3	
12 39	1164	+ 1			— 6
13 8	1164		— 1	+ 3	
13 17	1164		— 1		— 1
13 45	1164	+ 1			— 7
20 0	1168	+ 1		+ 8	
21 0	1161	+ 8			— 1
22 3	1156	+ 10		+ 1	
22 25	1156	+ 6			— 1
22 46	1155	+ 2		+ 3	
22 55	1155		— 5		— 2
23 4	1155		— 14	+ 7	

The disturbance of Horizontal Force is thus separated into two well-distinguished parts. One part consists of five long waves, alternately — and +. The other part consists of irregularities of short period, which do not show the least symptom of disappearing at the disappearance of the waves, and appear to have nothing in common with them except the connexion of both with the same general Magnetic Storm.

6. For fully understanding the import of these numbers, it will perhaps be necessary to study the succession of numbers in each individual instance. In this First Analysis, I have proceeded, as the first step, to take the means that appear to be most valuable. As regards the Waves, I have taken separately the mean of the wave-disturbances through each wave. But as this quantity gives little information unless taken in conjunction with the time through which it acts, I have multiplied it by the length of the wave in hours; and this product I have distinguished by the technical term *Fluctuation*. The

following is now an Epitome of the Magnetic Storm which we have had under consideration.

Epitome of Disturbances of Northerly Force in the Magnetic Storm of 1854, March 6.

Times of beginning and end of wave.		Length of wave in hours.	Mean Wave-disturbance.	Fluctuation.	Aggregate Fluctuation.	Sum of Hours.	Mean Disturbance.	Number of Irregularities.	Mean Period of Irregularity.	Mean value of Irregularity.
h m	h m	h				h			h	
0 0	11 1	11·02	—·0007	—·0077	—·0033	23·98	—·0001	22	0·50	±·0006
11 1	12 54	1·88	+ 5	+ 9				5	0·38	8
12 54	13 31	0·62	— 1	— 1				2	0·31	2
13 31	22 51	9·33	+ 5	+ 47				6	1·56	4
22 51	23 59	1·13	— 10	— 11				2	0·57	4

The disturbances of Westerly Force and Nadir Force are treated in the same way—the values of disturbance, &c. being converted, at convenient stages, into values expressed in terms of whole Northerly Force.

The numbers contained in these Epitomes serve as bases for the investigations which follow. The Epitomes themselves, though greatly reduced from the voluminous calculations on which they are founded, are far too extensive to be included in this Memoir: they will probably be printed in the Greenwich Observations.

7. Treating the Waves as the first subject, I take in the first instance the algebraical aggregate of the Fluctuations for each separate Magnetic Storm. In Table I., the first or longest of the three Tables which follow, every recorded storm is included; and in the second, or Table II., these are all collected to form annual aggregates. But as the days of record do not strictly coincide for the three instruments, partly from accidents in the chemical preparation of the photographic paper, &c., but more particularly from the experimental state of the Vertical-Force Instrument during a part of the year 1848, I have thought it desirable to form Table III. from the observations which are strictly comparable. In regard to the last columns of each department of Table I., and the last lines of Tables II. and III., it will be remarked that the "Fluctuation" is a product of number of hours by Magnetic Disturbance, and therefore, for the Mean Disturbance, the Aggregate of Fluctuations must be divided by the Sum of Hours.

TABLE I.—Algebraic Sums of Magnetic Fluctuations (in terms of Horizontal Force) on Days of Great Magnetic Disturbance.

Year, Month, and Day.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.
1841.									
Sept. 24	13·9	−0·0022	− 2	12·0	−0·0456	−38	14·0	−0·0392	− 28
25	22·0	− ·0200	− 9	12·9	− ·0054	− 4	11·3	+ ·2580	+229
27	8·2	− ·0270	−33	8·2	− ·0097	−12	8·2	+ ·0670	+ 82
Oct. 25	22·0	− ·0417	−19	22·0	− ·0484	−22	20·2	+ ·0226	+ 11
Nov. 18	17·9	− ·0735	−41	17·9	− ·0125	− 7	18·0	− ·0323	− 18
19	22·8	+ ·0016	+ 1	24·0	− ·0276	−12	23·7	− ·0379	− 16
Dec. 3	12·7	+ ·0088	+ 7	12·7	− ·0205	−16	10·9	+ ·0424	+ 39
14	10·0	− ·0296	−30	10·0	− ·0130	−13	10·0	+ ·0621	+ 62
1842.									
Jan. 1	6·7	−0·0240	−36	6·7	+0·0387	+58	6·1	+0·0061	+ 10
Feb. 24	8·0	− ·0090	−11	8·0	− ·0400	−50	8·0	+ ·0014	+ 2
April 14	7·6	+ ·0214	+28	7·4	− ·0423	−57	8·0	− ·0784	− 98
15	23·1	+ ·0087	+ 4	24·0	− ·1416	−59	22·2	− ·0061	− 3
July 1	7·7	+ ·0008	+ 1	7·7	− ·0178	−23	8·0	+ ·0135	+ 17
2	13·6	− ·0523	−39	13·4	− ·0138	−10	13·2	+ ·0608	+ 46
3	9·7	− ·0003	0	10·0	− ·0650	−65	10·0	− ·0289	− 29
Nov. 10	14·2	− ·0340	−24	14·2	− ·0710	−50	14·2	+ ·0185	+ 13
21	12·0	− ·0054	− 5	12·0	− ·0132	−11	12·0	− ·0312	− 26
Dec. 9	10·0	− ·0220	−22	10·0	− ·0187	−19	10·0	+ ·0311	+ 31
1843.									
Jan. 2	10·0	+0·0180	+18	10·0	−0·0180	−18	10·0	−0·0261	− 26
Feb. 6	6·0	+ ·0002	0	......	....	......	......	....	......
16	4·0	− ·0044	−11	4·0	− ·0048	−12	......	....	......
24	11·6	− ·0129	−11	11·6	− ·0189	−16	11·6	+ ·0031	+ 3
May 6	4·4	− ·0216	−49	4·1	− ·0226	−55	4·2	− ·0064	− 16
July 24	13·7	+ ·0145	+11	13·7	− ·0227	−17	14·0	+ ·0140	+ 10
25	6·0	+ ·0210	+35	6·0	+ ·0002	0	5·6	+ ·0329	+ 59
1844.									
Mar. 29	15·7	−0·0140	− 9	15·7	−0·0305	−19	16·0	−0·0448	− 28
30	12·0	− ·0097	− 8	12·0	− ·0126	−11	11·6	+ ·0017	+ 2
Oct. 1	6·0	− ·0156	−26	6·0	− ·0198	−33	6·0	+ ·0018	+ 3
20	......	....	......	8·0	− ·0224	−28	8·0	− ·0904	−113
Nov. 16	10·0	+ ·0112	+11	10·0	− ·0280	−28	9·7	+ ·0398	+ 41
22	8·0	+ ·0248	+31	8·0	− ·0196	−25	8·0	− ·0052	− 7
1845.									
Jan. 9	10·0	−0·0290	−29	10·0	−0·0440	−44	10·0	+0·080	+ 8
Feb. 24	15·7	− ·0198	−13	16·2	− ·0177	−11	16·2	− ·0211	− 13
Mar. 26	14·0	− ·0210	−15	14·0	− ·0090	− 6	14·0	− ·0070	− 5
Aug. 29	6·2	− ·0037	− 6	6·1	− ·0024	− 4	6·2	− ·0062	− 10
Dec. 3	14·1	− ·0022	− 2	14·2	− ·0667	−47	14·2	+ ·0439	+ 31
1846.									
May 12	10·0	−0·0009	− 1	10·0	−0·0044	− 4	10·0	−0·0040	− 4
July 11	......	....	......	10·0	− ·0092	− 9	3·4	− ·0044	− 13
Aug. 6	11·9	+ ·0099	+ 8	11·9	− ·0037	− 3	12·0	− ·0015	− 1
7	22·0	+ ·0286	+13	22·0	− ·0013	− 1	21·9	+ ·0051	+ 2
24	14·0	− ·0107	− 8	12·0	− ·0036	− 3	16·0	− ·0160	− 10
25	16·0	− ·0096	− 6	16·0	+ ·0050	+ 3	14·2	− ·0071	− 5

TABLE I. (continued).

Year, Month, and Day.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.
1846 (cont$^{\text{d}}$).									
Aug. 28	8·8	−0·0058	− 7	8·7	−0·0075	− 9	8·8	−0·0114	− 13
Sept. 4	15·9	+ ·0091	+ 6	15·8	− ·0116	− 7	16·0	+ ·0208	+ 13
5	13·0	+ ·0056	+ 4	12·9	− ·0062	− 5	12·3	+ ·0274	+ 22
10	13·9	+ ·0005	0	13·8	+ ·0029	+ 2	14·0	+ ·0140	+ 10
11	23·8	− ·0030	− 1	23·8	− ·0148	− 6	23·7	+ ·0292	+ 12
21	19·9	− ·0286	−14	19·8	− ·0183	− 9	20·0	− ·0100	− 5
22	14·0	− ·0158	−11	14·0	− ·0305	−22	13·9	+ ·0225	+ 16
Oct. 2	6·0	− ·0156	−26	6·0	− ·0102	−17	6·0	+ ·0060	+ 10
7	17·7	− ·0073	− 4	17·7	− ·0509	−29	18·0	− ·0378	− 21
8	12·0	+ ·0059	+ 5	11·8	− ·0227	−19	11·8	+ ·0905	+ 77
Nov. 26	16·2	− ·0069	− 4	14·6	− ·0279	−19	16·2	− ·0002	0
Dec. 23	10·0	− ·0160	−16	10·0	+ ·0170	+17	10·0	+ ·0074	+ 7
1847.									
Feb. 24	10·0	−0·0167	−17	10·0	−0·0110	−11	9·9	−0·0030	− 3
Mar. 1	8·0	+ ·0082	+10	8·0	− ·0047	− 6	8·0	+ ·0616	+ 78
19	20·0	− ·0121	− 6	20·0	− ·0846	−42	18·2	− ·0783	− 43
April 3	8·0	− ·0240	−30	8·0	− ·0061	− 8	8·0	+ ·0264	+ 33
7	16·0	− ·0046	− 3	16·0	− ·0493	−31	16·0	− ·0171	− 11
21	6·0	+ ·0004	+ 1	5·5	− ·0120	−22	6·0	+ ·0156	+ 26
May 7	8·0	+ ·0344	+43	8·0	− ·0032	− 4	10·0	− ·0100	− 10
June 24	4·0	+ ·0109	+27	......	....	......	......	....	......
July 9	......	....	......	4·0	− ·0352	−88	4·0	− ·0464	−116
Sept. 24	18·0	+ ·0082	+ 5	18·0	+ ·0332	+18	17·0	+ ·0435	+ 26
26	9·8	− ·0159	−16	9·8	− ·0401	−41	10·0	− ·0260	− 26
27	10·0	+ ·0008	+ 1	10·0	− ·0300	−30	9·7	+ ·0603	+ 62
Oct. 22	5·8	+ ·0033	+ 6	5·8	− ·0403	−70	6·0	− ·0108	− 18
23*(1st)	12·0	+ ·0091	+ 8	12·0	− ·0132	−11	11·6	+ ·0988	+ 85
23 (2nd)	2·0	− ·0025	−13	1·9	− ·0030	−16	2·0	+ ·0016	+ 8
24	23·3	+ ·0137	+ 6	23·3	− ·2088	−90	23·7	+ ·0538	+ 23
25	10·0	− ·0093	− 9	10·0	− ·0150	−15	9·5	+ ·0654	+ 69
Nov. 22	14·0	+ ·0120	+ 9	14·0	− ·0304	−22	15·2	− ·0421	− 28
Dec. 17	22·0	+ ·0157	+ 7	22·0	− ·0268	−12	14·0	+ ·1260	+ 90
18	12·0	− ·0120	−10	12·0	− ·0193	−16	......	....	......
19	10·0	+ ·0175	+17	10·0	− ·0910	−91	......	....	......
20	18·0	− ·0132	− 7	18·0	− ·0581	−32	......	....	......
1848.									
Jan. 16	14·2	−0·0047	− 3	10·3	−0·0340	−33	......	....	......
28	14·0	− ·0210	−15	19·1	+ ·0217	+11	......	....	......
Feb. 20	22·5	− ·0192	− 9	9·1	− ·0335	−37	......	....	......
21	16·9	+ ·0047	+ 3	22·8	− ·0742	−33	......	....	......
22	4·0	− ·0043	−10	4·0	− ·0125	−31	......	....	......
23	18·0	− ·0113	− 6	8·6	− ·0024	− 3	......	....	......
24	20·8	+ ·0335	+16	22·8	− ·0503	−22	......	....	......
Mar. 17	3·3	+ ·0077	+23	5·2	− ·0067	−13	......	....	......
20	14·2	− ·0126	− 9	11·5	− ·0309	−27	......	....	......
April 7	11·4	+ ·0045	+ 4	4·1	− ·0074	−18	......	....	......
May 18	9·1	− ·0016	− 2	8·4	+ ·0105	+12	......	....	......
July 11	16·4	− ·0054	− 3	19·4	− ·0415	−21	......	....	......
Oct. 18	11·5	+ ·0024	+ 2	10·6	− ·0271	−26	7·1	−0·0970	−136

* On October 23, 1847, all the observations were interrupted during 10 hours.

TABLE I. (continued).

Year, Month, and Day.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.
1848 (cont^{d}.)									
Oct. 23 ……	10·6	−0·0060	− 6	9·9	−0·0066	− 7	……	………	……
25 ……	17·5	− ·0034	− 2	18·5	+ ·0025	+ 1	……	………	……
29 ……	16·1	− ·0041	− 2	4·4	·0000	0	……	………	……
Nov. 17 ……	20·0	− ·0003	− 0	19·3	− ·1955	−101	18·9	+0·0601	+ 32
18 ……	14·2	− ·0266	−19	10·3	− ·0201	− 20	4·2	+ ·0208	+ 50
Dec. 17 ……	9·5	+ ·0011	+ 1	5·5	− ·0194	− 35	10·3	− ·0744	− 72
1849.									
Oct. 30 ……	22·9	−0·0129	− 6	22·8	−0·0160	− 7	22·9	−0·3484	−152
Nov. 27 ……	23·1	+ ·0291	+13	22·4	− ·0258	− 12	……	………	……
1850.									
Feb. 22 ……	23·7	−0·0076	− 3	23·5	−0·0088	− 4	23·3	+0·2030	+ 87
23 ……	23·6	+ ·0034	+ 1	23·3	− ·0327	− 14	23·5	− ·0279	− 12
Mar. 31 ……	23·9	− ·0104	− 4	23·5	− ·0375	− 16	23·3	·0000	0
May 7 ……	……	………	……	23·9	− ·0021	− 1	……	………	……
June 13 ……	24·0	− ·0249	−10	23·4	− ·0062	− 3	23·7	− ·3882	−164
Oct. 1 ……	23·2	+ ·0487	+21	22·7	− ·0522	− 23	22·0	− ·1188	− 54
2 ……	23·5	+ ·0401	+17	23·6	− ·0495	− 21	22·6	− ·0098	− 4
1851.									
Jan. 16 ……	23·6	−0·0244	−10	23·4	+0·0125	+ 5	22·9	−0·1009	− 44
19 ……	24·0	+ ·0175	+ 7	24·0	+ ·0328	+ 14	23·2	− ·1367	− 59
Feb. 18 ……	23·1	+ ·0082	+ 4	23·1	− ·0287	− 12	23·4	+ ·1382	+ 59
Sept. 3 ……	18·6	+ ·0451	+24	23·4	− ·0426	− 18	23·3	+ ·1004	+ 43
4 ……	23·4	+ ·0168	+ 6	23·9	− ·0232	− 9	23·0	+ ·1769	+ 77
6 ……	24·0	− ·0465	−19	23·8	− ·0238	− 10	23·4	− ·0508	− 22
7 ……	23·0	− ·0052	− 2	23·9	− ·0576	− 24	23·8	− ·0481	− 20
29 ……	22·7	− ·0550	−24	23·9	− ·0474	− 20	22·4	− ·3838	−171
Oct. 2 ……	23·7	+ ·0037	+ 2	24·0	− ·0632	− 26	23·6	− ·1364	− 58
28 ……	23·1	+ ·0152	+ 7	22·7	− ·0244	− 11	22·9	− ·1834	− 80
Dec. 6 ……	23·3	− ·0307	−13	23·4	− ·1264	− 54	22·6	+ ·0815	+ 36
28 ……	23·2	+ ·0083	+ 4	23·9	− ·0217	− 9	23·2	− ·0950	− 41
29 ……	18·5	− ·0360	−20	22·4	− ·0627	− 28	21·4	+ ·0191	+ 9
1852.									
Jan. 4 ……	23·8	+0·0245	+10	22·0	+0·0968	+ 44	23·5	−0·0137	− 6
19 ……	22·6	+ ·0073	+ 3	23·2	− ·0336	− 14	22·3	− ·1206	− 54
Feb. 14 ……	22·3	− ·0073	− 3	23·2	+ ·0771	+ 33	22·1	− ·0229	− 10
15 ……	23·7	+ ·0006	0	23·7	− ·0150	− 6	23·4	− ·1763	− 75
17 ……	23·3	− ·0021	− 1	23·5	− ·0449	− 19	23·0	+ ·2517	+109
18 ……	23·9	+ ·0031	+ 1	23·7	− ·0587	− 25	23·9	+ ·4422	+185
19 ……	21·0	− ·0261	−12	23·5	− ·0492	− 21	23·0	− ·2596	−113
20 ……	22·9	− ·0078	− 3	23·7	− ·0371	− 16	23·1	− ·1594	− 69
21 ……	20·0	− ·0186	− 9	23·8	− ·0604	− 25	22·5	− ·1735	− 77
April 20 ……	23·9	+ ·0226	+ 9	24·0	− ·0790	− 33	22·5	− ·1508	− 67
May 19 ……	16·0	− ·0485	−30	23·5	+ ·0047	+ 2	21·3	+ ·0595	+ 28
20 ……	6·8	− ·0068	−10	23·4	− ·0068	− 3	13·8	+ ·0605	+ 44
June 11 ……	22·5	− ·0030	− 1	23·5	− ·0310	− 13	22·3	− ·4354	−195
16 ……	23·3	− ·0126	− 5	23·5	+ ·0177	+ 8	……	……	……
July 10 ……	21·1	− ·0362	−17	21·3	+ ·0042	+ 2	22·3	− ·0302	− 14
Nov. 11 ……	23·9	+ ·0057	+ 2	23·2	− ·0235	− 10	23·0	− ·3236	−141
13 ……	23·7	+ ·0114	+ 5	23·3	− ·0352	− 15	21·3	− ·1638	− 77

TABLE I. (continued).

Year, Month, and Day.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.
1853.									
Jan. 10	22·7	−0·0063	− 3	22·6	−0·0200	− 9	......	....	......
Mar. 7	23·8	− ·0171	− 7	23·9	− ·0224	− 9	23·8	+0·3353	+141
8	23·0	− ·0073	− 3	23·9	− ·0072	− 3	22·5	+ ·5298	+236
11	......	....	......	23·9	− ·0253	−11	23·2	+ ·3529	+152
May 2	22·0	− ·0101	− 5	23·5	− ·0657	−28	22·9	+ ·3595	+157
3	23·7	− ·0074	− 3	23·7	− ·0552	−23	23·0	+ ·3424	+149
24	23·3	+ ·0092	+ 4	23·6	+ ·0300	+13	23·7	+ ·2269	+ 96
June 22	23·8	− ·0157	− 7	23·7	− ·0029	− 1	23·2	− ·1487	− 64
July 12	23·8	− ·0019	− 1	24·0	− ·0090	− 4	23·4	+ ·0097	+ 4
Aug. 21	......	....	......	......	....	......	23·7	− ·0617	− 26
Sept. 1	23·5	+ ·0220	+ 9	22·8	− ·0032	− 1	23·5	− ·0550	− 23
2	23·6	+ ·0037	+ 2	23·9	− ·0616	−26	23·7	+ ·0331	+ 14
Oct. 1	......	....	......	24·0	− ·0312	−13	......	....	......
2	......	....	......	24·0	− ·0336	−14	......	....	......
25	23·5	− ·0119	− 5	24·0	− ·0092	− 4	24·0	+ ·3093	+129
Nov. 9	23·5	+ ·0037	+ 2	23·7	− ·0474	−20	23·5	− ·0578	− 25
Dec. 6	23·5	+ ·0134	+ 6	24·0	− ·1079	−45	23·3	+ ·0183	+ 8
21	23·4	+ ·0044	+ 2	23·0	− ·0071	− 3	23·3	− ·1790	− 77
1854.									
Jan. 8	23·8	+0·0029	+ 1	23·4	+0·0246	+11	23·7	−0·1089	− 46
20*	23·9	− ·0043	− 2	23·5	− ·0096	− 4	7·0	− ·0104	− 15
(resumed) 20	......	....	......	......	....	......	14·1	− ·0550	− 39
Feb. 16	23·8	− ·0209	− 9	24·0	− ·0337	−14	23·8	− ·1165	− 49
24	23·9	− ·0145	− 6	23·3	+ ·0020	+ 1	23·7	− ·1414	− 60
25	23·9	+ ·0049	+ 2	23·9	+ ·0119	+ 5	23·7	+ ·0812	+ 34
Mar. 6	23·9	− ·0084	− 4	24·0	− ·0033	− 1	23·9	− ·1049	− 44
15	23·7	− ·0099	− 4	24·0	− ·0261	−11	23·3	− ·0030	− 1
16	23·5	− ·0034	− 1	24·0	− ·0408	−17	23·7	+ ·0498	+ 21
28	23·8	− ·0114	− 5	24·0	− ·1271	−53	22·7	+ ·1451	+ 64
April 10	23·9	− ·0076	− 3	23·9	− ·0123	− 5	22·6	− ·0851	− 38
23	23·6	+ ·0103	+ 4	24·0	− ·0196	− 8	23·7	+ ·0207	+ 9
May 25	23·6	+ ·0004	0	24·0	+ ·0176	+ 7	23·9	− ·0653	− 27
1855.									
Mar. 12	24·0	−0·0117	− 5	23·4	−0·0506	−22	23·5	−0·2111	− 90
April 4	23·6	− ·0028	− 1	23·6	− ·0108	− 5	20·3	− ·0018	− 1
July 19	......	....	......	22·8	− ·0263	−12	23·5	− ·0101	− 4
Oct. 18	23·7	− ·0052	− 2	24·0	− ·0477	−20	23·8	+ ·1049	+ 44
† 1857.									
Feb. 26	22·6	+0·0014	+ 1	22·6	−0·0086	− 4	23·2	−0·1368	− 59
Mar. 13	23·2	+ ·0052	+ 2	......	....	......	......	....	......
May 7	24·0	+ ·0207	+ 9	24·0	− ·0418	−17	22·6	− ·3191	−141
10	23·8	+ ·0056	+ 2	22·1	+ ·0270	+12	24·0	− ·0147	− 6
Sept. 3	24·0	− ·0124	− 5	24·0	− ·0259	−11	24·0	− ·4177	−174
Nov. 12	23·3	+ ·0163	+ 7	23·3	− ·0006	0	......	....	......
16	21·3	− ·0073	− 3	23·3	− ·0036	− 2	......	....	......
17	22·8	− ·0049	− 2	22·5	− ·0277	−12	......	....	......
Dec. 16	24·0	− ·0021	− 1	24·0	− ·0304	−13	24·0	− ·2230	− 93
17	22·8	− ·0086	− 4	22·6	− ·0881	−39	24·0	+ ·0427	+ 18

* On Jan. 20, 1854, the observations of the Vertical-Force Instrument were interrupted during 3 hours.

† In 1856 there were no days of Great Magnetic Disturbance throughout the year.

The last figure in the "Algebraic Mean of Disturbance" is in the fourth decimal place of Horizontal Force.

Table II.—Algebraic Sums of Magnetic Fluctuations (in terms of Horizontal Force) for each Year from 1841 to 1857, including all days of Record of Great Magnetical Disturbance.

Year.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.
1841	129·47	—·1836	—14	119·63	— ·1827	—15	116·19	+ ·3427	+ 29
1842	112·57	—·1161	—10	113·34	— ·3847	—34	111·74	— ·0132	— 1
1843	55·72	+·0148	+ 3	49·39	— ·0868	—18	45·40	+ ·0175	+ 4
1844	51·74	—·0033	— 1	59·70	— ·1329	—22	59·29	— ·0971	— 16
1845	60·00	—·0757	—13	60·41	— ·1398	—23	60·52	+ ·0176	+ 3
1846	244·86	—·0606	— 2	250·89	— ·1979	— 8	247·99	+ ·1305	+ 5
1847	246·75	+·0239	+ 1	246·29	— ·7489	—30	198·75	+ ·3193	+ 16
1848	264·18	—·0666	— 3	223·83	— ·5274	—24	40·65	— ·0905	— 22
1849	46·00	+·0162	+ 4	45·25	— ·0418	— 9	22·92	— ·3484	—152
1850	141·79	+·0493	+ 3	163·80	— ·1890	—12	138·34	— ·3417	— 25
1851	294·04	—·0830	— 3	305·70	— ·4764	—16	299·17	— ·6190	— 21
1852	364·65	—·0938	— 3	395·76	— ·2739	— 7	353·07	—1·2159	— 34
1853	327·14	—·0213	— 1	402·06	— ·4789	—12	350·67	+2·0150	+ 57
1854	285·10	—·0619	— 2	285·82	— ·2164	— 8	279·75	— ·3937	— 14
1855	71·37	—·0197	— 3	93·75	— ·1354	—14	91·03	— ·1181	— 13
1856	0·00	·0000	......	0·00	·0000	......	0·00	·0000	......
1857	231·53	+·0139	+ 1	208·37	— ·1997	—10	141·73	—1·0686	— 75
Sum	2926·91	—·6675		3023·99	—4·4126		2557·21	—1·4636	
Mean Disturbance	—·00023			— ·00146			— ·00057		

Table III.—Algebraic Sums of Magnetic Fluctuations (in terms of Horizontal Force) for each Year from 1841 to 1857, including only those days of Great Magnetic Disturbance in which Records were made by the three Instruments.

Year.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.	Number of Hours.	Algebraic Aggregate of Fluctuations.	Algebraic Mean of Disturbance.
1841	129·47	—·1836	—14	119·63	— ·1827	—15	116·19	+ ·3427	+ 29
1842	112·57	—·1161	—10	113·34	— ·3847	—34	111·74	— ·0132	— 1
1843	45·72	+·0190	+ 4	45·39	— ·0820	—18	45·40	+ ·0175	+ 4
1844	51·74	—·0033	— 1	51·70	— ·1105	—22	51·29	— ·0067	— 1
1845	60·00	—·0757	—13	60·41	— ·1398	—23	60·52	+ ·0176	+ 3
1846	244·86	—·0606	— 2	240·89	— ·1887	— 8	244·61	+ ·1349	+ 5
1847	202·69	+·0207	+ 1	202·19	— ·5453	—27	194·75	+ ·3657	+ 19
1848	55·17	—·0234	— 4	45·74	— ·2621	—58	40·65	— ·0905	— 22
1849	22·92	—·0129	— 6	22·84	— ·0160	— 7	22·92	— ·3484	—152
1850	141·79	+·0493	+ 3	139·88	— ·1869	—13	138·34	— ·3417	— 25
1851	294·04	—·0830	— 3	305·70	— ·4764	—16	299·17	— ·6190	— 21
1852	341·34	—·0812	— 2	372·27	— ·2612	— 7	353·07	—1·2159	— 34
1853	304·41	—·0150	— 1	308·40	— ·3688	—12	303·72	+1·7238	+ 57
1854	285·10	—·0619	— 2	285·82	— ·2164	— 8	279·75	— ·3937	— 14
1855	71·37	—·0197	— 3	70·97	— ·1091	—14	67·58	— ·1080	— 16
1856	0·00	·0000	0	0·00	·0000	0	0·00	·0000	0
1857	141·04	+·0046	0	139·26	— ·1678	—12	141·73	—1·0686	— 75
Sum	2504·23	—·6428		2524·42	—3·6984		2471·43	—1·6035	
Mean Disturbance	—·00026			— ·00147			— ·00065		

8. The most remarkable of the results of these Tables is, not only that upon the whole the Algebraic Aggregate of Fluctuations for the Northerly Force is negative (which has been previously recognized), but that it is negative in every separate year. It will be seen in Table I. that on some separate days the Aggregate of Fluctuations is positive, but the number of days is only 22, in opposition to 155 with negative Aggregates.

The Aggregate for the Westerly Force is also negative: and though the different years do not consent in the same way as for the Northerly Force, yet their discordance is not so great as to justify us in setting aside this indication, although there may be greater doubt upon the accuracy of its value. This Aggregate (taken in comparison with that for the Northerly Force) appears to show that, on the whole, the direction of Disturbing Force is 10° to the East of South.

The Aggregate for the Nadir Force appears greater, but it is very uncertain; it might be nearly destroyed by the omission of a single year.

9. These characteristics of the directions of the disturbing forces will appear also in the following enumeration of the instances in which the first and last waves of each Magnetic Storm are affected in different ways. In comparing the numbers it must be borne in mind that, when there is only one wave, that wave is considered, in different places, both as the first and the last.

	Westerly Force.	Northerly Force.	Nadir Force.
Whole number of positive fluctuations	340	177	118
of negative fluctuations	302	277	120
Number of instances in which the first wave is +	106	58	81
in which the first wave is —	62	114	64
in which the last wave is +	100	15	63
in which the last wave is —	68	157	82

Number of Storms beginning with Westerly Force+ and Northerly Force +	35
beginning with Westerly Force+ and Northerly Force —	68
beginning with Westerly Force— and Northerly Force +	21
beginning with Westerly Force— and Northerly Force —	40
ending with Westerly Force+ and Northerly Force +	7
ending with Westerly Force+ and Northerly Force —	90
ending with Westerly Force— and Northerly Force +	8
ending with Westerly Force— and Northerly Force —	58
Number of Storms beginning with Northerly Force+and Nadir Force +	26
beginning with Northerly Force+and Nadir Force —	21
beginning with Northerly Force—and Nadir Force +	55
beginning with Northerly Force—and Nadir Force —	42
ending with Northerly Force+and Nadir Force +	6
ending with Northerly Force+and Nadir Force —	7
ending with Northerly Force—and Nadir Force +	57
ending with Northerly Force—and Nadir Force —	74

10. The following Tables, Tables IV., V., and VI., exhibit the Aggregates of Fluctuations without regard of sign. They are required in order to give information on the Mean Value of Disturbance by Wave in each of the three directions.

TABLE IV.—Absolute Sums, without regard of sign, of Magnetic Fluctuations (in terms of Horizontal Force) on Days of Great Magnetic Disturbance.

Year, Month, and Day.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Waves.	Absolute Aggregate of Fluctuations.	Absolute Mean of Disturbance.	Number of Waves.	Absolute Aggregate of Fluctuations.	Absolute Mean of Disturbance.	Number of Waves.	Absolute Aggregate of Fluctuations.	Absolute Mean of Disturbance.
1841.									
Sept. 24	2	0·0292	21	1	0·0456	38	1	0·0392	28
25	6	·0608	28	2	·0846	66	1	·2580	228
27	1	·0270	33	3	·0101	12	1	·0670	82
Oct. 25	4	·0427	19	1	·0484	22	2	·0434	21
Nov. 18	2	·0961	54	1	·0125	7	3	·0517	29
19	5	·0214	9	3	·0292	12	2	·0505	21
Dec. 3	3	·0152	12	3	·0207	16	1	·0424	39
14	2	·0312	31	1	·0130	13	1	·0621	62
1842.									
Jan. 1	1	0·0240	36	1	0·0387	58	1	0·0061	10
Feb. 24	3	·0148	19	1	·0400	50	2	·0044	5
April 14	1	·0214	28	1	·0423	57	1	·0784	98
15	3	·0311	13	1	·1416	59	2	·0465	21
July 1	4	·0100	13	1	·0178	23	3	·0137	17
2	1	·0523	31	5	·0292	22	1	·0608	46
3	2	·0283	29	1	·0650	65	2	·0545	55
Nov. 10	1	·0340	24	1	·0710	50	1	·0185	13
21	2	·0320	27	3	·0248	21	1	·0312	26
Dec. 9	1	·0220	22	3	·0189	19	1	·0311	31
1843.									
Jan. 2	1	0·0180	18	1	0·0180	18	1	0·0261	26
Feb. 6	2	·0060	10	...	......	...	...	......	...
16	1	·0044	11	1	·0048	12	...	......	...
24	3	·0131	11	3	·0201	17	2	·0093	8
May 6	1	·0216	49	1	·0226	55	2	·0110	26
July 24	2	·0149	11	2	·0247	18	1	·0140	10
25	1	·0210	35	5	·0026	4	1	·0329	59
1844.									
Mar. 29	2	0·0314	20	3	0·0309	20	1	0·0448	28
30	4	·0169	14	3	·0126	10	2	·0161	14
Oct. 1	1	·0156	26	1	·0198	33	1	·0018	3
20	...	......	...	1	·0224	28	1	·0904	113
Nov. 16	2	·0200	20	1	·0280	28	1	·0398	41
22	1	·0248	31	3	·0220	28	2	·0092	11
1845.									
Jan. 9	1	0·0290	29	1	0·0440	44	1	0·0080	8
Feb. 24	2	·0200	13	3	·0185	11	1	·0211	13
Mar. 26	1	·0210	15	3	·0104	7	1	·0070	5
Aug. 29	3	·0053	9	1	·0024	4	1	·0062	10
Dec. 3	4	·0310	22	1	·0667	47	1	·0439	31
1846.									
May 12	3	0·0073	7	2	0·0100	10	2	0·0118	12
July 11	...	......	...	2	·0118	12	1	·0044	13
Aug. 6	3	·0209	18	2	·0133	11	2	·0147	12
7	1	·0286	13	7	·0089	4	3	·0123	6
24	3	·0109	8	1	·0036	3	1	·0160	10
25	1	·0096	6	3	·0070	4	1	·0071	5
28	3	·0122	14	3	·0075	9	1	·0114	13

TABLE IV. (continued).

Year, Month, and Day.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Waves.	Absolute Aggregate of Fluctuations.	Absolute Mean of Disturbance.	Number of Waves.	Absolute Aggregate of Fluctuations.	Absolute Mean of Disturbance.	Number of Waves.	Absolute Aggregate of Fluctuations.	Absolute Mean of Disturbance.
1846 (cont^{d}).									
Sept. 4	2	0·0201	13	2	0·0156	10	1	0·0208	13
5	4	·0148	12	3	·0226	17	2	·0304	23
10	2	·0187	14	3	·0047	3	1	·0140	10
11	5	·0342	14	5	·0226	10	3	·0316	13
21	2	·0474	24	3	·0201	10	1	·0100	5
22	6	·0352	25	3	·0425	30	2	·0645	46
Oct. 2	1	·0156	26	1	·0102	17	1	·0060	10
7	2	·0295	17	3	·0523	29	1	·0378	21
8	5	·0185	15	3	·0235	20	1	·0905	77
Nov. 26	2	·0237	14	3	·0281	19	2	·0210	13
Dec. 23	1	·0160	16	1	·0170	17	2	·0112	11
1847.									
Feb. 24	2	0·0223	22	1	0·0110	11	1	0·0030	3
Mar. 1	4	·0152	19	2	·0167	21	1	·0616	77
19	9	·0339	17	2	·0960	48	2	·1269	70
April 3	1	·0240	30	3	·0067	8	1	·0264	33
7	2	·0224	14	2	·0515	32	2	·0205	13
21	3	·0028	5	1	·0120	22	1	·0156	26
May 7	1	·0344	43	2	·0114	14	1	·0100	10
June 24	1	·0109	27	...	......	...	...	......	......
July 9	...	......	...	1	·0352	88	1	·0464	116
Sept. 24	10	·0550	31	5	·1912	106	2	·1277	75
26	3	·0163	17	1	·0401	41	2	·0338	34
27	3	·0056	6	1	·0300	30	1	·0603	62
Oct. 22	4	0·0043	7	2	·0409	71	1	·0108	18
23*(1st)	10	·0235	20	3	·0564	47	2	·1158	100
23 (2nd)	1	·0025	13	1	·0030	16	1	·0016	8
24	7	·0845	36	5	·2554	110	7	·1144	48
25	5	·0103	10	1	·0150	15	1	·0654	69
Nov. 22	6	·0226	16	2	·0572	41	2	·1057	70
Dec. 17	12	·0315	14	2	·0552	25	1	·1260	90
18	4	·0162	13	1	·0193	16	...	......	......
19	2	·0433	43	1	·0910	91	...	......	......
20	10	·0434	24	6	·1277	70	...	......	......
1848.									
Jan. 16	2	0·0179	12	1	0·0340	33	...	......	......
28	2	·0306	22	4	·0375	20	...	......	......
Feb. 20	6	·0380	17	1	·0335	37	...	......	......
21	4	·0267	16	3	·0962	42	...	......	......
22	2	·0045	11	1	·0125	31	...	......	......
23	2	·0235	13	3	·0028	3	...	......	......
24	3	·0375	18	3	·0525	23	...	......	......
Mar. 17	1	·0077	23	2	·0093	18	...	......	......
20	4	·0208	15	1	·0309	27	...	......	......
Apr. 7	3	·0109	10	2	·0084	21	...	......	......
May 18	4	·0096	11	2	·0123	14	...	......	......
July 11	6	·0184	11	2	·0689	36	...	......	......
Oct. 18	4	·0288	25	3	·0323	30	1	0·0970	136
23	4	·0200	19	3	·0234	24	...	......	......
25	3	·0158	9	4	·0157	9	...	......	......
29	2	·0129	8	1	·0000	0	...	......	......

* On October 23, 1847, all the observations were interrupted during 10 hours.

TABLE IV. (continued).

Year, Month, and Day.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Waves.	Absolute Aggregate of Fluctuations.	Absolute Mean of Disturbance.	Number of Waves.	Absolute Aggregate of Fluctuations.	Absolute Mean of Disturbance.	Number of Waves.	Absolute Aggregate of Fluctuations.	Absolute Mean of Disturbance.
1848 (cont^{d}).									
Nov. 17	7	0·0683	34	3	0·1961	103	7	0·0769	41
18	3	·0276	19	3	·0271	26	1	·0208	50
Dec. 17	2	·0161	17	1	·0194	35	1	·0744	72
1849.									
Oct. 30	3	0·0209	9	2	0·0228	10	1	0·3484	152
Nov. 27	3	·0295	13	3	·0280	12	......	......	......
1850.									
Feb. 22	4	0·0172	7	4	0·0244	10	1	0·2030	87
23	6	·0186	8	2	·0333	14	2	·0303	13
Mar. 31	4	·0214	9	1	·0375	16	1	·0000	0
May 7	......	......	......	3	·0249	10	......	......	......
June 13	2	·0285	12	2	·0534	23	1	·3882	164
Oct. 1	1	·0487	21	1	·0522	23	1	·1188	54
2	3	·0421	18	1	·0495	21	2	·0212	9
1851.									
Jan. 16	3	0·0432	19	2	0·0517	22	1	0·1009	44
19	4	·0293	12	2	·0468	19	1	·1367	59
Feb. 18	5	·0252	11	2	·0505	22	1	·1382	59
Sept. 3	5	·0473	25	3	·0522	22	3	·1114	48
4	7	·0218	9	5	·0444	19	1	·1769	77
6	2	·0595	25	3	·0378	15	3	·0648	28
7	7	·0462	20	7	·1108	46	3	·1805	76
29	6	·0720	32	3	·0972	41	2	·3864	172
Oct. 2	5	·0343	14	5	·0636	27	3	·1370	58
28	2	·0772	33	4	·0506	22	1	·1834	80
Dec. 6	4	·0465	20	1	·1264	54	1	·0815	36
28	5	·0247	11	3	·0243	10	1	·0950	41
29	4	·0420	23	1	·0627	28	3	·0233	11
1852.									
Jan. 4	3	0·0357	15	1	0·0968	44	3	0·0523	22
19	7	·0183	8	5	·0462	20	1	·1206	54
Feb. 14	4	·0217	10	3	·0823	37	4	·0585	26
15	7	·0412	17	3	·0712	30	3	·1879	80
17	6	·0283	12	4	·0475	20	3	·2525	110
18	7	·0277	12	4	·1041	43	1	·4422	185
19	3	·0919	43	5	·1042	44	1	·2596	113
20	7	·0186	8	3	·0441	18	1	·1594	69
21	3	·0370	18	3	·0660	28	1	·1735	77
April 20	7	·0570	24	3	·0796	33	1	·1508	67
May 19	3	·0487	30	4	·0289	12	1	·0595	28
20	1	·0068	10	3	·0140	6	1	·0605	44
June 11	3	·0514	23	2	·0606	26	1	·4354	195
16	4	·0310	13	5	·0275	12	......	......	......
July 10	2	·0490	23	2	·0336	15	3	·0660	29
Nov. 11	3	·0393	17	2	·0367	16	1	·3236	141
13	3	·0386	16	3	·0352	15	1	·1638	77

TABLE IV. (continued).

Year, Month, and Day.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Waves.	Absolute Aggregate of Fluctuations.	Absolute Mean of Disturbance.	Number of Waves.	Absolute Aggregate of Fluctuations.	Absolute Mean of Disturbance.	Number of Waves.	Absolute Aggregate of Fluctuations.	Absolute Mean of Disturbance.
1853.									
Jan. 10	3	0·0229	10	3	0·0208	9	......	......	......
Mar. 7	8	·0269	11	6	·0302	13	1	0·3353	141
8	6	·0293	13	5	·0242	10	1	·5298	236
11	......	......	......	5	·0279	12	1	·3529	152
May 2	3	·0359	16	1	·0657	28	1	·3595	157
3	6	·0210	9	3	·0552	23	1	·3424	149
24	7	·0280	12	5	·0886	38	3	·2273	96
June 22	3	·0253	11	3	·0259	11	1	·1487	64
July 12	8	·0299	13	3	·0646	27	4	·0777	33
Aug. 21	......	......	......	......	......	......	1	·0617	26
Sept. 1	7	·0264	11	4	·0838	37	2	·0648	28
2	5	·0299	13	5	·0738	31	2	·0717	30
Oct. 1	......	......	......	1	·0312	13	......	......	......
2	......	......	......	1	·0336	14	......	......	......
25	4	·0153	6	3	·0160	7	1	·3093	129
Nov. 9	5	·0503	21	1	·0474	20	2	·0766	33
Dec. 6	5	·0354	15	1	·1079	45	2	·0633	27
21	5	·0176	8	3	·0097	42	1	·1790	77
1854.									
Jan. 8	3	0·0249	10	2	0·0374	16	1	0·1089	46
20*......	8	·0245	10	3	·0108	4	1	·0104	15
20	......	......	......	......	......	......	1	·0550	39
Feb. 16	5	·0219	9	5	·0383	16	1	·1165	49
24	6	·0335	14	4	·0230	10	3	·1460	62
25	3	·0135	5	6	·0235	10	2	·1310	55
Mar. 6	4	·0228	10	5	·0145	6	1	·1049	44
15	9	·0265	11	2	·0407	17	4	·0514	22
16	7	·0294	12	1	·0408	17	1	·0498	21
28	4	·0284	12	1	·1271	53	1	·1451	64
April 10	6	·0458	19	4	·0687	29	3	·0855	38
23	3	·0233	10	4	·0334	14	2	·0897	38
May 25	6	·0112	5	3	·0188	8	4	·0759	32
1855.									
Mar. 12	5	0·0395	16	2	0·0574	25	1	0·2111	90
April 4	3	·0248	10	7	·0154	7	2	·0282	14
July 19	......	......	......	2	·0653	29	2	·0719	31
Oct. 18	3	·0234	10	2	·0499	21	1	·1049	44
† 1857.									
Feb. 26	3	0·0204	9	2	0·0180	8	1	0·1368	59
Mar. 13	3	·0194	8	......	......	......	......	......	......
May 7	3	·0689	29	4	·0726	30	1	·3191	141
10	9	·0118	5	4	·0388	18	2	·0893	37
Sept. 3	6	·0372	15	4	·0515	21	2	·4199	175
Nov. 12	3	·0353	15	4	·0162	7	......	......	......
16	2	·0271	12	4	·0216	9	......	......	......
17	3	·0605	26	3	·0339	15	......	......	......
Dec. 16	5	·0405	17	3	·0768	32	1	·2230	93
17	11	·0134	6	1	·0881	39	2	·0543	23

* 1854, Jan. 20. The Vertical-Force observations were interrupted during 3 hours.

† In 1856 there were no days of Great Magnetic Disturbance throughout the year.

The last figure in the "Absolute Mean of Disturbance" is in the fourth decimal place of Horizontal Force.

TABLE V.—Sums, without regard of sign, of Magnetic Fluctuations (in terms of Horizontal Force) for each Year from 1841 to 1857, including all days of Record of Great Magnetical Disturbance.

Year.	Number of Storms.	Westerly Force.			Northerly Force.			Nadir Force.		
		Number of Waves.	Number of Hours.	Absolute Sum of Fluctuations.	Number of Waves.	Number of Hours.	Absolute Sum of Fluctuations.	Number of Waves.	Number of Hours.	Absolute Sum of Fluctuations.
1841	8	25	129·47	·3236	15	119·63	·2641	12	116·19	·6143
1842	10	19	112·57	·2699	18	113·34	·4893	15	111·74	·3452
1843	7	11	55·72	·0990	13	49·39	·0928	7	45·40	·0933
1844	6	10	51·74	·1087	12	59·70	·1357	8	59·29	·2021
1845	5	11	60·00	·1063	9	60·41	·1420	5	60·52	·0862
1846	18	46	244·86	·3632	50	250·89	·3213	28	247·99	·4155
1847	21	100	246·75	·5249	45	246·29	1·2229	30	198·75	1·0719
1848	19	64	264·18	·4356	43	223·83	·7128	10	40·65	·2691
1849	2	6	46·00	·0504	5	45·25	·0508	1	22·92	·3484
1850	7	20	141·79	·1765	14	163·80	·2752	8	138·34	·7615
1851	13	59	294·04	·5692	41	305·70	·8190	24	299·17	1·8160
1852	17	73	364·65	·6422	55	395·76	·9785	27	353·07	2·9661
1853	18	75	327·14	·3941	53	402·06	·8065	24	350·67	3·2000
1854	12	64	285·10	·3057	40	285·82	·4770	25	279·75	1·1701
1855	4	11	71·37	·0877	13	93·75	·1880	6	91·03	·4161
1856	0	0	0·00	·0000	0	0·00	0000	0	0·00	·0000
1857	10	48	231·53	·3345	29	208·37	·4175	9	141·73	1·2424
Sums	177	642	2926·91	4·7915	455	3023·99	7·3934	239	2557·21	15·0182
Means of Absolute Disturbances ...				·00164			·00244			·00587

TABLE VI.—Sums, without regard of sign, of Magnetic Fluctuations (in terms of Horizontal Force) for each Year from 1841 to 1857, including only those days of Great Magnetic Disturbance in which Records were made by the three Instruments.

Year.	Number of Storms.	Westerly Force.			Northerly Force.			Nadir Force.		
		Number of Waves.	Number of Hours.	Absolute Sum of Fluctuations.	Number of Waves.	Number of Hours.	Absolute Sum of Fluctuations.	Number of Waves.	Number of Hours.	Absolute Sum of Fluctuations.
1841	8	25	129·47	·3236	15	119·63	·2641	12	116·19	·6143
1842	10	19	112·57	·2699	18	113·34	·4893	15	111·74	·3452
1843	5	8	45·72	·0886	12	45·39	·0880	7	45·40	·0933
1844	5	10	51·74	·1087	11	51·70	·1133	7	51·29	·1117
1845	5	11	60·00	·1063	9	60·41	·1420	5	60·52	·0862
1846	17	46	244·86	·3632	48	240·89	·3095	27	244·61	·4111
1847	16	83	202·69	·4111	36	202·19	·9497	29	194·75	1·0255
1848	4	16	55·17	·1408	10	45·74	·2749	10	40·65	·2691
1849	1	3	22·92	·0209	2	22·84	·0228	1	22·92	·3484
1850	6	20	141·79	·1765	11	139·88	·2503	8	138·34	·7615
1851	13	59	294·04	·5692	41	305·70	·8190	24	299·17	1·8160
1852	16	69	341·34	·6112	50	372·27	·9510	27	353·07	2·9661
1853	13	72	304·41	·3712	43	308·40	·6930	22	303·72	2·7854
1854	12	64	285·10	·3057	40	285·82	·4770	25	279·75	1·1701
1855	3	11	71·37	·0877	11	70·97	·1227	4	67·58	·3442
1856	0	0	0·00	·0000	0	0·00	·0000	0	0·00	·0000
1857	6	37	141·04	·1922	18	139·25	·3458	9	141·73	1·2424
Sums	140	553	2504·23	4·1468	375	2524·42	6·3124	232	2471·43	14·3905
Means of Absolute Disturbances ...				·00166			·00250			·00582

11. In examining the last line of these Tables, it must be borne in mind that the numbers are affected by the constant part of the Disturbance which appears as "Mean Disturbance" at the end of Table III. The value of mean disturbance for Nadir Force (as has been remarked) is uncertain, and that for Westerly Force is small; but that for Northerly Force is important. A constant term $-{\cdot}00147$, combined with variable quantities whose mean value is $\pm{\cdot}00250$, and whose actual value even at the maximum of its wave will very frequently be far less, will destroy some waves entirely. It will also increase the apparent Mean of Absolute Disturbances, even when the number of waves is not diminished. Thus: suppose, as a simple case, that the pure disturbance is represented by $a\sin\theta$, but that, when affected with a constant term, it is $a\sin\theta-b$. (As has been stated, when a is smaller than b, the addition of $-b$ will make every value $-$, and will destroy the alternation of $+$ waves and $-$ waves, and thus the just number of waves will be apparently diminished.) When a is greater than b, if Θ be the first value of θ which makes $a\sin\theta-b=0$, the positive Fluctuation will be found by integrating from $\theta=\Theta$ to $\theta=\pi-\Theta$, and the negative Fluctuation by integrating from $\theta=\pi-\Theta$ to $\theta=2\pi+\Theta$. The general value of the integral is $-a\cos\theta-b\theta$; the first limited integral is $2a\cos\Theta-b(\pi-2\Theta)$: the second is $-2a\cos\Theta-b(\pi+2\Theta)$, or (with sign changed, to make it positive) $+2a\cos\Theta-b(-\pi-2\Theta)$; and the sum of these, or aggregate of absolute fluctuations, is $4a\cos\Theta+4b\,.\,\Theta$. Now Θ is determined by the condition $a\sin\Theta-b=0$, or $\sin\Theta=\frac{b}{a}$. If b be small, $\Theta=\frac{b}{a}$ nearly, $\cos\Theta=1-\frac{b^2}{2a^2}$ nearly, and the aggregate of absolute fluctuations $=4a+\frac{2b^2}{a}$. The second term is the increase of the aggregate arising from the introduction of the term b.

If then we conceive the numbers in the last line of Table VI. to be affected with the correction which ought to be introduced in order to neutralize the effect of the large constant term in Northerly Force, it is certain that the number 375 would be considerably increased, and that the number 6·3124 would be considerably diminished. A very extensive examination of details would be necessary to enable us to say what would be the exact proportion of the changes: but it appears to me extremely probable (though at present far from certain) that the corrected Numbers of Waves are sensibly equal, the corrected Absolute Sums of Fluctuations are sensibly equal, and the corrected Means of Absolute Disturbances are sensibly equal, for Westerly Force and for Northerly Force.

The Number of Waves for Nadir Force is less than half that for the other forces; and the Absolute Sum of Fluctuations is about three times as great as that for the others.

12. It would be very important to ascertain any correspondence in the times of the waves in the different directions. I have not yet succeeded in discovering any satisfactory or certain relation.

First, in comparison of the Waves of Westerly and Northerly Forces, the coincidences of times of wave are so rare that it seems evident that nothing can be inferred from the few which can be found. From 1849 to 1857, when the photographic apparatus recorded equally the disturbances at all hours, I do not find one. In a less rigorous examination of the storms from 1841 to 1847, I find that on Nov. 19, 1841, there were contemporaneous waves from $12^h\ 17^m$ to $13^h\ 17^m$, W. F.+, No. F.+; and on Jan. 1, 1842, when the storm consisted of a single wave, $6^h\ 0^m$ to $12^h\ 41^m$, the forces were W. F.—, No. F.+. And the second W. F.— on Jan. 16, 1848, corresponds nearly with the sole No. F.—. Sometimes two waves in one direction correspond nearly with one in the other direction: thus in the beginning of the storm 1854, April 10, the W. F.+ from $0^h\ 7^m$ to $5^h\ 21^m$ and — from $5^h\ 21^m$ to $13^h\ 16^m$ occupy the same time as No. F.+ from $0^h\ 5^m$ to $13^h\ 9^m$: but this relation is not supported in the remainder of the same storm. A more frequent relation appears to be, that the evanescence of one wave corresponds with the maximum of the other: thus on February 21, 1852, and March 7, 1853, the waves stand in this order:

	Westerly Force.		Northerly Force.	
	Limits of Waves.	Character of Waves.	Limits of Waves.	Character of Waves.
1852. Feb. 21……	0·27	+	0·12	—
	4· 9	—	3·14	+
	15·15		5·16	—
			23·59	
1853. Mar. 7……	0·10	—	3·13	—
	4· 5	+	5·32	+
	6·25	—	7·19	
	12·20			

which relation, however, in the latter instance, is not maintained through the storm. And, generally, this relation does not appear to hold through the whole of any one storm consisting of numerous waves.

13. As the number of Nadir Waves approximates to half the number of Westerly Waves, it might seem worthy of inquiry whether the maximum of Nadir Wave corresponds to a change of Westerly Wave. The following instances have been remarked.

Time of Maximum of Nadir Wave.		Sign of Nadir Wave.	Change of Westerly Wave.	Time of Maximum of Nadir Wave.		Sign of Nadir Wave.	Change of Westerly Wave.
	h m				h m		
1841. Sept. 25.	3 35	+	+ to —	1852. Feb. 18.	4 37	+	+ to —
	4 17	+	— to +	June 11.	14 28	—	— to +
	6 19	+	+ to —	Nov. 11.	8 18	+	+ to —
1847. Sept. 24.	5 51	+	+ to —	1853. Mar. 8.	6 28	+	+ to —
	10 21	—	— to +		14 24	+	+ to —
Oct. 23.	5 27	+	+ to —	May 2.	17 35	+	— to +
	7 1	+	+ to —	3.	3 33	+	+ to —
Oct. 24.	13 4	—	— to +	24.	10 10	+	— to +
Dec. 17.	6 15	+	— to +	July 12.	11 37	—	+ to —
	8 13	+	— to +		15 57	+	— to +
1851. Sept. 4.	7 19	+	— to +	Sept. 1.	15 37	—	+ to —
7.	4 14	+	+ to —	2.	5 18	+	+ to —
	6 30	+	— to +	Oct. 25.	13 47	+	+ to —
	7 34	—	+ to —	1854. Apr. 10.	17 56	—	— to +
	10 19	—	+ to —	1857. Dec. 17.	6 10	+	+ to —
1852. Feb. 18.	2 56	+	+ to —				

I am unable to draw any inference from these.

14. The classification in Article 9 appears to lead to no result as to the effect of connexion of special signs of the first or last waves of the different forces. The inequalities shown in the first Table of Article 9 (of which the difference of numbers of last wave + and numbers of last wave − for the Northerly Force is the most remarkable) are quite sufficient to explain the inequalities in the combinations exhibited in the latter part of Article 9. And, on the whole, the principal conclusions which can be deduced from the examination of the Waves appear to me to be the following:—

That, while on the whole the Westerly Force is −, yet the number of + waves is the greater; and at the beginnings and ends of storms the number of + waves is greater than the number of − waves in a proportion exceeding 3 : 2.

That, the Northerly Force being on the whole −, in two instances out of three the first Northerly wave is −, and in ten instances out of eleven the last Northerly wave is −.

That, due regard being had to the effect of the constant − Northerly Force, it appears probable that the number of waves and the mean value of wave-disturbance are nearly the same for Westerly Force and for Northerly Force; but

That for the Nadir Force the number of waves is less than one-half the number for the other forces, while the mean value of disturbance is more than double that for the other forces.

15. I now proceed with the Irregularities. The following Tables (VII., VIII., IX.) exhibit their aggregates under the same divisions as those for the Waves. It will be remarked that, from the nature of the process by which the Irregularities are found, their algebraic sum in each storm is sensibly $=0$; and therefore they are treated here only as numbers without sign.

Table VII.—Absolute Sums, without regard of sign, of Coefficients of Magnetic Irregularity (in terms of Horizontal Force), on Days of Great Magnetic Disturbance.

Year, Month, and Day.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Irregularities.	Absolute Sum of Coefficients of Irregularity.	Mean Coefficient of Irregularity.	Number of Irregularities.	Absolute Sum of Coefficients of Irregularity.	Mean Coefficient of Irregularity.	Number of Irregularities.	Absolute Sum of Coefficients of Irregularity.	Mean Coefficient of Irregularity.
1841.									
Sept. 24	10	0·0133	13	6	0·0060	10	2	0·0031	15
25	70	·1417	20	73	·1226	17	61	·1760	29
27	6	·0086	14	12	·0090	8	3	·0021	7
Oct. 25	33	·0437	13	36	·0354	10	14	·0157	11
Nov. 18	25	·0329	13	28	·0325	12	18	·0208	12
19	19	·0252	13	26	·0213	8	13	·0139	11
Dec. 3	7	·0134	19	13	·0127	10	3	·0018	6
14	8	·0145	18	9	·0146	16	6	·0072	12
1842.									
Jan. 1	6	0·0068	11	8	0·0038	5	5	0·0021	4
Feb. 24	7	·0132	19	9	·0162	18	3	·0013	4
April 14	12	·0152	13	11	·0168	15	6	·0090	15
15	20	·0291	15	35	·0373	11	15	·0134	9
July 1	9	·0137	15	15	·0198	13	10	·0113	11
2	23	·0349	15	35	·0502	14	10	·0134	13
3	29	·0437	15	42	·0502	12	20	·0236	12
Nov. 10	11	·0197	18	14	·0139	10	4	·0021	5
21	14	·0132	9	15	·0204	14	1	·0008	8
Dec. 9	19	·0209	11	36	·0176	5	6	·0036	6
1843.									
Jan. 2	5	0·0059	12	6	0·0056	9	2	0·0005	3
Feb. 6	3	·0024	8	...	......	...	...	......	...
16	7	·0008	1	6	·0015	3	...	......	...
24	12	·0118	10	37	·0166	4	6	·0041	7
May 6	17	·0206	12	22	·0196	9	9	·0105	12
July 24	4	·0047	12	6	·0058	10	5	·0013	3
25	14	·0151	11	13	·0141	11	5	·0015	3
1844.									
Mar. 29	21	0·0230	11	24	0·0159	7	9	0·0046	5
30	18	·0246	14	29	·0335	12	7	·0041	6
Oct. 1	9	·0056	6	9	·0070	8	1	·0005	5
20	...	......	...	11	·0113	10	3	·0046	15
Nov. 16	28	·0290	10	19	·0190	10	9	·0049	5
22	22	·0234	11	31	·0300	10	9	·0072	8
1845.									
Jan. 9	15	0·0167	11	9	0·0105	12	4	0·0033	8
Feb. 24	16	·0163	10	26	·0123	5	13	·0072	6
Mar. 26	12	·0125	10	16	·0124	8	4	·0028	7
Aug. 29	19	·0065	3	11	·0087	8	5	·0015	3
Dec. 3	57	·0698	12	61	·0708	12	27	·0242	9
1846.									
May 12	13	0·0161	12	15	0·0130	9	4	0·0044	11
July 11	...	......	...	14	·0178	13	7	·0057	8
Aug. 6	26	·0172	7	35	·0172	5	7	·0036	5
7	64	·0207	3	55	·0308	6	15	·0090	6
24	9	·0075	8	9	·0055	6	5	·0015	3
25	5	·0033	7	5	·0059	12	2	·0015	8
28	28	·0150	5	24	·0178	7	3	·0023	8

TABLE VII. (continued).

Year, Month, and Day.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Irregularities.	Absolute Sum of Coefficients of Irregularity.	Mean Coefficient of Irregularity.	Number of Irregularities.	Absolute Sum of Coefficients of Irregularity.	Mean Coefficient of Irregularity.	Number of Irregularities.	Absolute Sum of Coefficients of Irregularity.	Mean Coefficient of Irregularity.
1846 (cont^{d}).									
Sept. 4 ……	26	0·0178	7	29	0·0156	5	5	0·0028	6
5 ……	32	·0255	8	36	·0285	8	7	·0093	13
10 ……	6	·0049	8	6	·0056	9	3	·0008	3
11 ……	28	·0311	11	31	·0378	12	12	·0123	10
21 ……	23	·0162	7	18	·0158	9	7	·0041	6
22 ……	68	·0771	11	59	·0692	12	28	·0244	9
Oct. 2 ……	8	·0089	11	11	·0100	9	3	·0018	6
7 ……	25	·0343	14	28	·0295	11	3	·0049	16
8 ……	29	·0213	8	29	·0245	9	5	·0031	6
Nov. 26 ……	28	·0253	9	29	·0235	9	7	·0080	11
Dec. 23 ……	12	·0163	14	9	·0133	17	7	·0039	6
1847.									
Feb. 24 ……	20	0·0132	7	15	0·0107	7	4	0·0026	7
Mar. 1 ……	42	·0416	10	43	·0384	9	16	·0126	8
19 ……	49	·0835	17	36	·0518	14	24	·0283	12
April 3 ……	15	·0214	14	18	·0232	13	3	·0039	13
7 ……	19	·0225	12	22	·0306	14	4	·0044	11
21 ……	12	·0142	12	8	·0095	12	2	·0018	9
May 7 ……	6	·0088	15	4	·0047	12	2	·0010	5
June 24 ……	3	·0046	15	…	……	…	…	……	…
July 9 ……	…	……	…	8	·0134	17	5	·0075	15
Sept. 24 ……	148	·2666	18	128	·3262	26	119	·2192	18
26 ……	12	·0128	11	15	·0142	9	9	·0087	10
27 ……	16	·0167	10	12	·0124	10	10	·0201	20
Oct. 22 ……	29	·0232	8	30	·0406	14	24	·0157	6
23*(1st)	86	·1132	13	73	·1332	18	58	·0882	15
23(2nd)	3	·0016	5	1	·0021	21	2	·0088	44
24 ……	113	·2034	18	128	·3134	24	94	·1722	18
25 ……	20	·0225	11	17	·0184	11	7	·0121	17
Nov. 22 ……	34	·0428	13	46	·0462	10	15	·0375	25
Dec. 17 ……	86	·1400	16	39	·0577	15	33	·0540	16
18 ……	29	·0297	10	21	·0236	11	…	……	…
19 ……	66	·0937	14	44	·0963	22	…	……	…
20 ……	97	·2546	26	64	·2191	34	…	……	…
1848.									
Jan. 16 ……	21	0·0570	27	21	0·0381	18	…	……	…
28 ……	18	·0361	20	19	·0422	22	…	……	…
Feb. 20 ……	35	·0573	16	16	·0329	21	…	……	…
21 ……	35	·1182	34	49	·1857	38	…	……	…
22 ……	4	·0099	25	5	·0087	17	…	……	…
23 ……	16	·0283	18	12	·0248	21	…	……	…
24 ……	24	·0431	18	21	·0407	19	…	……	…
Mar. 17 ……	4	·0036	9	7	·0141	20	…	……	…
20 ……	28	·0553	20	20	·0470	23	…	……	…
April 7 ……	21	·0390	19	9	·0241	27	…	……	…
May 18 ……	20	·0233	12	12	·0252	21	…	……	…
July 11 ……	33	·0544	16	25	·0608	24	…	……	…
Oct. 18 ……	21	·0675	32	18	·0666	37	14	0·0524	37
23 ……	23	·0518	23	19	·0396	21	…	……	…
25 ……	20	·0284	14	22	·0300	14	…	……	…
29 ……	11	·0185	17	1	·0018	18	…	……	…

* On Oct. 23, 1847, all the observations were interrupted during 10 hours.

TABLE VII. (continued).

Year, Month, and Day.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Irregularities.	Absolute Sum of Coefficients of Irregularity.	Mean Coefficient of Irregularity.	Number of Irregularities.	Absolute Sum of Coefficients of Irregularity.	Mean Coefficient of Irregularity.	Number of Irregularities.	Absolute Sum of Coefficients of Irregularity.	Mean Coefficient of Irregularity.
1848 (cont^{d}).									
Nov. 17 ……	38	0·1225	32	77	0·2394	31	41	0·2362	58
18 ……	17	·0272	16	17	·0306	18	1	·0008	8
Dec. 17 ……	19	·0396	21	12	·0213	18	14	·0167	12
1849.									
Oct. 30 ……	19	0·0232	…	8	0·0192	…	4	0·0046	12
Nov. 27 ……	11	·0158	…	7	·0166	…	…	……	…
1850.									
Feb. 22 ……	27	0·0219	8	26	0·0356	14	5	0·0113	23
23 ……	35	·0506	15	28	·0612	22	3	·0129	43
Mar. 31 ……	29	·0249	9	17	·0249	15	1	·0072	72
May 7 ……	…	……	…	13	·0174	13	…	……	…
June 13 ……	13	·0180	14	14	·0202	14	4	·0123	31
Oct. 1 ……	34	·0384	11	30	·0405	14	8	·0123	15
2 ……	25	·0390	16	25	·0400	16	7	·0087	12
1851.									
Jan. 16 ……	43	0·0544	13	36	0·0429	12	4	0·0090	23
19 ……	37	·0341	9	35	·0420	12	6	·0077	13
Feb. 18 ……	22	·0297	13	39	·0410	11	20	·0165	8
Sept. 3 ……	19	·0311	16	28	·0231	8	40	·0355	9
4 ……	29	·0512	18	63	·0843	13	42	·0460	11
6 ……	18	·0320	18	40	·0558	14	47	·0388	8
7 ……	89	·1659	19	106	·1899	18	86	·1367	16
29 ……	63	·1426	23	122	·1828	15	67	·1115	17
Oct. 2 ……	33	·0489	15	43	·0602	14	29	·0414	14
28 ……	24	·0448	19	46	·0509	11	20	·0180	9
Dec. 6 ……	40	·0697	17	51	·0615	12	30	·0404	13
28 ……	36	·0381	11	37	·0313	9	15	·0144	10
29 ……	47	·0463	10	52	·0452	9	12	·0098	8
1852.									
Jan. 4 ……	38	0·0343	9	22	0·0208	9	18	0·0087	5
19 ……	31	·0358	12	59	·0540	9	31	·0177	6
Feb. 14 ……	20	·0255	13	19	·0562	30	17	·0195	11
15 ……	101	·0987	10	62	·0888	14	53	·0398	7
17 ……	90	·1440	16	92	·1924	21	124	·1354	11
18 ……	73	·0965	13	66	·1295	20	54	·0576	11
19 ……	73	·1630	22	71	·1397	20	100	·1789	18
20 ……	45	·0457	10	60	·0641	11	17	·0198	12
21 ……	50	·0739	15	70	·0785	11	23	·0226	10
April 20 ……	52	·0690	13	52	·1515	29	41	·0440	11
May 19 ……	25	·0207	8	36	·0322	9	12	·0121	10
20 ……	3	·0031	10	37	·0466	13	14	·0077	6
June 11 ……	31	·0573	18	37	·0586	16	32	·0352	11
16 ……	41	·0373	9	39	·0464	12	…	……	…
July 10 ……	29	·0352	12	25	·0411	16	15	·0111	7
Nov. 11 ……	37	·0483	13	38	·0435	11	20	·0224	11
13 ……	43	·0506	12	25	·0301	12	12	·0080	7
1853.									
Jan. 10 ……	19	0·0195	10	16	0·0146	9	…	……	…
Mar. 7 ……	66	·0423	6	63	·0423	7	11	0·0201	18
8 ……	72	·0621	9	57	·0415	7	11	·0147	13

TABLE VII. (concluded).

Year, Month, and Day.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Irregu-larities.	Absolute Sum of Coeffi-cients of Irregularity.	Mean Coefficient of Irregu-larity.	Number of Irregu-larities.	Absolute Sum of Coeffi-cients of Irregularity.	Mean Coefficient of Irregu-larity.	Number of Irregu-larities.	Absolute Sum of Coeffi-cients of Irregularity.	Mean Coefficient of Irregu-larity.
1853 (cont^d).									
Mar. 11	......	....	......	54	0·0411	8	11	0·0175	16
May 2	59	0·0367	6	80	·0528	7	15	·0165	11
3	63	·0391	6	61	·0556	9	21	·0157	7
24	77	·0646	8	97	·1206	12	37	·0555	15
June 22	50	·0361	7	51	·0454	9	17	·0170	10
July 12	123	·1097	9	129	·1231	10	34	·0524	15
Aug. 21	......	....	......	......	....	......	8	·0118	15
Sept. 1	42	·0260	6	46	·0418	9	13	·0190	15
2	70	·0665	9	90	·0959	11	36	·0391	11
Oct. 1	......	....	......	12	·0036	3	......	....	......
2	......	....	......	9	·0037	4	......	....	......
25	22	·0187	9	27	·0156	6	10	·0105	10
Nov. 9	49	·0407	9	49	·0376	8	19	·0118	6
Dec. 6	60	·0489	8	41	·0461	11	26	·0321	12
21	35	·0298	8	28	·0221	8	8	·0067	8
1854.									
Jan. 8	33	0·0207	6	24	0·0218	9	13	0·0090	7
20*	49	·0279	6	35	·0206	6	4	·0023	6
20	......	....	......	......	....	......	4	·0059	15
Feb. 16	56	·0460	8	67	·0527	8	26	·0170	7
24	53	·0460	9	67	·0481	7	21	·0175	8
25	56	·0405	7	63	·0487	8	22	·0208	9
Mar. 6	33	·0178	5	37	·0204	6	16	·0216	14
15	59	·0463	8	65	·0425	7	28	·0229	8
16	58	·0556	10	69	·0513	7	24	·0188	8
28	62	·0591	9	77	·0549	7	49	·0249	5
April 10	49	·0527	11	79	·0688	9	52	·0357	7
23	38	·0206	6	49	·0322	7	21	·0108	5
May 25	38	·0229	6	52	·0342	6	32	·0301	9
1855.									
Mar. 12	55	0·0361	6	59	0·0320	5	23	0·0157	7
April 4	55	·0355	6	53	·0390	7	19	·0111	6
July 19	......	....	......	80	·0451	6	21	·0152	7
Oct. 18	40	·0267	7	60	·0311	5	13	·0111	8
†									
1857.									
Feb. 26	41	0·0128	3	21	0·0119	6	10	0·0126	13
Mar. 13	37	·0155	4	......	....	......	......	....	......
May 7	90	·0778	9	102	·0883	9	58	·0504	9
10	60	·0196	3	65	·0309	5	13	·0129	10
Sept. 3	55	·0501	9	92	·0629	7	37	·0296	8
Nov. 12	47	·0256	5	58	·0292	5	......	....	......
16	41	·0265	6	56	·0191	3	......	....	......
17	42	·0329	8	68	·0307	4	......	....	......
Dec. 16	66	·0847	13	82	·1496	18	19	·0147	8
17	78	·0626	8	93	·0771	8	30	·0221	7

In the column "Mean Coefficient of Irregularity," the last figures correspond to the fourth decimal place of Horizontal Force.

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* In 1854, Jan. 20, the Vertical Force observations were interrupted during 3 hours.

† In 1856 there were no days of Great Magnetic Disturbance throughout the year.

TABLE VIII.—Sums, without regard of sign, of Coefficients of Magnetic Irregularity (in terms of Horizontal Force), for each Year from 1841 to 1857, including all days of Record of Great Magnetical Disturbance.

Year.	Westerly Force.			Northerly Force.			Nadir Force.		
	Number of Storms.	Number of Irregularities.	Sum of Coefficients.	Number of Storms.	Number of Irregularities.	Sum of Coefficients.	Number of Storms.	Number of Irregularities.	Sum of Coefficients.
1841	8	178	·2933	8	203	·2541	8	120	·2406
1842	10	150	·2104	10	220	·2462	10	80	·0806
1843	7	62	·0613	6	90	·0632	5	27	·0179
1844	5	98	·1056	6	123	·1167	6	38	·0259
1845	5	119	·1218	5	123	·1147	5	53	·0390
1846	17	430	·3585	18	442	·3813	18	130	·1034
1847	20	905	1·4306	20	772	1·4857	17	431	·6986
1848	19	408	·8810	19	382	·9736	4	70	·3061
1849	2	30	·0390	2	15	·0358	1	4	·0046
1850	6	163	·1928	7	153	·2398	6	28	·0647
1851	13	500	·7888	13	698	·9109	13	418	·5257
1852	17	782	1·0389	17	810	1·2740	16	583	·6405
1853	14	807	·6407	17	910	·8034	15	277	·3404
1854	12	584	·4561	12	684	·4962	12	312	·2373
1855	3	150	·0983	4	252	·1472	4	76	·0531
1856	0	0	·0000	0	0	·0000	0	0	·0000
1857	10	557	·4081	9	637	·4997	6	167	·1423
Sums	168	5923	7·1252	173	6514	8·0425	146.	2814	3·5207
Mean Coefficient ...			·00120			·00123			·00125

TABLE IX.—Sums, without regard of sign, of Coefficients of Magnetic Irregularity (in terms of Horizontal Force), for each Year from 1841 to 1857, including only those days of Great Magnetic Disturbance in which Records were made by the three Instruments.

Year.	Number of Storms.	Westerly Force.		Northerly Force.		Nadir Force.	
		Number of Irregularities.	Sum of Coefficients.	Number of Irregularities.	Sum of Coefficients.	Number of Irregularities.	Sum of Coefficients.
1841	8	178	·2933	203	·2541	120	·2406
1842	10	150	·2104	220	·2462	80	·0806
1843	5	52	·0581	84	·0617	27	·0179
1844	5	98	·1056	112	·1054	35	·0213
1845	5	119	·1218	123	·1147	53	·0390
1846	17	430	·3585	428	·3635	123	·0977
1847	16	710	1·0480	635	1·1333	426	·6911
1848	4	95	·2568	124	·3579	70	·3061
1849	1	19	·0232	8	·0192	4	·0046
1850	6	163	·1928	140	·2224	28	·0647
1851	13	500	·7888	698	·9109	418	·5257
1852	16	741	1·0016	771	1·2276	583	·6405
1853	13	788	·6212	819	·7404	258	·3111
1854	12	584	·4561	684	·4962	312	·2373
1855	3	150	·0983	172	·1021	55	·0379
1856	0	0	·0000	0	·0000	0	·0000
1857	6	390	·3076	455	·4207	167	·1423
Sums ...	140	5167	5·9421	5676	6·7763	2759	3·4584
Mean Coefficient...			·00115		·00119		·00125

16. The most striking particulars in the last line of these Tables are the following: First, the almost exact equality of the Mean Coefficients of Irregularity in the three

elements. And this remarkable agreement proves that the Irregularities as measured here are real objective facts. For they are measured from photographic sheets in which the scales are very different: on the Westerly and Northerly records, 0·01 of Horizontal Force is represented by 2·87 inches and 2·55 inches, while on the Nadir record 0·01 of Horizontal Force is represented by 0·88 inch. Yet the eye of the Reader of the Photographs has caught the Irregularities when shown on this small scale as certainly as when shown on the larger scale. With reference to their physical import, I think it likely that the equality of Coefficients of Irregularity may hereafter prove to be one of the most important of the facts of observation.

Second, the near agreement in the number of Irregularities for Westerly Force and for Northerly Force.

Third, the near agreement in the number of Irregularities for Nadir Force with half the number of Irregularities for Westerly or for Northerly Force.

17. I have not succeeded in discovering any clear relation between the times of occurrence of Irregularities of Westerly Force and of Northerly Force. They certainly do not coincide. In their intermixture, I cannot assert that an Irregularity of one element always occurs between two of the other element, though there is a general appearance of that law.

18. It appeared to me possible that an Irregularity of Nadir Force might occur at the change between + and − Irregularities of Westerly Force; and the following examination seems to show a certain degree of plausibility in the supposition:—

Day.	Total Number of Nadir Irregularities.	Number of Nadir Irregularities corresponding to changes of sign for Westerly Irregularities.
1841. Sept. 25	61	52
1847. Sept. 24	119	76
Oct. 23	60	36
24	94	66
Dec. 17	36	20
1851. Sept. 4	42	26
7	86	68
29	67	50
1852. Feb. 15	53	42
17	124	101
18	54	42
19	100	68
June 11	32	22
Dec. 11	20	14
1853. Mar. 8	11	8
May 2	15	12
3	21	13
24	37	25
July 12	34	25
Sept. 1	13	9
2	36	25
Oct. 25	10	9
Dec. 6	26	23
1854. Feb. 24	21	16
April 10	52	35
1855. Mar. 12	23	16
1857. May 7	58	39
Sept. 3	37	31
Dec. 17	30	21
Total	1372	990

19. The investigations which I had proposed to myself as more peculiarly the object of this paper are now terminated, in so far as their results can be comprehended in tables of numerical values and remarks on the relations between the numbers. But I think it desirable to subjoin Tables tending to exhibit the laws of frequency of the great wave-disturbances and the irregularities, with respect to the months of the year and with respect to the hours of the day.

20. First, for the months of the year. The following numbers are formed by simply collecting from Tables I., IV., and VII. all the numbers arranged in groups under each nominal month. It will be seen at once that the distribution of magnetic storms through the year is so irregular that, even in the long period of seventeen years, no inference can be drawn connecting the Magnetic Storms with the Seasons.

TABLE X.—Aggregates of Fluctuations and Inequalities, arranged by Months, in terms of the Horizontal Force.

Month.	Westerly Force.			Northerly Force.			Nadir Force.		
	Algebraical Aggregate of Fluctuations.	Absolute Aggregate of Fluctuations.	Sum of Irregularities.	Algebraical Aggregate of Fluctuations.	Absolute Aggregate of Fluctuations.	Sum of Irregularities.	Algebraical Aggregate of Fluctuations.	Absolute Aggregate of Fluctuations.	Sum of Irregularities.
January	− ·0435	·3183	·3492	+ ·0679	·4827	·3169	− ·5582	·6250	·0662
February ...	− ·1425	·6275	1·2093	− ·5521	1·0223	1·3985	− ·1176	2·4732	·5974
March	− ·1279	·3905	·6038	− ·5193	·6071	·5640	+ 1·0271	2·0367	·2158
April	+ ·0289	·2635	·3192	− ·4074	·4596	·4330	− ·2766	·5416	·1341
May	− ·0266	·3052	·3533	− ·0554	·4638	·5411	+ ·6293	1·9545	·2291
June	− ·0453	·1471	·1533	− ·0224	·1674	·1706	− ·9723	·5841	·0522
July	− ·0598	·2238	·3114	− ·2361	·4187	·4414	+ ·0109	·4423	·1430
August	+ ·0087	·0875	·0702	− ·0135	·0427	·0859	− ·0988	·1294	·0312
September...	− ·1198	·7046	1·1977	− ·4614	1·0812	1·3994	− ·1785	2·2337	·9391
October......	+ ·0066	·5864	·8836	− ·8129	·9881	1·0282	− ·2781	1·8979	·4866
November ...	− ·0431	·6511	·6016	− ·6096	·7150	·6836	− ·5549	·9893	·3744
December ...	− ·1032	·4860	1·0726	− ·7603	·9448	·9799	− ·0969	1·1105	·2516

The disproportion of Irregularities to Fluctuations in the Nadir Force, as compared to those in the other Forces, is very remarkable.

21. Secondly, for the hours of the day. For each hour, on a day of storm, the nearest value of wave-disturbance (not of fluctuation) and the nearest value of irregularity were taken from the sheets in which the reductions described in Article 5 were made; and all the numbers thus found were collected for each hour, the + and − values of wave-disturbance being placed in separate columns. Thus the following Table is formed.

TABLE XI.—Sums of Wave-disturbances and of Irregularities, arranged by hours of Göttingen Solar Time, in terms of Horizontal Force.

Hour of Göttingen Time.	Westerly Forc.				Northerly Force.				Nadir Force.			
	Number of Measures.	Sums of Wave-disturbance.		Sums of Irregularities.	Number of Measures.	Sums of Wave-disturbance.		Sums of Irregularities.	Number of Measures.	Sums of Wave-disturbance.		Sums of Irregularities.
		+	−			+	−			+	−	
0	25	·0201	·0103	·0213	29	·0136	·0717	·0323	5	·0285	·0000	·0090
1	56	·0558	·0106	·0416	57	·0339	·0726	·0674	19	·0681	·0306	·0236
2	77	·0658	·0203	·0658	82	·0617	·0900	·0954	33	·1434	·0455	·0370
3	76	·0881	·0224	·0725	92	·1060	·0807	·1060	40	·1773	·1131	·0563
4	98	·1051	·0334	·1144	108	·1201	·0823	·1462	63	·3094	·1187	·0774
5	95	·0831	·0437	·1179	103	·1407	·1019	·1233	60	·2832	·1113	·0681
6	105	·0752	·0713	·1327	114	·1276	·1291	·1290	74	·3701	·0856	·0794
7	104	·0593	·1079	·1353	108	·0806	·1422	·1344	77	·3976	·0974	·0915
8	122	·0331	·1759	·1746	136	·0570	·2171	·1754	79	·3092	·1280	·0853
9	126	·0276	·1848	·1743	119	·0479	·2393	·1439	80	·2866	·1575	·1169
10	123	·0165	·2191	·1976	130	·0553	·2612	·1750	86	·2529	·2061	·1241
11	116	·0267	·1841	·1531	111	·0544	·2747	·1524	77	·2110	·2837	·0889
12	121	·0278	·2070	·1429	122	·0449	·2917	·1422	74	·1629	·2716	·1007
13	111	·0277	·2036	·1606	108	·0307	·2470	·1429	63	·1097	·2830	·0799
14	112	·0442	·1574	·1442	109	·0308	·2897	·1260	74	·1768	·3133	·0941
15	99	·0601	·1324	·1604	100	·0362	·2194	·1443	59	·1329	·2598	·0717
16	102	·0537	·0951	·1359	97	·0160	·2428	·1287	59	·0966	·2881	·0825
17	84	·0695	·0508	·0926	86	·0120	·2137	·1117	54	·0910	·2963	·0619
18	87	·1016	·0315	·0970	93	·0101	·2043	·1169	46	·1010	·2038	·0532
19	76	·1008	·0193	·0793	85	·0112	·2531	·0990	44	·0830	·1889	·0470
20	75	·1170	·0107	·0826	81	·0076	·2646	·0713	39	·0614	·1295	·0427
21	58	·0613	·0083	·0527	65	·0087	·1919	·0694	29	·0619	·0740	·0306
22	59	·0647	·0179	·0520	69	·0038	·2241	·0694	26	·0355	·0460	·0270
23	51	·0460	·0214	·0346	57	·0052	·1463	·0441	24	·0491	·0396	·0177

It must be remarked here that the number of measures at 0^{h} is made in this Table unfairly small. This arises partly from the interruptions which are almost unavoidable in the operation of changing the photographic sheets at 0^{h}, and partly from the manner in which the measured quantities have been treated in the discussion of Storms. When a storm has evidently occupied a part of a day, it has been usual to treat by rule the measures of the entire sheet of that day, from 0^{h} to 24^{h}; and in that process, as is described in the beginning of Article 5, the two first and two last measures are lost; and some of these ought, in a great number of cases, to be referred to 0^{h}. The best value that can be taken for 0^{h} will be the mean of the values for 23^{h} and for 1^{h}.

22. It will be seen that, at the same hour, the mean value of Irregularity is nearly the same for the three Forces, but that, from hour to hour, the mean Irregularities are largest where the number of measures is greatest, that is, where storms are most frequent. In regard to the Wave-disturbance; for Westerly Force, the aggregate is + from 17^{h} to 6^{h}, − from 7^{h} to 16^{h}; for Northerly Force, the aggregate is + from 3^{h} to 5^{h}, − from 6^{h} to 2^{h}; and for Nadir Force, the aggregate is + from 23^{h} to 10^{h}, − from 11^{h} to 22^{h}. In regard to the modification which these Wave-disturbances might be supposed to produce on the laws of Diurnal Inequality, when it is remarked that each

of the hours 0^h, 1^h, 2^h, &c. has been repeated 17 × 365 times, it will be seen that the introduction of these Storm Days into the general mass of observations will in no instance alter the mean Diurnal Inequality by a unit in the fourth decimal place. In a year of very great disturbance, as 1853, they may possibly introduce a correction of one unit, or perhaps two units, in the fourth decimal of some of the Diurnal numbers.

23. The import of the numbers of the last Table will be best seen by the following treatment. If for either of the three directions of force, at any one hour, we form the Algebraic sum of the + and − sums of wave-disturbances, and divide by the number of measures, we obtain the mean wave-disturbance whenever a storm occurs at that hour. If we form the Absolute sum, and divide it similarly, we obtain the double average departure from that mean whenever a storm occurs at that hour. The mean Irregularity is obtained by simple division.

TABLE XII.—Frequency of Storms, mean Wave-disturbance, average departure from the mean, and mean Irregularity, in terms of the Horizontal Force, at each hour of Göttingen Solar Time.

Hour of Göttingen Time.	Westerly Force.				Northerly Force.				Nadir Force.			
	Frequency of Storms.	Mean Wave-disturbance.	Average departure from Mean. ±	Mean Irregularity. ±	Frequency of Storms.	Mean Wave-disturbance.	Average departure from Mean. ±	Mean Irregularity. ±	Frequency of Storms.	Mean Wave-disturbance.	Average departure from Mean. ±	Mean Irregularity. ±
0	54	+ ·00039	·00061	·00085	57	− ·00200	·00147	·00112	22	+ ·00570	·00285	·00180
1	56	+ 81	59	74	57	− 68	93	118	19	+ 197	260	124
2	77	+ 59	56	86	82	− 35	93	116	33	+ 297	286	112
3	76	+ 86	73	95	92	+ 28	101	115	40	+ 161	363	140
4	98	+ 73	71	117	108	+ 35	94	135	63	+ 303	340	123
5	95	+ 42	67	124	103	+ 38	118	120	60	+ 287	329	114
6	105	+ 4	70	126	114	− 1	113	113	74	+ 385	308	107
7	104	− 47	80	130	108	− 57	103	124	77	+ 390	321	119
8	122	− 117	86	143	136	− 118	101	129	79	+ 229	276	108
9	126	− 125	84	138	119	− 161	121	121	80	+ 161	278	146
10	123	− 165	96	161	130	− 158	122	135	86	+ 54	267	144
11	116	− 136	91	132	111	− 198	148	137	77	− 94	321	116
12	121	− 148	97	118	122	− 202	138	117	74	− 147	294	136
13	111	− 159	104	145	108	− 200	129	132	63	− 275	312	127
14	112	− 101	90	129	109	− 238	147	116	74	− 185	331	127
15	99	− 73	97	162	100	− 183	128	144	59	− 215	333	122
16	102	− 41	73	133	97	− 234	133	133	59	− 325	326	140
17	84	+ 22	72	110	86	− 235	131	130	54	− 380	359	115
18	87	+ 81	77	112	93	− 209	115	126	46	− 224	331	116
19	76	+ 107	79	104	85	− 285	155	117	44	− 241	309	107
20	75	+ 142	85	110	81	− 317	168	88	39	− 175	245	110
21	58	+ 91	60	91	65	− 281	154	107	29	− 42	234	106
22	59	+ 79	70	88	69	− 319	165	101	26	− 40	157	104
23	51	+ 48	66	68	57	− 248	133	77	24	+ 40	185	74

The Soli-tidal character of the principal characteristics of the occasional Magnetic Storms, as to frequency, magnitude, inequalities of wave-disturbance, and Irregularities, is seen clearly in this Table.

24. I now come to the consideration of the physical inference from these numerical conclusions. And first I would remark that I do not think that they can be reconciled with the supposition of definite galvanic currents or definite magnets, suddenly produced, in any locality whatever, as sufficient to explain the disturbances observed here. On that hypothesis, it would seem necessary to believe that such sudden currents or magnets would produce simultaneous disturbances in the three co-ordinate directions, that, if the long period of a wave permitted some deviation from this rule, yet the short period of an inequality would admit of no such deviation, and that, on any supposition, the number of disturbances in the three directions would be approximately equal. Yet in fact we find that neither in Waves nor in Irregularities is there the least appearance of simultaneity, and that, though there is close equality of numbers between the Westerly and Northerly Forces, yet the Nadir Force (in which the Irregularities are as strongly marked as in the Westerly and Northerly, and the Wave-disturbances much more strongly marked) exhibits less than half the number. These considerations appear to me quite conclusive as showing that the observed disturbances cannot be produced by the forces of any suddenly created galvanic current or polar magnet.

25. To suggest instead of this an imperfect conjecture, based upon grounds so inadequate as those which we can at present use for its foundation, must be a delicate and dangerous, I may almost say an invidious enterprise. Yet the impression of an explanation of broad character, partly definite but generally indefinite, has, in the course of this investigation, forced itself so strongly on my mind, that I should think it wrong to omit to describe it. Its fundamental idea is, that there may be in proximity to the earth something which (to avoid unnecessary words) I shall call a Magnetic Ether; that under circumstances generally, but not always, having reference to the solar hour, and therefore probably depending on the sun's radiation or on its suppression, a current from N.N.W. to S.S.E., approximately, or from S.S.E. to N.N.W. (according to the boreal or austral nature of the ether) is formed in this Ether; that this current is liable to interruptions or perversions of the same kind as those which we are able to observe in currents of air and water; and that their effect is generally similar, producing eddies and whirls, of violence sometimes far exceeding that of the general current from which they are derived.

26. Our powers of observing the two elements to which I have referred for analogy are somewhat different, but both imperfect. We know that in a gale of wind, the direction of the wind is continually changing; the horizontal pressure and the barometric pressure also are continually changing; but the changes are so rapid that we cannot easily determine whether there is any correspondence between them. But, in the storms on a large scale, there is reason to think that some winds are radial, but far more are cyclonic; that in some instances the barometer rises in the centre, but in more it is depressed; and in many instances the disturbance of vertical pressure is enormous (for 1 inch of barometer corresponds to a pressure of about 70 lbs. per square foot). Of water, perhaps the best study is to be found in disturbed tidal currents, as those of the

Western Islands of Scotland; here, in some places, approximately circular spaces are to be seen which are quiet, but which appear to the eye to be elevated above the rest; in some disturbed places the water is thrown upwards; in other places the sea is whirling round with great speed, in a good circular form, and with a funnel of considerable depth in the centre; in other places, boiling currents are running very fast in opposite directions, though separated by no great space; the general impression however is that of circularity*; great circles and small circles coexisting. Though these circular forms may be more prevalent in one part of the sea than another, they are not fixed, but wander irregularly, sometimes suddenly disappearing, and sometimes as suddenly created anew. In like manner, in the course of a river, travelling funnels may be seen, whose depth sometimes exceeds their breadth.

27. Now it appears to me that if a sentient and reasoning being were immersed either in the air or in the water through which these circles are wandering, he would perceive actions nearly similar to those which we have found to exist in the magnetic storms. The large and slowly-displaced circles would produce Wave-disturbances, slowly changing their direction, and thus having different times of evanescence in the N. and S. direction (on the one hand) and in the E. and W. direction (on the other hand); the smaller circles, in like manner, would produce the rapid Irregularities. And in the relation between E. and W. disturbances and vertical disturbances, there is a point which well deserves attention. When a water-funnel passed nearly over the observer, travelling (suppose) in a N. direction, he would first experience a strong current to the E., afterwards a strong current to the W. (or *vice versâ*), and between these there would be a very strong vertical pressure in one direction, not accompanied by one in the opposite direction; thus he would have half as many vertical as horizontal impulses. This state of things corresponds to the proportion which we have found throughout for the magnetic disturbances, and to the relation found in Article 18. I may also add that the rule at which we have arrived, that the waves of vertical force are few, but that their power, when they do occur, is very great, seems to correspond to what is reported of the whirlwinds of great atmospheric storms; which, violent and even frequent as they may be, occur very rarely at any assigned place.

28. It seems to me that there is so much plausibility in these suppositions as to justify me in expressing a wish that some effort might be made to verify them. The immediate object of observations would be, to ascertain through a locality of considerable extent the times and magnitudes of Wave-disturbances and of Irregularities on the same days throughout, with the view of discovering whether they could be collectively represented as the effects of such travelling vortices as I have suggested. In regard to the extent of the locality, I should think that a portion of the Continent of Europe would suffice, and that five or six magnetic observatories would decide the points under inquiry. In regard to the mode of observation, though eye-observation is, for a limited time, the most accurate, yet self-registering record is the only method which can insure the

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* I have been upon these currents, and in close proximity to these whirlpools.

observation of all that is required; only, I would specially observe, it is indispensable that eye-observations be used to check the zeros of time and of measure, and that the photographic traces be so strong that they will not be lost in rapid motions of the magnet. In regard to the mode of primary reduction, I imagine that the method followed in this Memoir (with such small alterations as experience may suggest) will be found best.

⁂ The computations for the "Diurnal Inequalities" were performed by computers under the immediate superintendence of Mr. JOHN LUCAS; some portions of them were revised and corrected by JAMES GLAISHER, Esq., F.R.S., Superintendent of the Magnetical and Meteorological Department of the Royal Observatory. The curves were drawn under Mr. GLAISHER's superintendence by Mr. W. C. NASH, and reduced to scale by Mr. JAMES CARPENTER, Assistant in the Astronomical Department of the Royal Observatory. The computations of the present Memoir were made under the superintendence of Mr. GLAISHER, by Mr. NASH and junior computers.

XXX. *Results of hourly Observations of the Magnetic Declination made by Sir* Francis Leopold M^cClintock, *and the Officers of the Yacht 'Fox,' at Port Kennedy, in the Arctic Sea, in the Winter of* 1858–59*; and a Comparison of these Results with those obtained by Captain* Rochfort Maguire, *and the Officers of Her Majesty's Ship 'Plover,' in* 1852, 1853, *and* 1854, *at Point Barrow. By Major-General* Edward Sabine, *R.A., President of the Royal Society.*

Received December 21, 1863,—Read January 7, 1864.

In the spring of 1857 Captain Francis Leopold M^cClintock, of the Royal Navy, being about to proceed to the Arctic Seas in the 'Fox' Yacht in search of the ships which had formed Sir John Franklin's Expedition, applied to the President and Council of the Royal Society "to afford him such information and instructions as might enable him to make the best use of the opportunity afforded by the voyage for the prosecution of meteorological, magnetical, and other observations."

A committee having been appointed to communicate with Captain M^cClintock, I, as one of the Members of that Committee, drew up a memorandum respecting the magnetical observations which he might have an opportunity of making, and supplied him with suitable instruments belonging to the Government Establishment under my superintendence. With the sanction of the Committee of the Kew Observatory, Lieutenant W. R. Hobson, R.N., and Captain Allen Young, two of the Officers who proposed to accompany Captain M^cClintock, were instructed in the use of these instruments at the Kew Observatory.

As this communication is limited to a notice of the hourly observations of the *Magnetic Declination*, which Captain M^cClintock and his Officers were enabled to make in the winter of 1858–59, it will be sufficient at present to extract from the memorandum, adverted to in the preceding paragraph, the portion which relates to that branch of the inquiry, as the most suitable introduction to the account of the observations themselves.

"The results of the hourly observations of the declination made at Point Barrow in 1853 and 1854, by Captain Rochfort Maguire, R.N., and the Officers of Her Majesty's Ship 'Plover,' when compared with the hourly observations at the Toronto Observatory, have brought into view, in accompaniment with many circumstances of a highly interesting *resemblance*, some features, in the magnetic disturbances at Point Barrow, which appear as if they were the *converse* of those of the corresponding phenomena at Toronto. Now, Toronto in lat. 43° 40′ and long. 79° 22′ W., and Point Barrow in lat. 71° 21′ and long. 156° 15′ W., are situated on the same continent; and it seems probable that there may exist some intermediate locality where the phenomena of the disturbances may be of a *critical* character. The more precise determination of this locality is full of interest, both as respects terrestrial magnetism and geographical physics generally. It is for this reason very desirable that we should learn, by similar observations to those made at Toronto and Point Barrow, what are the corresponding periodical laws of the disturbances of the declination at stations which either in latitude or longitude may be intermediate between those places. It is highly probable that, if either the 'Erebus' or 'Terror' be still existing, there may be found in one or the other, or in both ships, the records of observations in at least two intermediate localities, in which the Expedition may have been stationary in different years; because both ships were furnished with the proper instruments, and some of the Officers had attended at Woolwich to practise with them before the Expedition sailed: to this it may be added, that both Sir JOHN FRANKLIN and Captain CROZIER were strongly impressed with the desirability of making the observations, and letters are extant from both, written from Davis Strait, after they had sailed from England, expressing their full intention to set up the instruments wherever the ships should be detained for a sufficient period to give the observations value. The possible existence of such records is here referred to with the view of impressing on the attention of Captain M^CCLINTOCK the scientific importance of recovering these records, if possible, and of bringing them safely home.

"The station where Captain McCLINTOCK's ship will probably remain during the months preceding the departure of the sledge-parties, as well as during the still longer period when they will be employed in the search which forms the object of the Expedition, will not be far distant, in all probability (whether that station be in Peel's or in Regent's Inlet), from the latitude of Point Barrow and longitude of Toronto. It is a locality, therefore, at which observations similar to those at Point Barrow, which have proved in many respects very important, are extremely desirable. The duration of Captain MAGUIRE's hourly observations was *eight* months in 1853, and *nine* months in 1854. The accord in the conclusions drawn from the observations of either year taken separately, with their joint results when taken together, shows that eight or nine months is sufficient for the purposes adverted to, if a longer duration be inconvenient. Even less than eight months would suffice for a *general* indication, though of course the longer the observations can be continued, under equal circumstances of care, &c., the more precise is the information acquired. Captain McCLINTOCK has stated to the writer of this memorandum that he anticipates no difficulty in maintaining hourly observations between the time when the ship is laid up in autumn and the departure of the sledge-parties in the spring, and that it might be possible that, when once become a routine, they might be kept on still by those few persons who will remain with the ship. The observations are in themselves extremely simple, and it happens fortunately that one of the Officers who expects to accompany Captain McCLINTOCK, Mr. GREY, was also with Captain MAGUIRE at Point Barrow, and is therefore acquainted with magnetic observations, which were remarkably well conducted in Captain MAGUIRE's Expedition *. Amongst the instruments at Woolwich which have been returned from the dismantled observatories, there is a Declinometer which will be suitable for the purpose, when it has received small repairs, which in anticipation of this opportunity are already in progress †.

"At the station where the ship will be laid up, the amount of Dip may possibly exceed 89°; but experience has shown that until the Dip is nearer 90° than 89°, there may still be found a sufficient horizontal directive force to give consistent results with a Declinometer."

On the return of the Expedition from the Arctic Seas in the summer of 1859, Captain M^cCLINTOCK placed in my hands the hourly observations of the Declinometer, which had been made at Port Kennedy, in lat. 72° 0′ 49″ and long. 94° 19′ W., from November 1858 to March 1859 inclusive, together with remarks, from which the following extracts are made.

"The ship took up her winter position in Port Kennedy on the 27th of September, in thirteen fathoms water, and about 500 yards from the land. The country is very rugged, in many places precipitous to the sea, and is composed of gneiss and granite with masses of trap. No low or level spot could be found sufficiently far from overlooking hills to suit as the site of a magnetic observatory. There remained, however, the alternative of building upon the ice when sufficiently strong. About the middle of October, the ship being now firmly frozen in, I selected a large hummock of old ice, elevated about 2 feet above the recently formed ice, as the best foundation to build upon. It bore magnetic south from the ship, distant 220 yards, and was about 400 yards from the land. I considered therefore its position to be satisfactory.

"Ice was now cut, and, being from 8 to 10 inches in thickness, served well to construct an observatory having an interior space of 7 feet square. The roof of the house, and also of the porch, was of loose planks, covered and cemented together by sludge (snow and water mixed), which also served as mortar for the slabs of ice. The porch was secured by a door, and a fearnought screen protected the entrance from the porch. A pedestal composed of slabs of ice cemented together stood in the centre of the room. A marble slab was placed thereon, and, after being levelled and adjusted at right angles to the

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* The application made to the Admiralty for permission to Mr. GREY to accompany the Expedition was unfortunately not successful. By the zeal of Captain M^cCLINTOCK and of Messrs. HOBSON and ALLEN YOUNG, the loss of Mr. GREY's services was in great measure supplied.

† The magnet of the Declinometer was of the same pattern as that of the Admiralty Standard Compass, consisting of four bars of steel clock-spring, fixed vertically and equidistant in a light framework of brass, carrying a very light metallic ring divided to 5′. The pair of central needles were 7·3 inches long, and the pair of external ones 3·5 inches. The magnet was suspended by a thread of untwisted silk passing over a pulley at the top of a suspension-tube. When not thus suspended, the magnet rested on a pivot of "native alloy"; its weight could be either partially or wholly relieved by means of the suspension-thread. In the hourly observations, the opposite divisions of the graduated circle were read by microscopes carried by the general framework, to which the suspension-tube was also attached.

magnetic meridian, was frozen upon the pedestal. A tripod table-top, with brass grooves to receive the levelling-screws, having been frozen upon the marble slab, the Declinometer was then mounted and levelled; and when all seemed to be in proper working order, the feet of the levelling-screws themselves were frozen to the table, so as to prevent all movement. The magnet carried a graduated circle of 6 inches diameter, divided to 5′, and rested on a pivot supported by an agate cup; its weight could be relieved, either partially or entirely, by a suspension-thread composed of fibres of untwisted silk; the divisions of the circle corresponding to the opposite ends of the magnet were read by fixed microscopes. When the declinometer was first set up and the hourly series commenced, the weight of the magnet was not entirely relieved by the suspension-thread: in this state, and after an interval of two days from the first adjustment, the torsion-force was observed as follows:—

Torsion Circle.	Reading.
	° ′
At zero	218 05
Turned 360° to the East	216 20
At zero	218 10
Turned 360° to the West	220 10
At zero	218 05

whence we should have 115′ as the effect of 360° of torsion, or about 0′·3 as the effect of 1° of torsion. At first, however, and as thus adjusted, the declinometer did not appear to work in a thoroughly satisfactory manner. This may have been occasioned by the levelling-adjustments of the magnet suited to the magnetic latitude having altered its centre of gravity and impaired the free action of the pivot in the cup. I therefore removed the supporting pivot altogether on the 4th of December, but in doing so I accidentally broke the suspension-fibre; this was replaced, and the magnet finally adjusted on the 6th of December, supported only by the silk thread; and from this date I consider there could have been nothing to interfere with the exactness of the observations.

"During the first few weeks of the series, accumulations of drift snow upon one side or other of the observatory would slightly alter the level of the ice; upon these occasions I always relevelled the instrument, if necessary, myself. It could not move in azimuth, as it remained frozen to its pedestal. Both ends of the needle were always read off and recorded. Whenever the magnet was either touched, or observed to be in a state of agitation, a note to that effect was entered on the margin of the observation paper.

"As auroras were of frequent occurrence, I have given a Table of those observed during the period of the hourly observations. There was nothing in or near the observatory which could possibly affect the magnet. Withinside the house there were only a wooden candlestick, a copper lamp, and a board upon which the observation paper was fastened with copper tacks."

In the discussion of the results, the means of the readings of the two ends of the magnet have been taken throughout as the position of the magnet corresponding to the time of the observation. The record of the hourly observations from November 1, 1858 to March 27, 1859, comprehending 3384 observations, was placed in the hands of the Non-Commissioned Officers of the Royal Artillery in the Woolwich Establishment to undergo the usual process of examination. After a careful consideration, I judged that a difference of 1° 10′ from the mean or normal position of the same month and hour afforded a suitable value for the standard of disturbance, as separating about a fifth part of the whole body of the observations. There were 748 observations which differed from their respective normals by that amount or more; and these have been accordingly regarded as "disturbed observations." They form about 1 in 4·5 of the whole number; their aggregate values in the different months were as follows:—

	Total aggregate values. ° ′	Ratios to the mean monthly aggregate value.
1858, Nov. 1 to 28 . . .	486 10	1·42
„ Dec. 1 to 31 . . .	397 10	1·16
1859, Jan. 1 to 31 . . .	189 34	0·55
„ Feb. 1 to 28 . . .	284 28	0·83
„ Mar. 1 to 27 . . .	354 47	1·04
Total in the 5 months . .	1712 09	

$$\text{Mean monthly value} \quad . \quad \frac{1712^\circ\ 09'}{5} = 342^\circ\ 26'.$$

Separated into their easterly and westerly constituents, and into the different hours of their occurrence, the Ratios of Easterly and Westerly disturbance at Port Kennedy to the mean hourly easterly and westerly disturbance were obtained in the manner which has been so frequently described; and by expanding these in sines and cosines of the hour-angle and its multiples, the following approximate formulæ are obtained:—

Port Kennedy, lat. 72° 01′ N., long. 94° 20′ W.

Easterly Disturbances.

$$1+{\cdot}90\sin(\alpha+89^\circ\ 18')+{\cdot}31\sin(2\alpha+86^\circ\ 32').$$

Probable Error of a single observed hourly Ratio ±0·11.

Westerly Disturbances.

$$1+{\cdot}318\sin(\alpha+272^\circ\ 56')+{\cdot}637\sin(2\alpha+71^\circ\ 58').$$

Probable Error of a single observed hourly Ratio ±0·17.

From Table III., in the discussion of the hourly observations at Point Barrow*, we have the corresponding formulæ at that Station as follows:—

Point Barrow, lat. 71° 21′ N., long. 156° 15′ W.

Easterly Disturbances.

$$1+1{\cdot}087\,(\sin\alpha+171^\circ\,20')+{\cdot}523\,(\sin 2\alpha+200^\circ\,31').$$

Probable Error of a single observed hourly Ratio $\pm 0{\cdot}23$.

Westerly Disturbances.

$$1+0{\cdot}673\,(\sin\alpha+264^\circ\,17')+{\cdot}568\,(\sin 2\alpha+94^\circ\,49').$$

Probable Error of a single observed hourly Ratio $\pm 0{\cdot}13$.

From these formulæ we have the Ratios of easterly and westerly disturbance at the several hours of local astronomical time at the two stations, as shown in the following Table:—

TABLE I.

Local Astron. Hours.	Port Kennedy.		Point Barrow.		Local Civil Hours.
	Easterly Ratios.	Westerly Ratios.	Easterly Ratios.	Westerly Ratios.	
(1)	(2)	(3)	(4)	(5)	(6)
0	2·20	1·29	0·98	0·89	Noon.
1	2·14	1·32	0·48	0·81	1 P.M.
2	1·95	1·20	0·08	0·63	2 P.M.
3	1·66	0·99	−0·13	0·43	3 P.M.
4	1·32	0·73	−0·18	0·29	4 P.M.
5	0·98	0·50	−0·08	0·26	5 P.M.
6	0·70	0·41	0·11	0·37	6 P.M.
7	0·51	0·48	0·32	0·64	7 P.M.
8	0·39	0·70	0·51	1·04	8 P.M.
9	0·35	1·04	0·61	1·48	9 P.M.
10	0·37	1·41	0·65	1·87	10 P.M.
11	0·39	1·74	0·65	2·14	11 P.M.
12	0·41	1·93	0·66	2·23	Midnight.
13	0·40	1·93	0·72	2·13	1 A.M.
14	0·39	1·75	0·88	1·85	2 A.M.
15	0·38	1·41	1·15	1·47	3 A.M.
16	0·40	1·01	1·52	1·07	4 A.M.
17	0·50	0·64	1·90	0·72	5 A.M.
18	0·68	0·57	2·25	0·51	6 A.M.
19	0·95	0·28	2·48	0·42	7 A.M.
20	1·27	0·36	2·53	0·48	8 A.M.
21	1·61	0·56	2·37	0·62	9 A.M.
22	1·91	0·85	2·01	0·77	10 A.M.
23	2·13	1·12	1·63	0·88	11 A.M.

The easterly and westerly deflections at Port Kennedy and Point Barrow present the same general features as at all other stations where the laws of the disturbances have been investigated. In Plate XLI., figs. 1 & 3 represent graphically the easterly ratios in Table I. columns 2 & 4, as do figs. 2 & 4 the westerly ratios in columns 3 & 5 of the same Table. It will be seen that figs. 1 & 3 show the conical form and single maximum, and the small and nearly equable amount of variation during the ten or eleven hours when the ratios are least, which characterize figs. 1, 4, 5, & 6 in Plate XIII. Phil. Trans. 1863, Art. XII. Similarly figs. 2 & 4 show a double maximum resembling that which is seen in fig. 2 in Plate XIII. Phil. Trans. 1863, Art. XII. In the case of the Easterly Disturbances, the conical summit or extreme easterly deflection occurs, as will be seen, approximately at the same *absolute* time at Port Kennedy and Point Barrow; and the principal maximum of westerly disturbance at the same *local* time at the two stations. The secondary maximum of westerly disturbance is less strongly marked in the Point Barrow than in the Port Kennedy curve, and its epoch is not so identically the same at both stations as is the case in the principal maximum. This may be due to the magnitude of the disturbances, and the shortness of the time during which the observations at either station were maintained; or there may be a real difference in the epoch and amount of the secondary maximum. The accord at the two stations of the principal easterly maximum in *absolute* time and of the principal westerly maximum in *local* time is too remarkable to be passed unnoticed, though it is certainly *possible* that the accord is in both cases simply an accidental coincidence. The stations at which the laws of the disturbances have been approximately investigated are as yet too few to make an attempt at a more extensive generalization, at present, either safe or advantageous. What seems most to be desired is, that stations for further research should be selected upon a systematic plan, and with reference especially to their geographical relations; and that the inquiry should not be limited to the disturbances of the declination, but should include those also of the dip and total force. By the combination of the facts which would be thus obtained, we might have a reasonable prospect of gaining an assured knowledge of the general laws by which these phenomena are governed in all parts of the globe. To initiate this scheme of research, which would have been at the same time important to science and honourable to our country, was the object of the recommendation made to Her Majesty's Government in 1858 by the two principal scientific institutions of Great Britain. Until some such systematic proceeding is adopted, the progress of this branch of magnetical science is likely to remain fragmentary.

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* Philosophical Transactions for 1857, Art. XXIV. p. 502.

Port Kennedy and Point Barrow have a common magnetical relation in being both situated to the geographical *North* of a critical locality in the magnetic system, viz. the locality of greatest total magnetic force in the northern hemisphere, or the centre of the larger loop of the isodynamic lemniscates. The geographical latitude is nearly the same, but in geographical longitude they differ 61°, or about four hours in time. Port Kennedy is situated on the eastern and Point Barrow on the western side of the American Continent and its adjacent islands. Their distance apart is about 1200 geographical miles. The normal direction of the magnet is widely different at the two stations, the Declination at Port Kennedy being N. 135° 47′ W. (1858), and at Point Barrow N. 41° E. (1854); the magnet therefore points in nearly opposite geographical directions at the two stations. There is also a considerable and an important difference in the amount of the Dip, and consequently in the antagonistic force by which the horizontal component of the earth's magnetism opposes the action of any disturbing force. At Point Barrow, where the dip was 81° 36′, the intensity of the terrestrial horizontal force had still an absolute value of about 1·88 in British units, being about half its value in our own islands; whilst at Port Kennedy, the dip being 88° 27′·4[*], the horizontal magnet was nearly *astatic*. It is evident that, from this great inferiority in the retaining force at Port Kennedy, we ought to be prepared for a generally much greater apparent amount of disturbance at that station than at Point Barrow; and accordingly we find that whilst at the latter a disturbance-value of 22′·87 caused the separation in the category of *large disturbances* of between one-fifth and one-sixth of the whole body of hourly observations, it required a disturbance-value of 70′ to separate a nearly equal proportion of the observations at Port Kennedy. On the hypothesis of the energy of the disturbing force being *equal* at the two stations, and taking, as a sufficient approximation, the statement that one in every five hourly observations at Point Barrow is in excess of its normal of the same month and hour by an amount equalling or exceeding 22′·87, a very simple calculation will show what the amount of the disturbance-value should be which should place the same proportion, or one-fifth, of the whole hourly observations at Port Kennedy in the category of large disturbances. For this purpose we may take from the most recent maps of the isodynamic lines the total terrestrial magnetic force, approximately the same at both stations, =12·9 in British units; then, having the Dip at Point Barrow 81° 36′, and at Port Kennedy 88° 27′, we have the terrestrial horizontal force 1·88 at Point Barrow, and 0·35 at Port Kennedy. Whence we find that, on the hypothesis of there being an equal energy of the disturbing force at the two stations, the disturbance-value corresponding to 22′·87 at Point Barrow should have been 123′ at Port Kennedy instead of 70′. Whilst, therefore, there is an increase in the *effect* of the disturbing action at Port Kennedy by reason of the diminution of the antagonistic horizontal terrestrial force, there is obviously also evidence of an actual and very considerable superiority in the energy of the *disturbing force itself* at Point Barrow as compared with Port Kennedy.

The inference which we thus derive from the direct comparison of the disturbances at Port Kennedy and Point Barrow is in accordance with the fact previously made known

[*] Observations of the Dip at Port Kennedy made on the Ice, far distant either from the Ship or the Land.

1858.	Needle.	Poles.		Means.	Observer.
		Direct.	Reversed.		
Oct. 9	A 1	88° 33′·1	88° 38′·2	88° 35′·5	Capt. M^{c}Clintock.
„ 9	A 2	88 30·1	88 25·1	88 27·5	„
„ 21	A 1	88 19·3	88 32·8	88 26·0	Capt. Allen Young.
„ 28	A 1	88 25·3	88 27·0	88 26·1	Capt. M^{c}Clintock.
„ 29	A 1	88 27·3	88 29·1	88 28·2	„
Nov. 2	A 1	88 20·2	88 29·2	88 24·5	„
„ 13	A 1	88 21·3	88 26·4	88 24·0	„
			Mean	88 27·4 N.	

to us by the hourly observations of the Aurora at Point Barrow, for which we are indebted to Captain Maguire and the officers of H.M. Ship 'Plover,' that the prevalence of that well-known concomitant of magnetic disturbance is far greater at Point Barrow than at any other part of the globe where observations have been made. The increased assurance which we now possess by direct comparison, that the maximum of the disturbing energy is not coincident with the *present* locality of the dip of 90°, or with that of the *present* maximum of the total terrestrial magnetic force, may have hereafter an important bearing on the theory of the physical causes which combine in producing the magnetic phenomena of the globe. The number of days on which the Aurora is recorded to have been seen at Port Kennedy in the five months and four days from October 28, 1858 to March 31, 1859, was 42, or little more than one day out of four; whereas at Point Barrow the Aurora is stated to have been seen, during two successive winters, six days out of seven*. The disparity thus shown is further enhanced and rendered more remarkable by the circumstance that, in the decennial disturbance-period, 1853 and 1854 (which were the years of observation at Point Barrow) are years of minimum, and 1858 and 1859 (which were the years of observation at Port Kennedy) are years of maximum disturbance.

Table II.—Auroras recorded at Port Kennedy in the winter months of 1858–59.

Date.	Direction of Aurora.	Date.	Direction of Aurora.
1858.		1859.	
Oct. 28	* S. to W.	Jan. 1	* W. to S.
29	* S.S.E. to W.N.W.	2	* S.W.
30	* S.W.	3	S.E.
31	N.W.	8	W.S.W. to S.E.
Nov. 6	S.E. to W.S.W.	9 A.M.	* W. to N.W.
7	* S.W.	9 P.M.	N. to S. through zenith.
8	* S.W.	10 A.M.	* N.W. to S.E^{y}, S.
9	* S. to W.	10 P.M.	N. to S. through zenith.
12	N. to zenith.	11	* S.E. to W.
14	* W.N.W. to S.W.	31 A.M.	* N.W. to S.E^{y}, S.
Dec. 3	* S.W.	31 P.M.	W.S.E. to zenith.
4	E. through S.W., N.W.	Feb. 1	* N W. to S.E^{y}, S.
5	* N.W. to S.E.	8	* S.W.
6	* W. to S.E.	19	N. to S. through zenith.
8	S.E.	20	S. to zenith.
12	* N.W. to S.E. through S.	23	N.E. to S.W.
13	* W.N.W. to S.S.E.	26	N. to S. through zenith.
14	* N.W. to E.S.E. through S.	Mar. 6	N.N.W. to S.S.E. through zenith.
15	N.W. through S. to E.	30	* W. to S.W.
24	All over the heavens.	31	* W.
28	* West^{y} to S.S.E.		
30	S.		

The following remarks by Dr. David Walker, R.N., by whom the record of the Auroras was kept, will be read with interest. "Of the 42 Auroras observed during our winter, 24 (marked with an asterisk) were in the direction of a space of water open throughout the winter, or of the vapour rising from it. More than this number might be traced to it; but of these 24 I am certain. On five occasions the Aurora caused an agitation of the Declinometer: on one of these (Dec. 24, 1858) I observed a deflection of 15°; on the other four times the vibration was not much more than a degree; four of the five occurred when the Aurora was from north to south, passing through the zenith."

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* Philosophical Transactions for 1857, Art. XXIV. p. 512.

Table III. contains a statement, taken from the record of the hourly observations, of the days and hours on which disturbances exceeding 5° from the normal of the same month and hour were observed.

Table III.—Port Kennedy. Differences exceeding 5° from the normal of the same month and hour shown by the Declinometer.

Day.	Hour.	Disturbance.
1858.		
Nov. 18 ……	21	5° 23′ E.
18 ……	22	5 35 E.
18 ……	23	5 55 E.
19 ……	22	7 25 E.
19 ……	23	6 47 E.
28 ……	20	6 08 E.
28 ……	21	5 47 E.
28 ……	22	6 46 E.
Dec. 4 ……	11	5 35 W.
4 ……	15	6 00 W.
4 ……	16	7 35 W.
4 ……	17	6 27 W.
4 ……	18	5 40 W.
12 ……	19	6 42 E.
22 ……	22	5 04 E.
1858.		
Dec. 23 ……	21	6° 59′ E.
23 ……	22	7 09 E.
23 ……	23	6 11 E.
24 ……	11	7 51 E.
1859.		
Jan. 14 ……	22	8 00 E.
16 ……	0	7 56 E.
Feb. 9 ……	0	9 31 E.
9 ……	1	6 12 E.
23 ……	1	10 10 E.
23 ……	2	5 10 E.
25 ……	21	5 01 E.
Mar. 25 ……	3	5 57 E.
26 ……	0	6 51 E.
26 ……	1	5 56 E.

December 4, 22, and 23, February 9 and 23, were also days of excessive disturbance at Kew (Phil. Trans. 1863, Art. XII. Table I.).

Disturbance-diurnal Variation.—Table IV. exhibits the disturbance-diurnal variation at Port Kennedy, or the average excess at the several hours of easterly over westerly, or of westerly over easterly, deflection.

Table IV.

Local Hours.	Deflections.		Disturbance-diurnal variation.	Local Hours.	Deflections.		Disturbance-diurnal variation.
	Easterly.	Westerly.			Easterly.	Westerly.	
6 A.M.	11′·7	6′·2	5′·5 E.	6 P.M.	10′·9	3′·1	7′·8 E.
7 A.M.	15·4	5·3	10·1 E.	7 P.M.	9·3	2·1	7·2 E.
8 A.M.	20·8	5·2	15·6 E.	8 P.M.	6·8	4·8	2·0 E.
9 A.M.	35·4	6·7	28·7 E.	9 P.M.	6·0	9·9	3·9 W.
10 A.M.	37·8	5·9	31·9 E.	10 P.M.	5·9	19·1	13·2 W.
11 A.M.	31·8	9·9	21·9 E.	11 P.M.	10·2	20·4	10·2 W.
Noon.	35·6	10·7	24·9 E.	Midnight.	6·2	24·1	17·9 W.
1 P.M.	36·6	17·6	19·0 E.	1 A.M.	7·5	18·8	11·3 W.
2 P.M.	43·1	12·3	30·8 E.	2 A.M.	6·5	15·8	9·3 W.
3 P.M.	30·6	9·6	21·0 E.	3 A.M.	6·2	11·0	4·8 W.
4 P.M.	21·2	13·7	7·5 E.	4 A.M.	6·8	10·9	4·1 W.
5 P.M.	18·3	4·0	14·3 E.	5 A.M.	10·2	7·0	3·2 E.

We have in this Table the opportunity of perceiving how effectually the disturbance-diurnal variation may operate in masking or disfiguring the regular progression of the solar-diurnal variation, when the disturbances are not eliminated. It is well known that the solar-diurnal variation produces generally in the extratropical parts of the northern hemisphere a maximum deflection to the East about 8 A.M., and a maximum deflection to the West about 2 P.M. (and this is found to be the case at Port Kennedy, as well as elsewhere, when the disturbances are eliminated). Now the Table shows that the Easterly extreme about 8 A.M. must be considerably more than doubled by the occurrence at the same hour of a large easterly disturbance-deflection, and that at 2 P.M. the Easterly disturbance-deflection has become so large as to far more than compensate the effect of the usual amount of the solar-diurnal westerly maximum belonging to that hour, making the joint deflection at that hour a considerable *easterly* one. It will also be seen that when the two kinds of variation are left unseparated, their joint effect produces a large nocturnal maximum of westerly deflection, which disappears in the solar-diurnal variation when the disturbances are eliminated.

Solar-diurnal Variation.—The almost extreme difference in the normal direction of the magnet, *geographically considered*, at Port Kennedy and Point Barrow, gives a more than ordinary importance to the comparison of the facts of the solar-diurnal variation at the two stations, rendering it an apt illustration, in an almost extreme case, of the laws by which this class of phenomena is regulated. Magnetically speaking, the mean direction of the magnet is necessarily the same at the two stations; that is to say, the mean pointing of the marked end of the magnet, which we usually term its north pole, is to the magnetic north; but in a geographical sense this direction is at Port Kennedy about 35° to the West of South, and at Point Barrow about 41° to the East of North. The localities afford therefore in this respect a contrast nearly as great as can exist in any part of the globe; since magnetically the directions are the same, whilst geographically they want only 6° of being 180° apart, or diametrically opposite. The value of this contrast appears when we proceed to consider the facts of the solar-diurnal variation at the two stations, and perceive rightly their important bearing on the correct understanding of its true nature and character, and of the physical relations which must be involved in any well-grounded explanation.

To prevent, as far as may be possible, misconception in the minds of those to whom the subject is not familiar, it may be premised that in speaking of the direction of the magnet the eye of the observer is here supposed to be at its middle, and directed towards the marked or north end; a change of direction towards the magnetic east will thus be to the observer's right, and a change towards the west to the observer's left. Now the most marked features of the solar-diurnal variation, and which are found to prevail universally in all the extratropical parts of the northern hemisphere, are, an extreme deflection to the observer's right (or towards the magnetic east) about 8 A.M., and an extreme deflection to the observer's left (or towards the magnetic west) about 2 P.M.

Whenever the phenomena are viewed within the aforesaid limits, the facts thus referred to are identical *when expressed magnetically.*

This description of the solar-diurnal variation, in which all geographical relations are put aside, applies with equal correctness to the phenomena at Port Kennedy and at Point Barrow; but when geographical relations are again introduced, the same phenomena have to be described in a very different manner, and the two stations become widely distinguished from each other*. The marked end of the magnet, when looked at at 8 A.M., is seen at Port Kennedy to have moved from its mean position of S. 35° W. towards the geographical *West*, and at Point Barrow to have moved from its mean position of N. 41° E. towards the geographical *East*; and correspondingly at 2 P.M. the marked end is at Port Kennedy to the geographical *East* of its mean position, and at Point Barrow to the geographical *West*. The bearing of this distinction between the magnetical and geographical aspects of the facts upon physical explanations will be evident if we advert to the hypothesis of currents of thermic origin, either in the earth or in the atmosphere, generated by the rotation of the earth in presence of the sun. It may be well therefore to take a more general view of the phenomena of which the two stations which have been here compared present a particular case, which fortunately is a very notable and instructive one. Let us imagine (as in the woodcut in the next page) two stations, *a* and *b*, both situated in the vicinity of the dip of 90°; and (to avoid questions of absolute and local time) let us assume them to be in the same geographical meridian, *a* being situated to the geographical north, and *b* to the geographical south of the locality where the dip is 90°. Then at both stations the magnet, when in its mean position, will point magnetically north; but at *a* this direction will be geographically south, whilst at *b* it will be geographically (as well as magnetically) north. Let us next consider the direction which the magnet will be found to have assumed at *a* and *b* respectively when at the extreme points of opposite deflection due to the solar-diurnal variation. These synchronize everywhere (as far as is yet known) in the extratropical parts of the northern hemisphere, approximately with the local hours of 8 A.M. and 2 P.M. (which for convenience we will call precisely 8 A.M. and 2 P.M.),—at 8 A.M. the north pole of the magnet being everywhere to the observer's right, or to the magnetic east of its mean position, and at 2 P.M. to the observer's left, or to the magnetic west of its mean position. At *a* the magnetic east is geographic west, and *vice versâ*; while at *b* the geographic and magnetic east are the same; therefore at 8 A.M. the magnet is deflected geographically to the west at *a* and to the east at *b*, and at 2 P.M. geographically to the east at *a* and to the west at *b*. Let us next consider the direction in which the north end of the magnet *moves* between 8 A.M. and 2 P.M. at *a* and at *b*: at both stations the movement is from the observer's right to his left, from the magnetic east to the magnetic west. But at *a* this movement, viewed geographically, is from west to east, whilst at *b* it is geographically as well as magnetically from east to west. Whilst, therefore, the direction of the movement is magnetically the same, geographically it is opposite.

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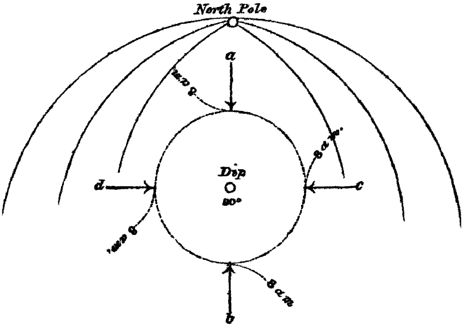
* Throughout this discussion regarding the solar-diurnal variation, the disturbances are, of course, assumed to have been eliminated.

Now let us take two other stations *c* and *d*, *c* to the east and *d* to the west of 90° of dip, and both situated, not as in the woodcut (where, for convenience in illustration, they are separated by a considerable meridional interval), but in its vicinity, so that the distance between them may not be such as to make an important difference in their local time. The mean direction of the north pole of the magnet will necessarily be at both stations to the magnetic north; but at *c* this will be to the geographic west, and at *d* to the geographic east. At 8 A.M. the deflection will be to the observer's right, or magnetic east at both stations; but this will be at *c* to the geographic north, and at *d* to the geographic south; whilst at 2 P.M. the deflections at both stations will be to the observer's left or magnetic west; but this will be at *c* to the geographic south, and at *d* to the geographic north. As in the former case, the direction of the movement between 8 A.M. and 2 P.M. is magnetically the same, but geographically opposite.



Now, keeping these facts in view, let us imagine a circle to be drawn round the point of 90° of dip, passing through *a*, *b*, *c*, and *d*; at every point in the periphery of that circle the mean direction of the marked end of the magnet will be *magnetic north*, but will have every possible diversity of *geographical direction*. At 8 A.M. the deflection due to the solar-diurnal variation will be everywhere to the magnetic east, and at 2 P.M. to the magnetic west; whilst at both hours it will have at different points in the periphery of the circle every possible diversity of geographical direction. Likewise the movement from 8 A.M. to 2 P.M. will be, at every point in the periphery, from the magnetic east to the magnetic west, whilst geographically it will have every possible diversity.

It is obvious that what is here stated of points taken in the periphery of the circle is equally true of every point taken in the interior of the circle, until the point of 90° of dip is so nearly approached as to render the horizontal magnet absolutely astatic.

The facts of the solar-diurnal variation at Port Kennedy and Point Barrow, after the elimination of the larger disturbances, furnish a practical exemplification of the justice of this description in all its details.

The magnitude of the disturbance-diurnal variation at Port Kennedy and Point Barrow, compared with that of the solar-diurnal variation about the hours when the diurnal inequality (which is the resultant of the two variations combined) is at its extreme eastern and western limits, affords an instructive example to those who employ *the magnitude of the diurnal range* in different years as a means of tracing the epochs of maximum and minimum of the magnetic variation in the decennial period. Referring to Table IV., we find that at 8 A.M., the usual hour in Europe of the easterly extreme of the diurnal inequality, that extreme is augmented at Port Kennedy by a disturbance-deflection amounting, on the average of the five months during which the observations were maintained, to above 15′ easterly; whilst at 2 P.M., the usual hour of the westerly extreme, there is the counteracting influence of a disturbance-deflection, which is still easterly, exceeding 30′. Now, as both these values, 15′ and 30′, very considerably exceed the ordinary deflections caused by the regular solar-diurnal variation, either to the East at 8 A.M. or the West at 2 P.M., it is obvious that, at stations where the energy of the disturbing force is considerable, the magnitude of the diurnal range at such stations must be mainly influenced by and dependent on the amount and hours of the disturbance-diurnal variation. Indeed, when we duly consider the extreme liability to variation in these last-named circumstances, we shall be prepared to find that, as magnetical researches are extended, stations present themselves where the effect of the increase of the amount of disturbance at the epochs of maximum of the decennial period is to cause the combination of the two variations to exhibit in such years a *decrease* instead of an increase in the magnitude of the diurnal range—actually causing the epochs of maximum and minimum in the cycle to apparently change places with each other; in such cases the *minima* of the range of the diurnal inequality will coincide with the maxima of the sun's spots and of the magnetic disturbances; whilst other stations will be found where the difference between the epochs will be apparently increased in amount; and others where it will be obliterated, and no cycle be traceable by this method of inquiry.

The method of tracing the epochs of maximum and minimum of the decennial period by a comparison of the aggregate values of the disturbing action in different years, as shown by the separation and analysis of the disturbances themselves, is not subject to the inconvenience which has been thus noticed: it has also the advantage that the proportionate increase in the amount of disturbance between the epochs of maximum and minimum, 2·5 : 1 (St. Helena Observations, vol. ii. p. cxxxi), is much greater than the difference in the range of the diurnal inequality, or of the solar-diurnal variation, and forms therefore a larger basis upon which the judgment may be grounded.

The Table (IV.) shows the very large amount of the average disturbance-diurnal variation at certain hours; the comparison of a similar Table prepared in the same way from the observations at Nertchinsk from 1851 to 1857 inclusive will show (when the Table shall be published) the liability at different stations to extreme *variation in the direction* of the disturbance-diurnal variation at the several hours of local time. At Nertchinsk, at 8 A.M. the *Westerly* disturbance-diurnal variation is nearly at its maximum, being a deflection in the contrary sense to that at Port Kennedy, as well as to that of the solar-diurnal variation at both places; whilst at 2 P.M. the average disturbance-deflection is almost null (the westerly being about to pass into the easterly). There is therefore no counterbalance to the diminution which has been effected in the 8 A.M. extreme; and thus at Nertchinsk the diurnal inequality is *lessened* by the effect of the disturbances, and is necessarily most *lessened* at the epoch when the disturbances are *greatest*. In this as in many other instances, we see how liable those are to mislead themselves, who disregard the advice contained in the Royal Society's Report of 1840, to eliminate the disturbances as the first and necessary step in the analysis of the complicated phenomena which constitute the "diurnal inequality."

In the observations which have supplied the subject-matter of this communication, the Royal Society will recognize another instance, added to the many which have preceded it, of the zeal and devotion with which recommendations proceeding from the Society are carried out by our naval officers. Even those who have not themselves experienced an arctic climate may readily imagine that it is no slight effort to maintain with the requisite regularity, for several months together, *hourly* observations which have to be made at a considerable distance from the ship, exposed to the severity of an arctic winter. I venture to think that such a service is well entitled to our thankful recognition.

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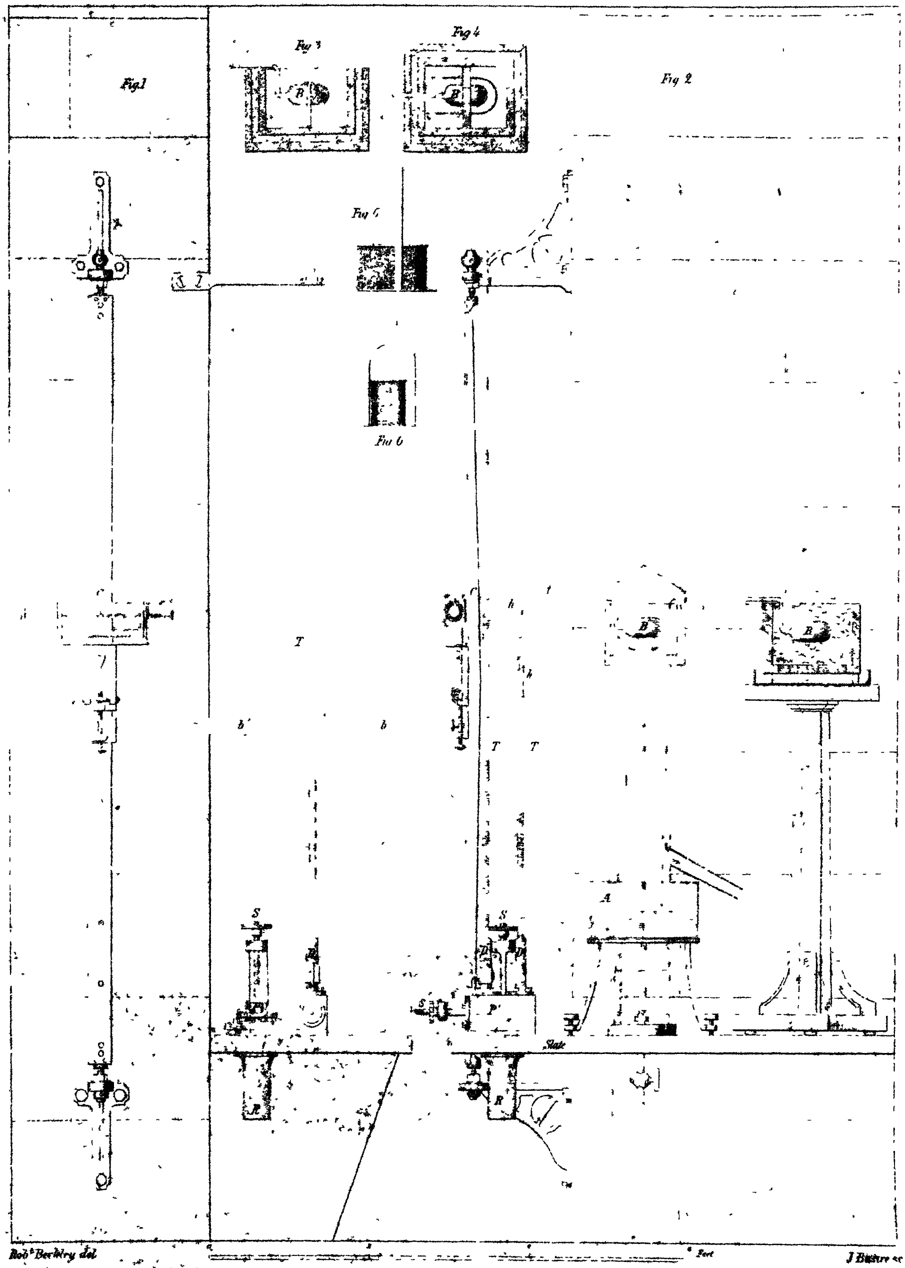
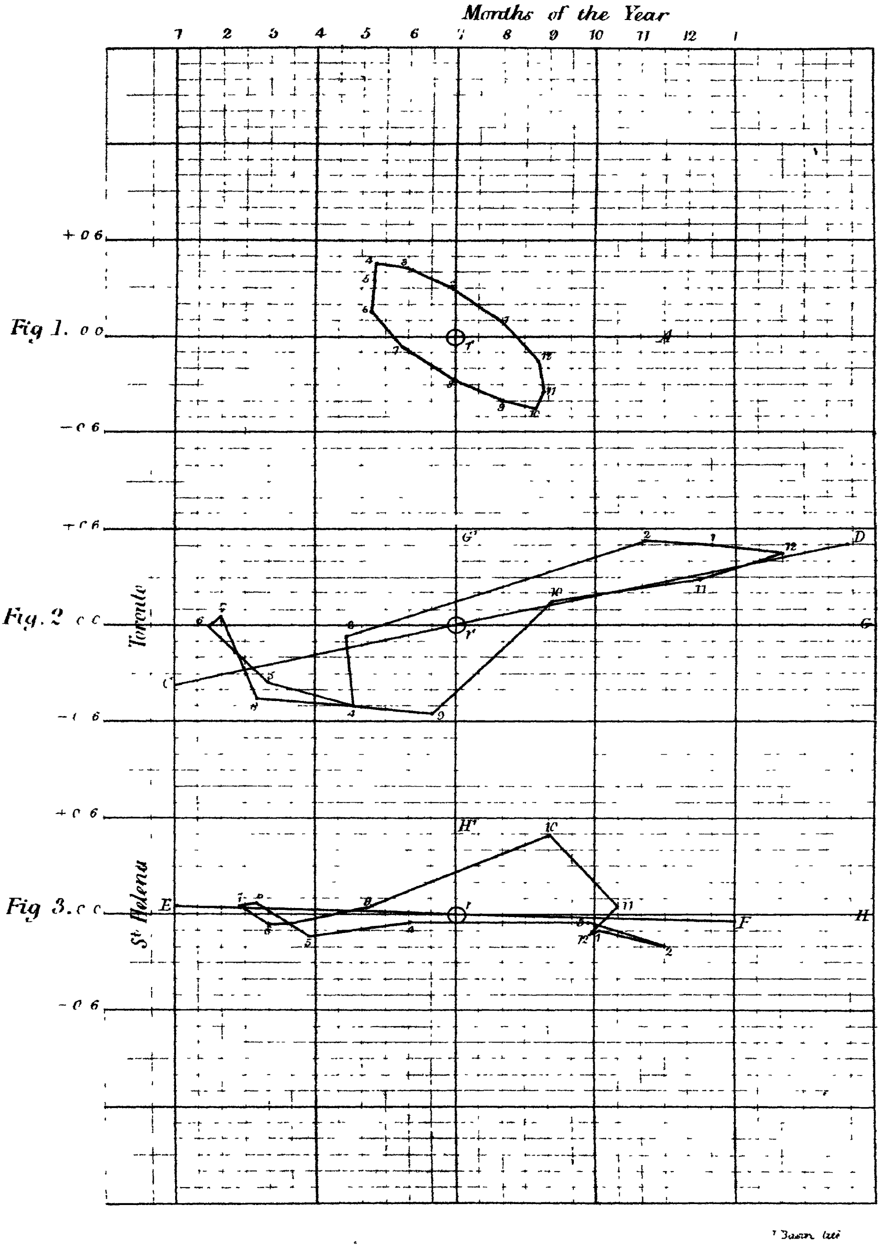
WHEATLEY (H. B.) Of Anagrams: a Monograph treating of their History, with an Introduction. 12mo. *London* 1862.	The Author.
WHITING (G.) The Products and Resources of Tasmania, as illustrated in the International Exhibition 1862. 8vo. *Hobart Town* 1862.	The Author.
WILLIAMS (J.) Account of a Deposit found in an Ancient Chinese Statue of Buddha, June 11, 1862. 8vo. *London* 1863.	The Author.
WORMS (H.) The Earth and its Mechanism: being an Account of the various proofs of the Rotation of the Earth. 8vo. *London* 1862.	Messrs. Longman.
WÜLLERSTORF-URBAIR (B. von) Reise der österreichischen Fregatte Novara um die Erde in den Jahren 1857–59. 3 vols. 8vo. *Wien* 1861.	The I. R. Navy Board, Vienna
———— Reise der osterreichischen Fregatte Novara: Nautisch-physicalischer Theil, Abth. 1 & 2. 4to. *Wien* 1862.	————
———— Indischer Ocean. Beilagen 1–7. (7 Maps.)	————
YATES (James, F.R.S.) On the Excess of Water in the Region of the Earth about New Zealand, its Causes and Effects. (Excerpt from Edinb. New Phil. Journ.) 8vo. 1862.	The Author.



Robt Beckley del

J Basire sc



J. Basire lith.

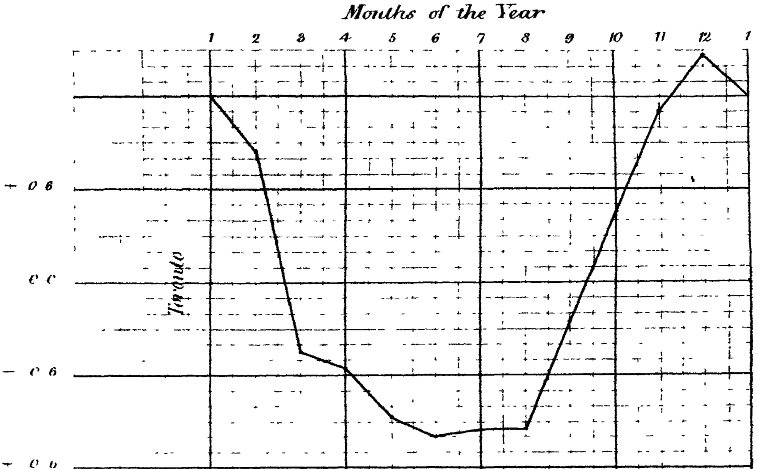


Fig. 5.

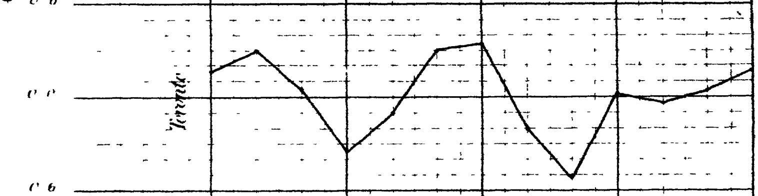


Fig 6

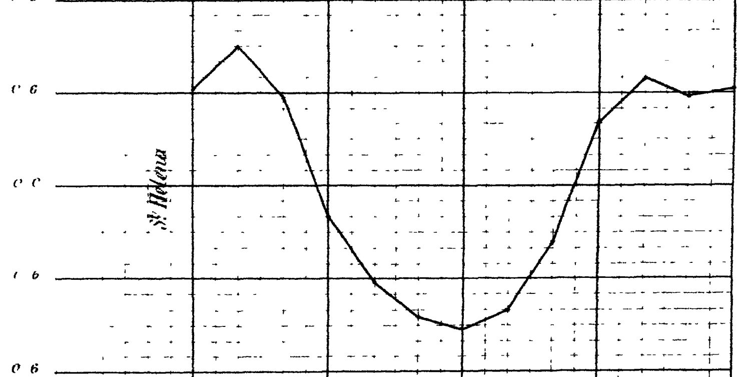


Fig 7.

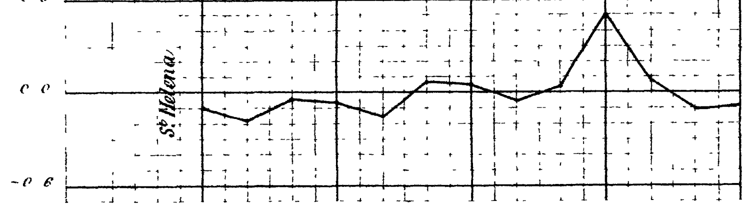


Fig. 8.

J. Basire lith.

Phil. Trans. MDCCCLXIII. Plate XXVI.

Fig 9

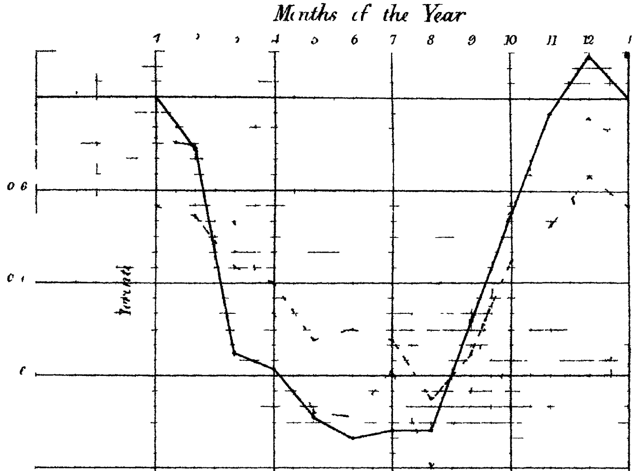


Fig 10

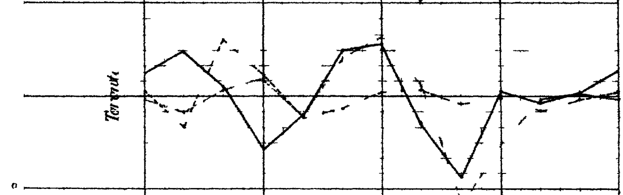


Fig 11

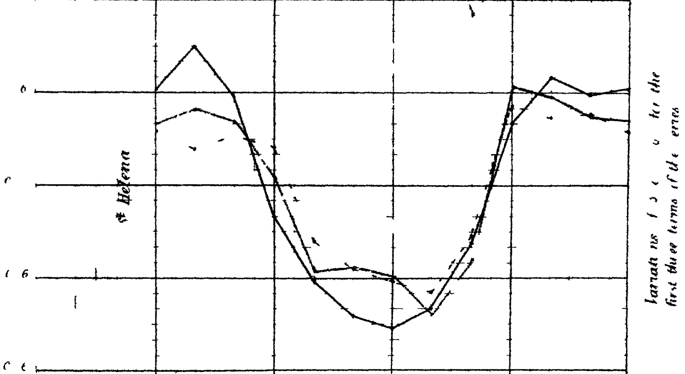
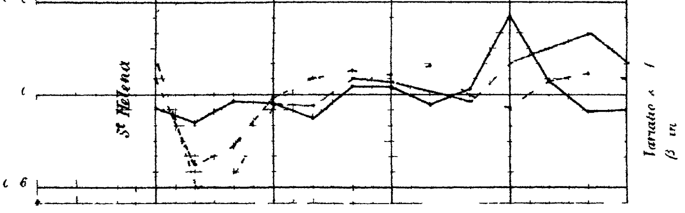
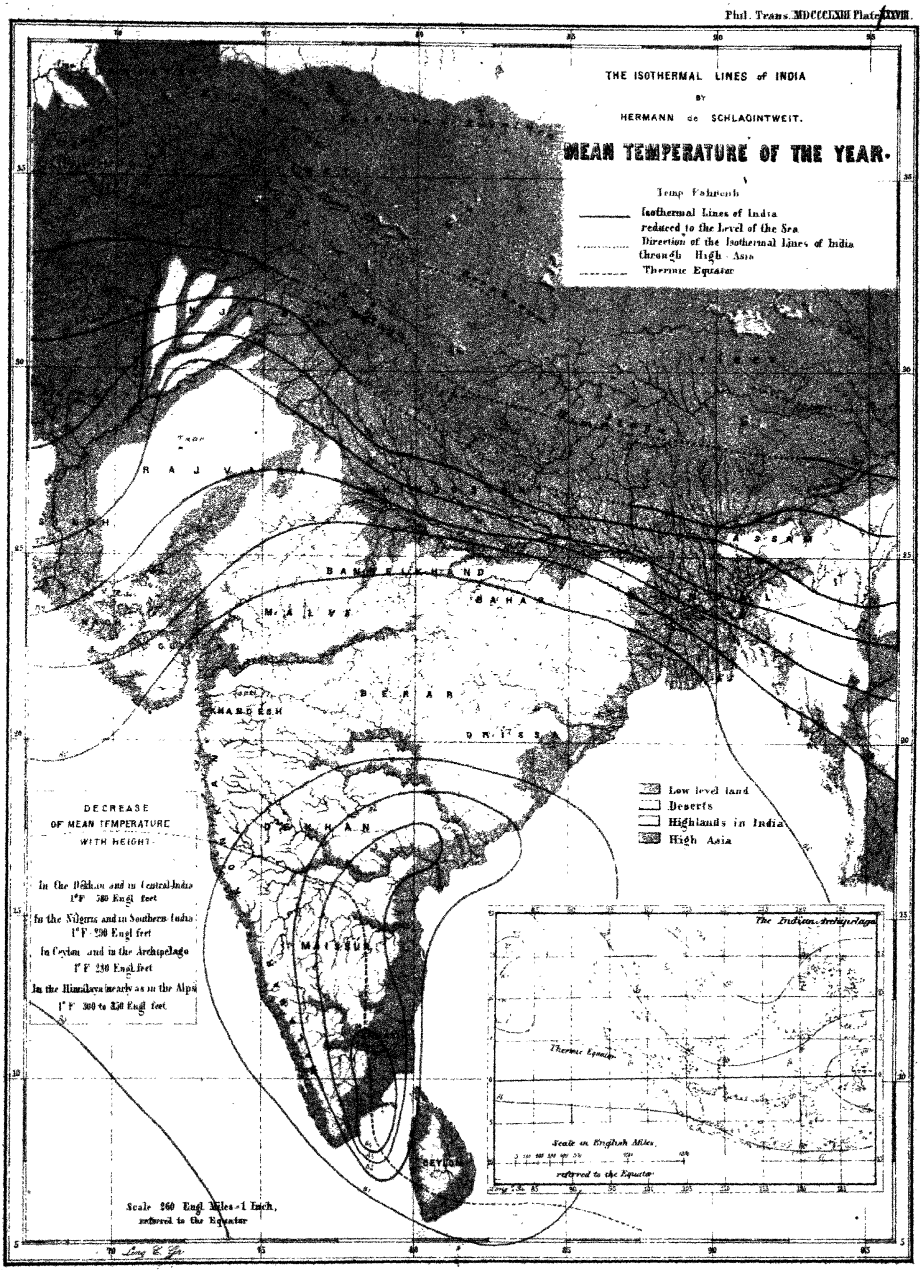


Fig 12



Basire lith.



THE ISOTHERMAL LINES of INDIA

BY

HERMANN de SCHLAGINTWEIT.

MEAN TEMPERATURE OF THE YEAR.

Temp. Fahrenh.

Isothermal Lines of India reduced to the Level of the Sea

Direction of the Isothermal lines of India through High - Asia

Thermic Equator

▨ Low level land

☐ Deserts

☐ Highlands in India

▨ High Asia

DECREASE
OF MEAN TEMPERATURE
WITH HEIGHT.

In the Dekhan and in Central-India
1° F. 580 Engl. feet

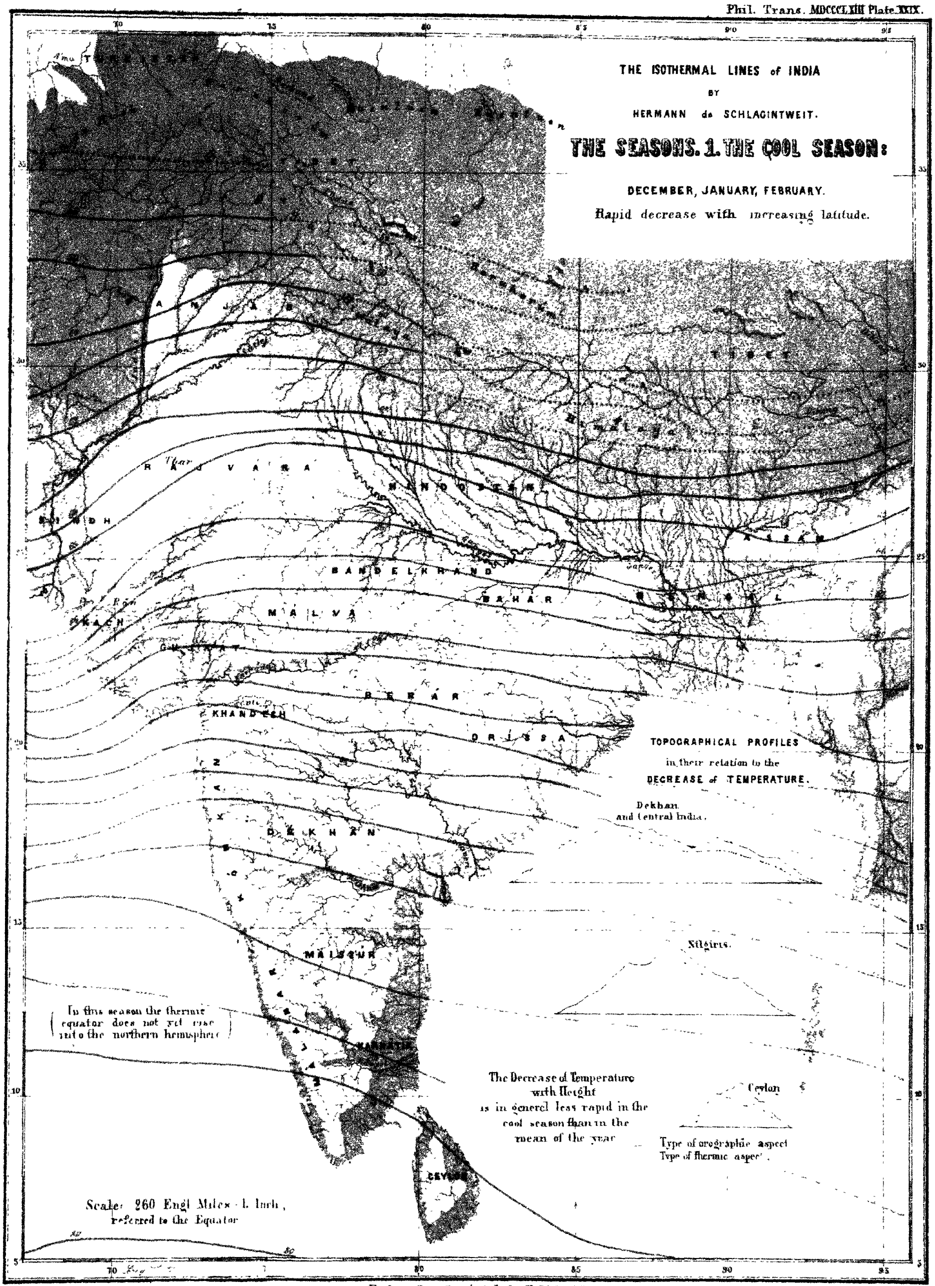
In the Nilgiris and in Southern-India
1° F. 290 Engl. feet

In Ceylon and in the Archipelago
1° F. 230 Engl. feet

In the Himalaya nearly as in the Alps
1° F. 300 to 350 Engl. feet

Scale 260 Engl. Miles = 1 Inch,
referred to the Equator

Engraved and printed by F. Wolf, Munich.



THE ISOTHERMAL LINES of INDIA

BY

HERMANN de SCHLAGINTWEIT.

THE SEASONS. 1. THE COOL SEASON:

DECEMBER, JANUARY, FEBRUARY.

Rapid decrease with increasing latitude.

TOPOGRAPHICAL PROFILES
in their relation to the
DECREASE of TEMPERATURE.

Dekhan
and Central India.

Nilgiris.

Ceylon

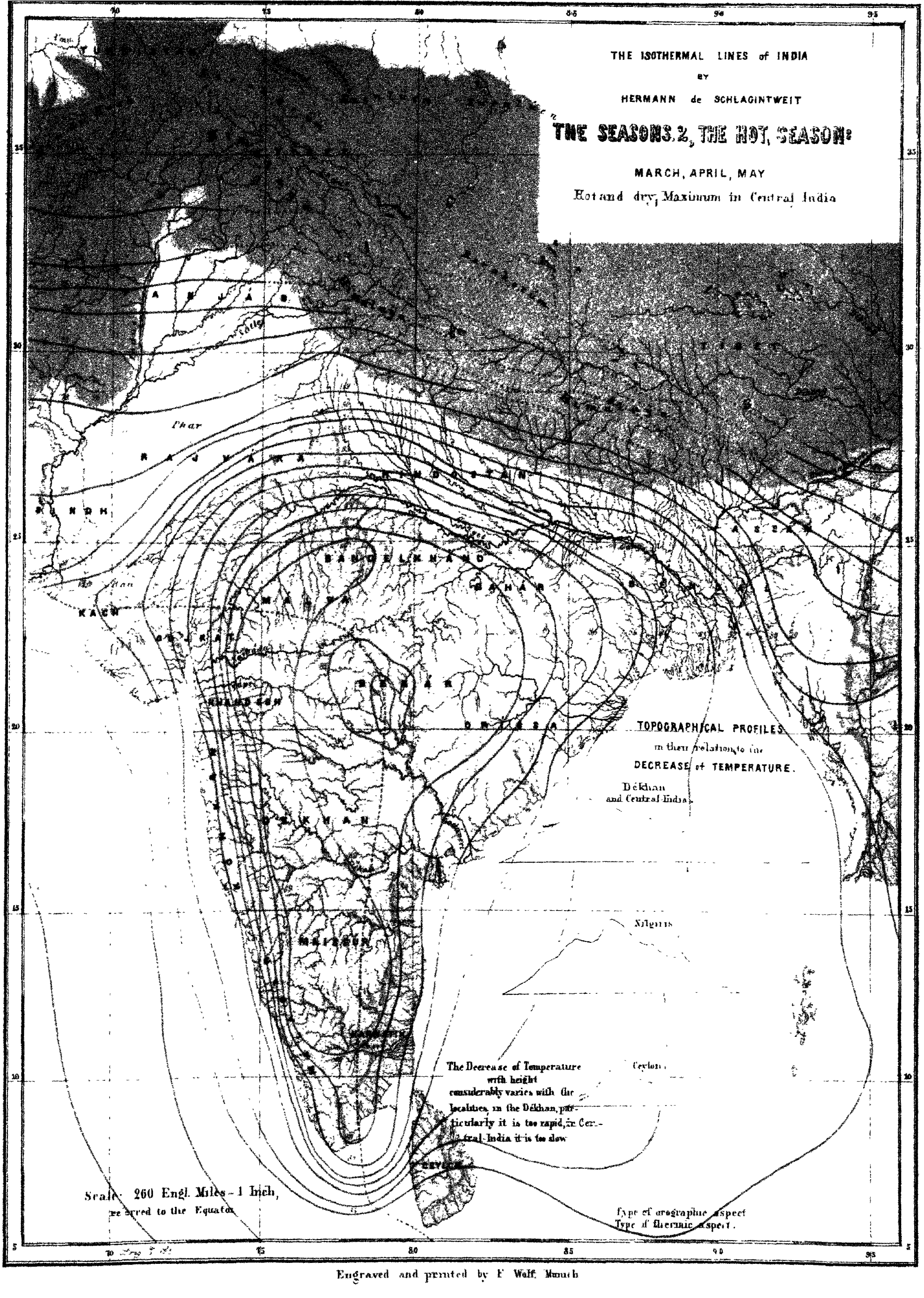
Type of orographic aspect
Type of thermic aspect.

The Decrease of Temperature
with Height
is in generel less rapid in the
cool season than in the
mean of the year

In this season the thermic
equator does not yet rise
into the northern hemisphere

Scale: 260 Engl Miles = 1 Inch,
referred to the Equator

Engraved and printed by F Wolf, Munich.



THE ISOTHERMAL LINES of INDIA

BY

HERMANN de SCHLAGINTWEIT

THE SEASONS. 2, THE HOT SEASON:

MARCH, APRIL, MAY

Hot and dry, Maximum in Central India

TOPOGRAPHICAL PROFILES

in their relation to the

DECREASE of TEMPERATURE.

Dékhan

and Central India.

Nilgiris

Ceylon

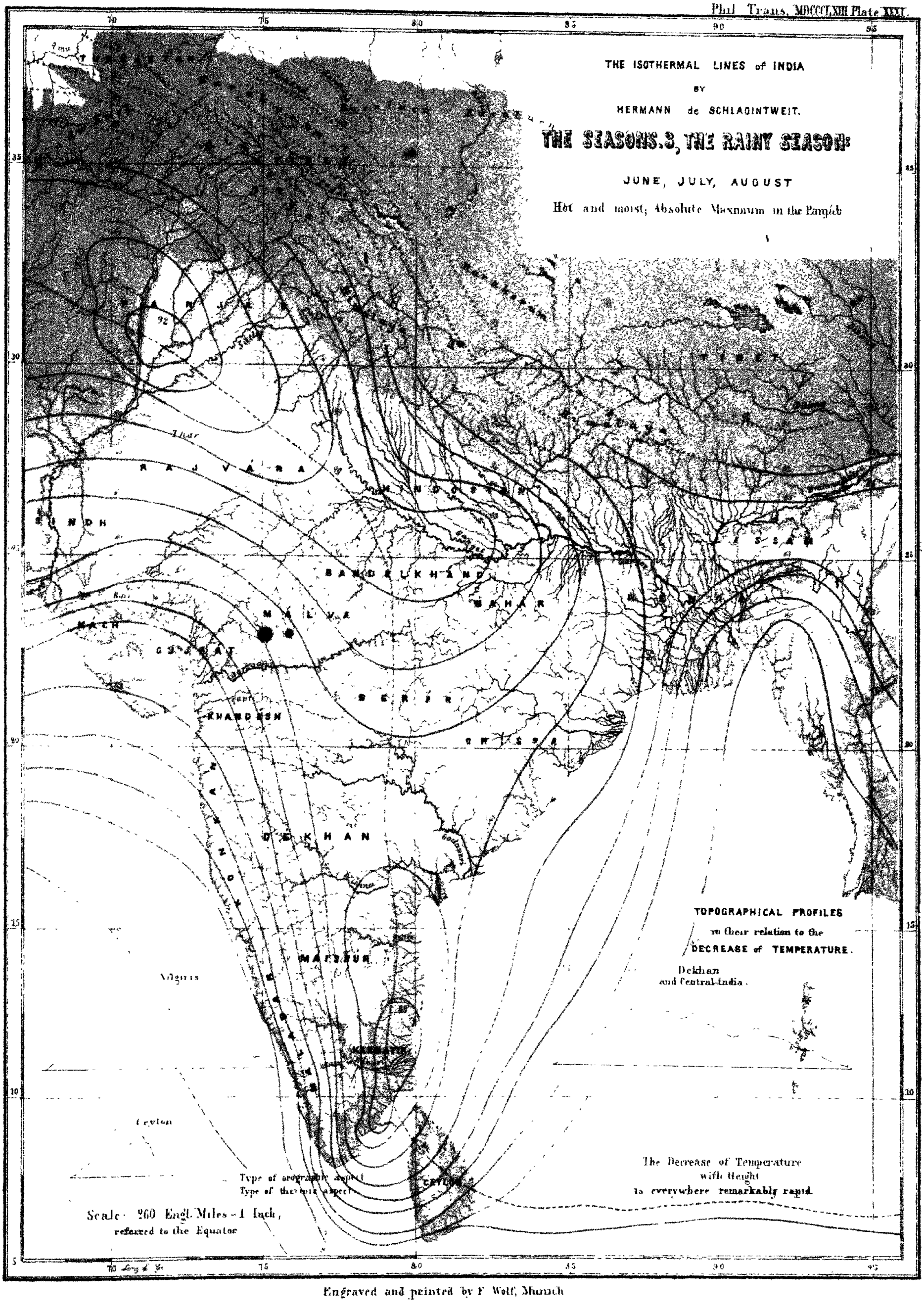
The Decrease of Temperature with height considerably varies with the localities, in the Dékhan, particularly it is too rapid, in Central India it is too slow

Scale 260 Engl. Miles - 1 Inch, referred to the Equator

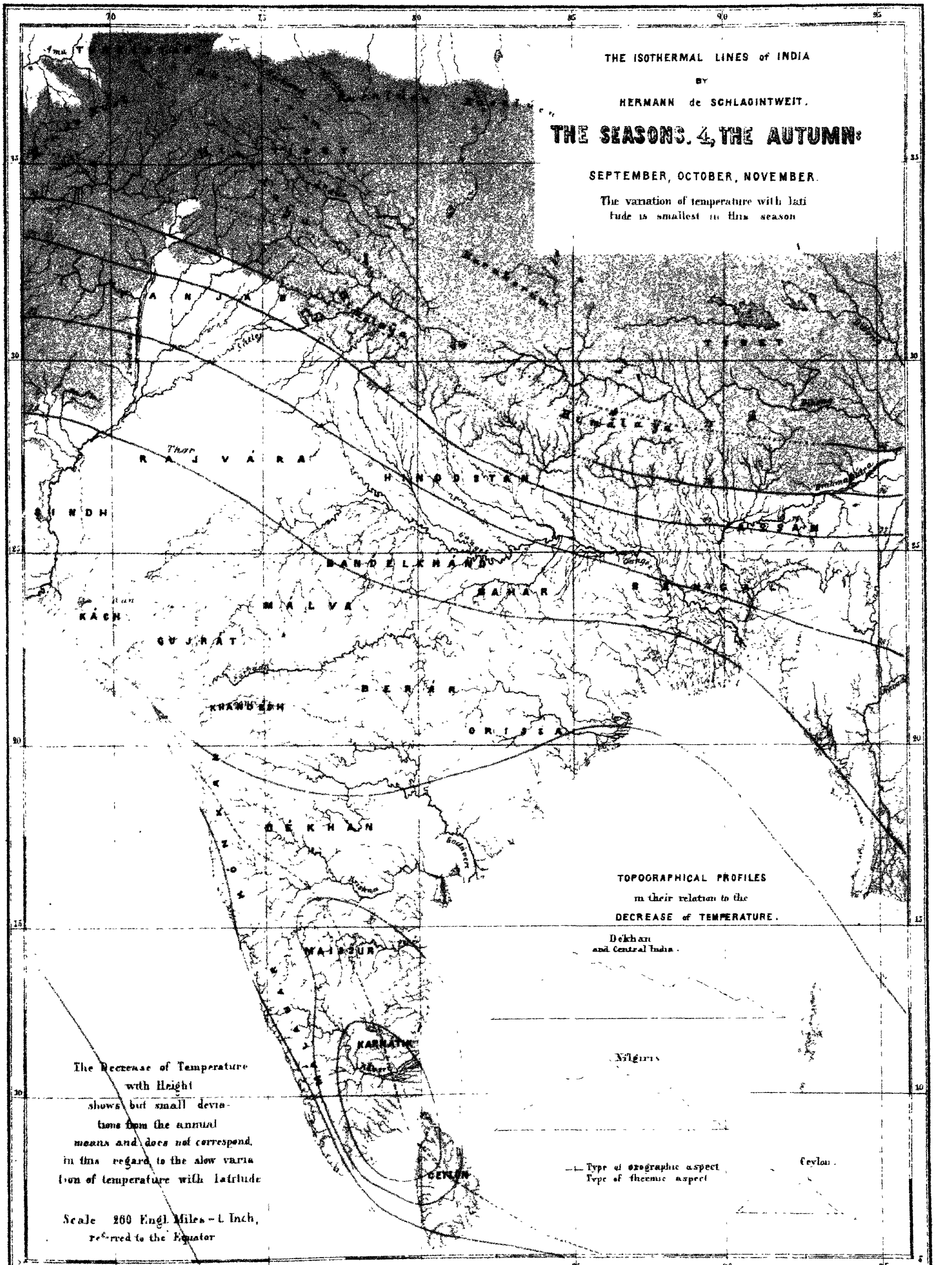
Type of orographic aspect

Type of thermic aspect.

Engraved and printed by F. Wolf, Munich

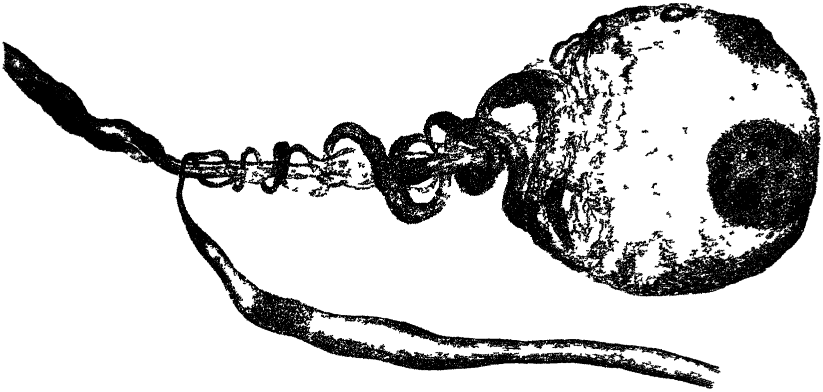


Engraved and printed by F. Wolf, Munich

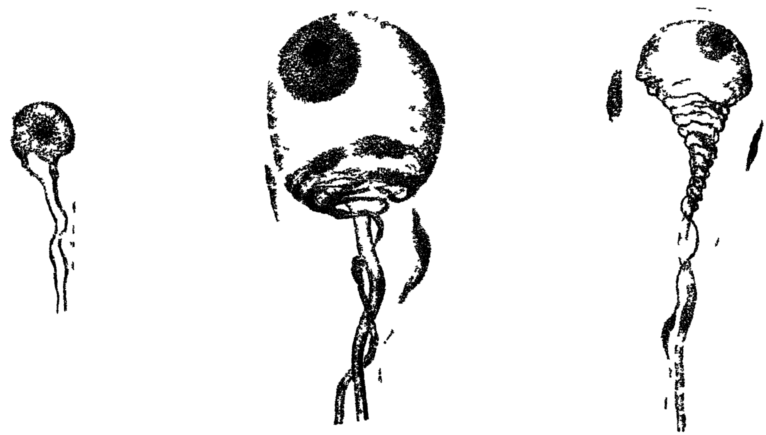

THE ISOTHERMAL LINES of INDIA
BY
HERMANN de SCHLAGINTWEIT.
THE SEASONS. 4, THE AUTUMN:
SEPTEMBER, OCTOBER, NOVEMBER.
The variation of temperature with latitude is smallest in this season
PANJAB
RAJVARA
Thar
SINDH
HINDOSTAN
ASSAM
TIBET
BANDELKHAND
BAHAR
MALVA
KACH
GUJRAT
BERAR
KHANDESH
ORISSA
DEKHAN
MAISUR
KARNATIK
CEYLON
Ganges
Satlej
Krishna
Godavery
TOPOGRAPHICAL PROFILES
in their relation to the
DECREASE of TEMPERATURE.
Dekhan
and Central India.
Nilgiris
Ceylon.
Type of orographic aspect
Type of thermic aspect
The Decrease of Temperature
with Height
shows but small deviations from the annual
means and does not correspond
in this regard to the slow variation of temperature with latitude
Scale 260 Engl. Miles = 1 Inch,
referred to the Equator
70
75
80
85
90
95
35
30
25
20
15
10

Engraved and printed by F. Wolf Munich

Fig. 1.



[illegible] Fig. [illegible] The [illegible] isolated [illegible]



Young ganglion cell. × 7[illegible]

Fully formed ganglion cell. × [illegible]00

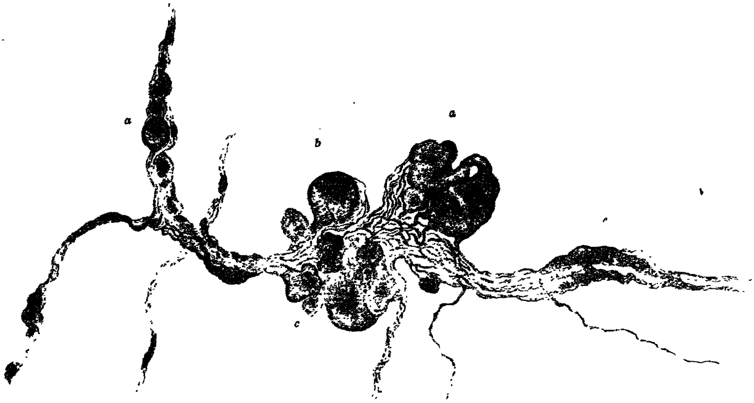
Old ganglion cell. × [illegible]

Three ganglion cells from a ganglion near the aorta of the common Frog, Rana temporaria. In the youngest cell the second fibre does not coil spirally round the first. In Fig. 3 several coils are observed, and in Fig. 4 the coils are very numerous. The spiral increases in extent as the cell advances in age.

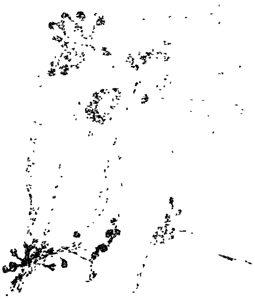
1000th of an inch — — — × 700

1000th of an inch ——————— × 1800

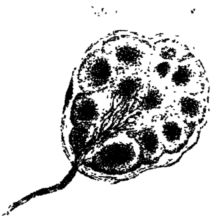
Fig. 6.



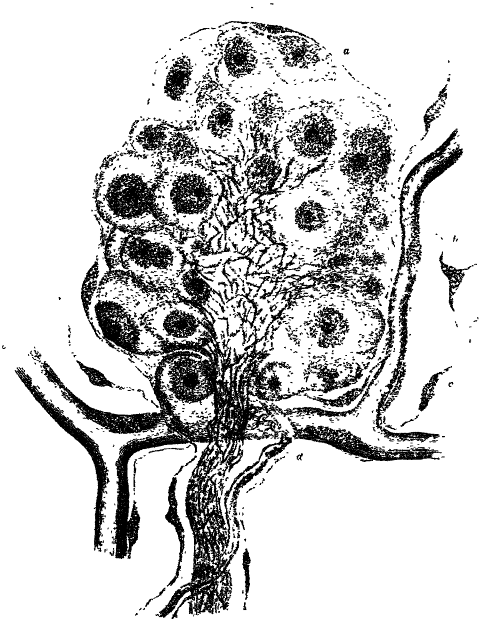
[illegible] At a, [illegible] cells in process of [illegible]. These cells [illegible] b a cell almost [illegible] At d, [illegible] fibres are being formed [illegible] The masses at a at [illegible] are but an early stage of such a ganglion as c, b, [illegible] × 700.



[illegible] nerve trunks [illegible] iliac artery.
Fig. [illegible] × 40



The ganglion cell marked a in Fig. 6. × 700 diameters. The mass is undergoing division, and from it a number of separate cells like the groups in Fig. 6 would result.

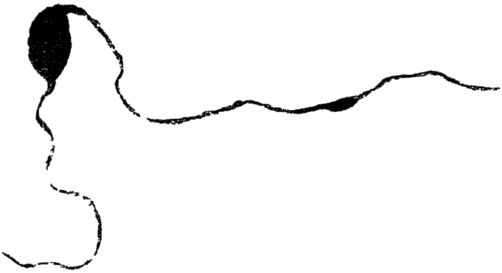


The mass of imperfectly developed ganglion cells at b, Fig. 6 a, a mass dividing into two. b, connective tissue corpuscle c, nerve distributed to capillary vessel, e. d, a more advanced ganglion cell exhibiting straight and spiral fibres, its straight fibre is much thicker than any in the bundle of fine fibres passing into the body of the ganglion × 1800.

1000th of an inch × 700

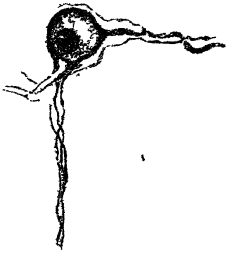
1000th of an inch × 1800

L. S. B. ad nat. del. Harris [illegible]

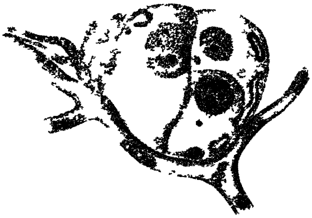


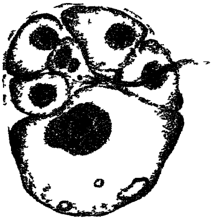
[illegible] from [illegible] ganglion [illegible] Hy [illegible] 180

Fig



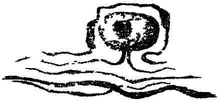
[illegible] g on ce [illegible] several fibres [illegible]
Near heart of Frog × 60

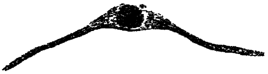




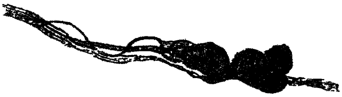
[illegible] F [illegible]

F

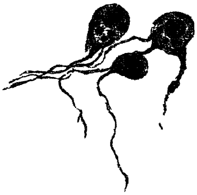




[illegible] H [illegible]



[illegible] ganglion [illegible] near [illegible]

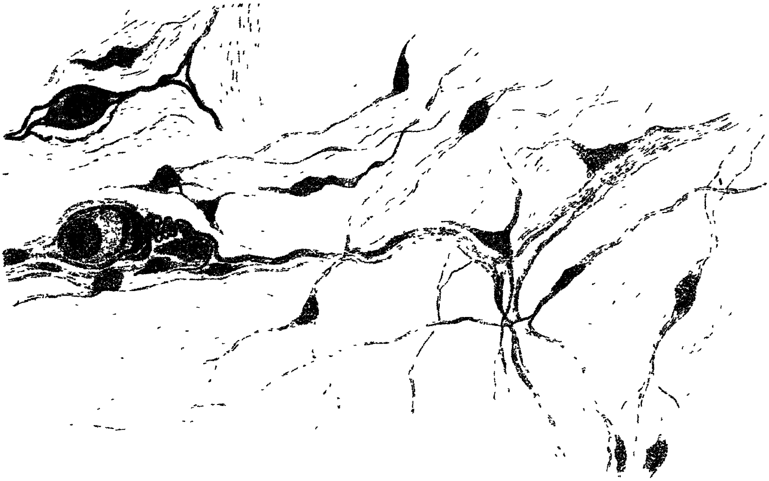


[illegible] gang [illegible] near [illegible] of th [illegible]
arterie of Abdomen Hy [illegible]

1000th of an inch × 00
1000th of an inch —— —— 1800

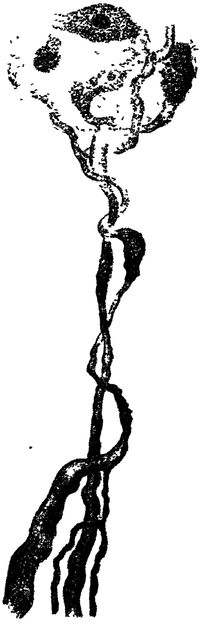
Fig 17

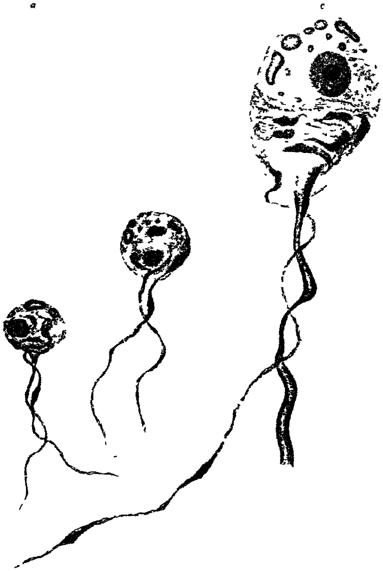
Fig 18



[illegible] Fig 17 Bladder Hyla. Fig 18 near Anterior [illegible] Fig 17 a younger cell [illegible] Fig 18 [illegible] this figure. The nerve fibres are *not connected with* the connective tissue corpuscles.

Fig





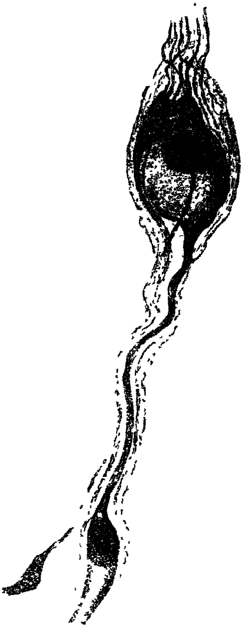
The lower part of a ganglion cell, with the nerve fibres running into it. The spiral fibre divides at the lower part of the figure into a *dark bordered fibre* and two fine fibres. Observe the nuclei in connection with the nerve fibres near their origin from the cell. From the same nerve as Fig I. × 1800

Three cells of different ages, from the same ganglion. Near lumbar nerves. Hyla arborea × 700. The arrangement and connections of the spiral fibre with numerous nuclei, are very distinct in the large figure. Observe the oil globules in the upper part of the cells.

1000th of an Inch × 700

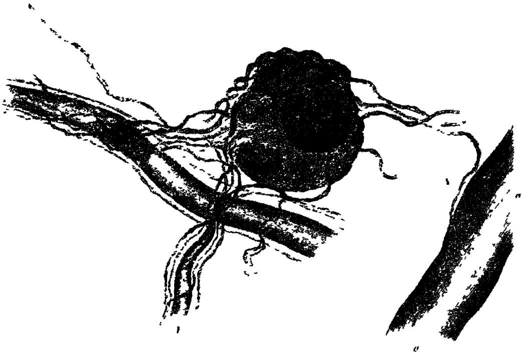
1000th of an Inch × 1800

Fig.

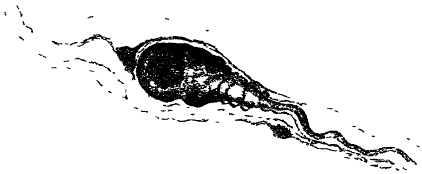


[illegible]

Fig.

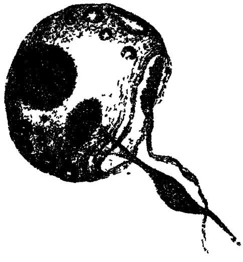


[illegible] with fibres connected with it passing off in three different directions. The large dark [illegible] of the fibres [illegible] a capillary [illegible] From the bladder of Hyla [illegible] 1800 [illegible] December, 1862



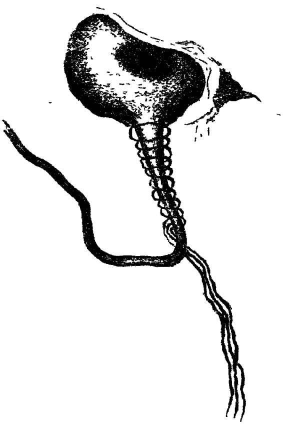
Ganglion cell and fibres in [illegible] 700 Frog 1861

Fig.



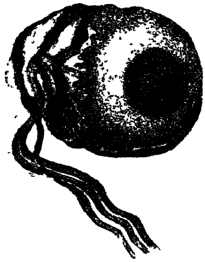
Ganglion cell from a ganglion close to lumbar nerves Hyla viridis × 1800

Fig.



Ganglion cell in connective tissue, near renal artery Frog
The fibres are seen to pursue opposite directions × 700

Fig.



Ganglion cell from ganglion. Posterior root of a spinal nerve Hyla.

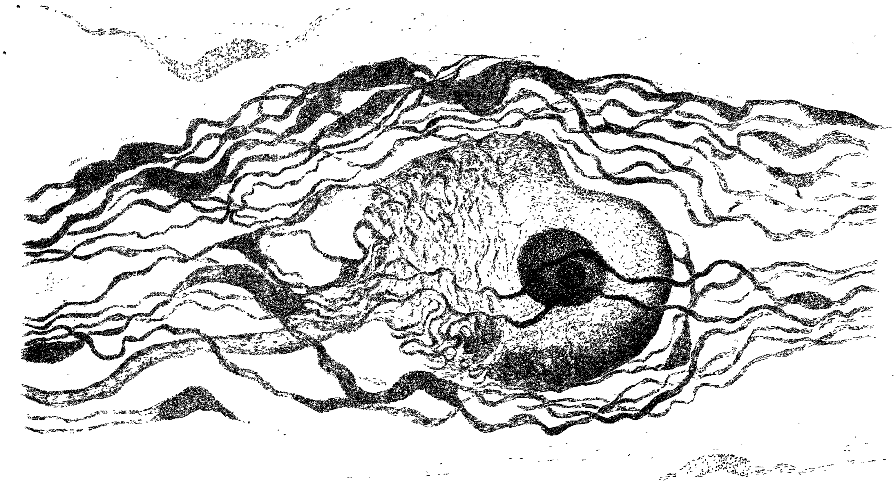
1000th of an inch ⊢——⊣ × 700

1000th of an inch ⊢——⊣ × 1800

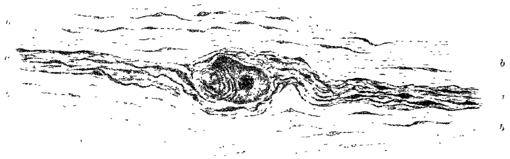
L. S. B. ad nat. del

[illegible]

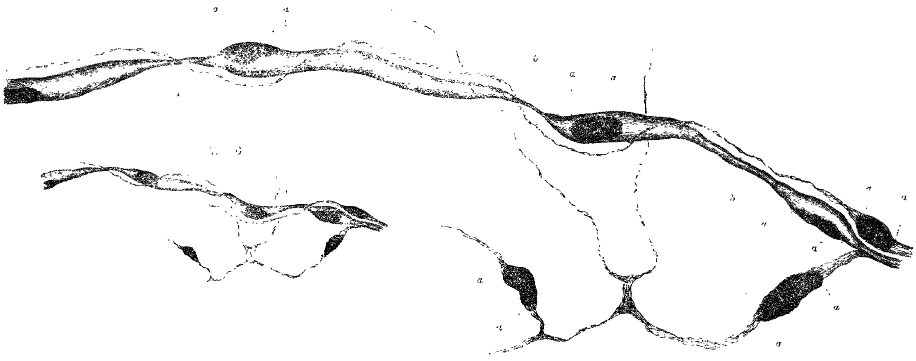
Fig. 27.



[illegible] lower part of the aorta. Frog. This is the central part of Fig. 28. × 1800



Ganglion cell, with [illegible] of very fine fibres, embedded in a [illegible] connective tissue [illegible] which was [illegible] a quarter of an inch in length. × 215



Dark-bordered fibre, with fine fibre passing *spirally* round it. Both fibres are nucleated. The 'nuclei' of the dark-bordered fibres are nearly equidistant, and much closer together than the nuclei of the spiral fibre. The dark bordered fibre might be said to consist of spindle-shaped cells or elementary parts continuous with each other. The fibre increases at the points marked *a*, *a*. The points marked *b* correspond to the parts of the fibre which were first formed, and are the oldest. Fig. 30 is supposed to represent the same fibres at an earlier period of development. From the bladder of the Hyla. × 1800. 1862

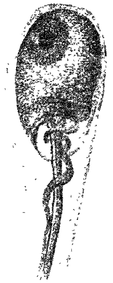
1000th of an Inch × 700
1000th of an Inch × 1800

L. S. B. ad nat. del.

Harrison's Impt.

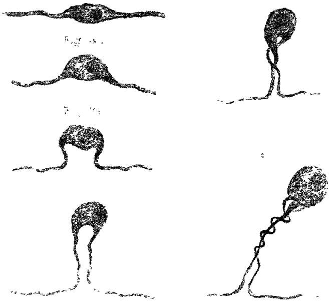
Phil. Trans. MDCCCLXIII. PLATE XXIII.

Fig

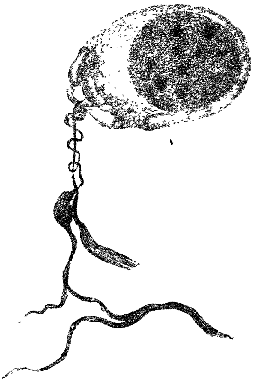


Ganglion cell in connective tissue, near the artery. Frog × 700

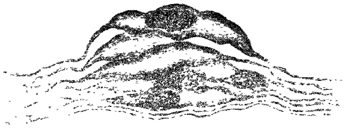
Fig 3?



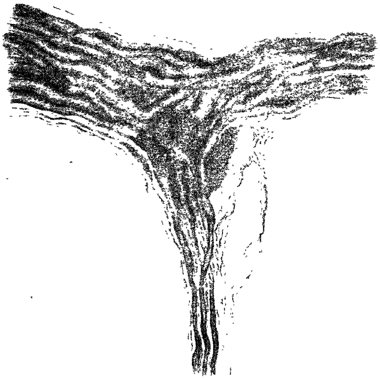
Drawings [illegible] which probably take place during the formation of the ganglion [illegible] straight and spiral fibre [illegible] Fig 32.



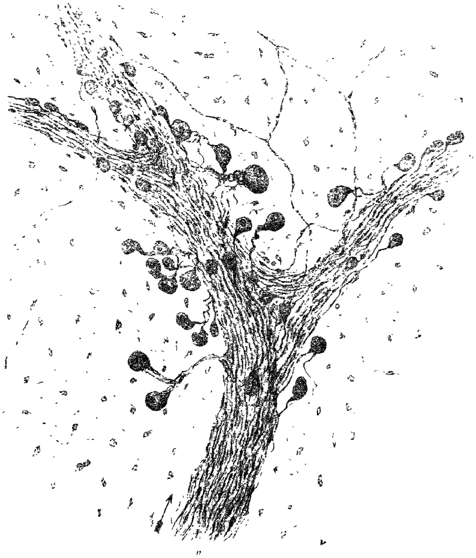
Young ganglion cell near lumbar nerves. Hyla. Observe the great size of the 'nucleus.' The spiral fibre is seen to divide into two branches, one of which runs with another nerve fibre × 1800



[illegible]



Fine compound nerve trunk, with a branch coming off at right angles, composed of fibres which pursue opposite courses in the trunk. From the submucous tissue of the Palate Frog × 700. 1862



Vagus nerve near the heart. Hyla. *a*, trunk of the nerve passing towards the heart. *b*, a branch which connects this with the nerve of the opposite side. The connective tissue, with its corpuscles, is also represented. *c*, two ganglion cells, more highly magnified in Fig. 42. *d*, two ganglion cells, the fibres of which, after meeting together, pursue different directions in the trunk of the nerve. × 215.

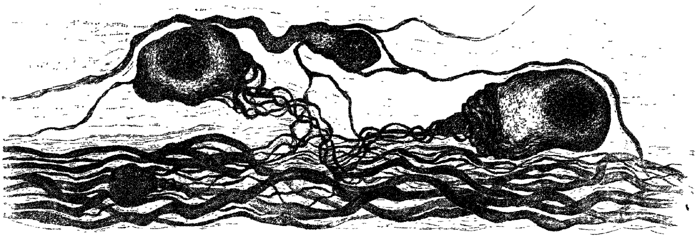
1000th of an inch × 700

1000th of an inch 1800

L. S. B. ad nat. del.

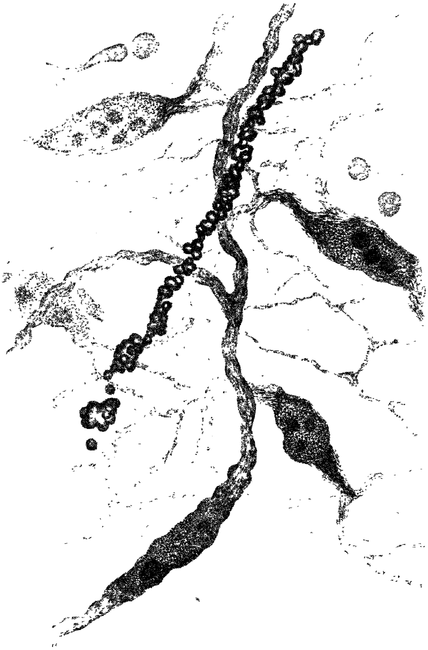
Harrisons Print

Fig 42



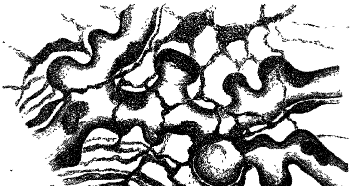
A portion of the trunk of the vagus, near the heart Hyla viridis With ganglion cells, the fibres of which are seen to pursue opposite directions in the trunk It is very difficult to follow the fibres for any distance In consequence of being so fine, their course is obscured by the larger fibres. × 1300.

Fig



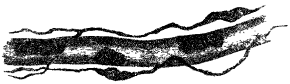
[illegible] fine nerve-fibre ramifying in the connective tissue at the base of the heart of the Hyla. [illegible] nerve-fibre with nucleus. Its relation to the connective tissue corpuscles and to the process of a pigment cell is well seen. × 2800.

Fig



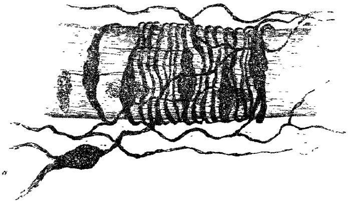
Capillary vessel with peculiar diverticula. Palate of Frog. Opposite a, the termination of a fine dark-bordered fibre is seen, and all over the field numerous branches of the terminal network of very fine pale nucleated nerve fibres are represented. × 700.

Fig

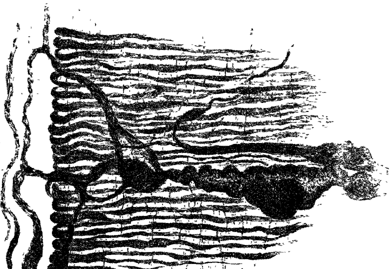


Fine nerves distributed external to a capillary. Bladder of Hyla. Observe the nuclei of the nerves and those of the capillary vessel. × 700.

Fig



A portion of a very small artery from the bladder of the Hyla. The muscular fibre cells, and the nerve fibres, a, are well seen. The fibres represented were traced from undoubted nerve fibres beyond a.



A portion of the coat of a branch of the Iliac artery of the Frog. Upon the surface external to the muscular fibres are seen some ganglion cells in process of development, with their fibres which ramify upon the muscular coat. Beneath a, a branch of a nerve fibre is seen to pass beneath a muscular fibre. × 700.

1000th of an inch × 700

I. A. R. I. 75.

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